1. Introduction

The main result of the paper is the following theorem:

**Theorem 1.2** (Dirac-Motzkin conjecture). Suppose that $P$ is a finite set of $n$ points in the plane, not all on one line. Suppose that $n \geq n_0$ for a sufficiently large absolute constant $n_0$. Then $P$ spans at least $\frac{n^2}{2}$ ordinary lines.

If $n$ is odd, Green and Tao show that there are actually at least $\frac{3}{4}n - O(1)$ ordinary lines, which is quite curious. The value of $n_0$ is not known though. It could be as low as 14, because the only configurations one has found with less than $\frac{n^2}{2}$ ordinary lines consist of 7 and 13 points. For 7 points, one can take the following configuration:

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Figure 1: 7 points with 3 ordinary lines
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It consists of a regular triangle together with the midpoints of its sides and its centroid. The ordinary lines are colored red. The configuration with 13 points and only 6 ordinary lines consists of two regular pentagons with one common side, the midpoint of that side and 4 points...
on the line at infinity (in the projective plane). It looks as follows:

Figure 2: 13 points with 6 ordinary lines

2. The Böröczky examples

In this section, we will study the Böröcky examples, which are very important for the study of ordinary lines. First, we define the following set of points in the projective plane:

\[ X_{2m} := \{ [\cos \frac{2\pi j}{m}, \sin \frac{2\pi j}{m}, 1] | 0 \leq j < m \} \cup \{ [\sin \frac{\pi j}{m}, \cos \frac{\pi j}{m}, 0] | 0 \leq j < m \} \]

So this means we have \( m \) points on the unit circle and \( m \) points on the line at infinity.
Let \( m \geq 2 \). There are four different kinds of Böröczky examples:

(i) The set \( X_{2m} \) contains \( 2m \) points and spans precisely \( m \) ordinary lines. It is straightforward to check this statement, so we will just illustrate the different kinds of Böröczky examples with some pictures, here for the case \( m = 4 \). The ordinary lines are exactly the tangent lines of the points on the unit circle to the unit circle.

(ii) The set \( X_{4m} \cup \{ [0, 0, 1] \} \) contains \( 4m + 1 \) points and spans precisely \( 3m \) ordinary lines, which consist of \( 2m \) tangent lines to the unit circle and \( m \) lines through the origin. Example: \( m = 2 \).

(iii) The set \( X_{4m} \setminus \{ [0, 1, 0] \} \) contains \( 4m - 1 \) points and spans precisely \( 3m - 3 \) ordinary lines, which consist of \( 2m - 2 \) tangent lines to the unit circle and \( m - 1 \) newly formed lines. Example: \( m = 3 \).
(iv) The set $X_{4m+2}$ minus any of the $2m+1$ points on the line at infinity contains $4m+1$ points and spans precisely $3m$ ordinary lines, which consist of $2m$ tangent lines to the unit circle and $m$ newly formed lines. Example: $m = 2$.

The Böröczky examples prove the following proposition:

**Proposition 2.1.** Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be the function defined by $f(2m) = m$, $f(4m + 1) = 3m$ and $f(4m - 1) = 3m - 3$. Then for $n \geq 6$ there is an example of a set of $n$ points in the projective plane which spans precisely $f(n)$ ordinary lines.

So the Böröczky examples give us for each $n$ a worst case example. But in fact, they are exactly the worst cases. Green and Tao prove the following theorem:

**Theorem 2.2** (Sharp threshold for Dirac-Motzkin conjecture). Let the function
Define \( f \) as above. Then there is a \( n_0 \) such that the following is true: If \( n \geq n_0 \) and if \( P \) is a set of \( n \) points in the projective plane, not all on a line, then \( P \) spans at least \( f(n) \) ordinary lines. Furthermore if equality occurs then up to a projective transformation, \( P \) is one of the B"or"oczky examples.

3. Melchior’s proof of the Sylvester-Gallai theorem

In this section, we will review Melchior’s proof of the Sylvester-Gallai theorem, since this will be the starting point for further considerations.

**Theorem 1.1 (Sylvester-Gallai).** Suppose that \( P \) is a finite set of points in the plane, not all on one line. Then \( P \) spans at least one ordinary line.

**Proof.** Let \( N_k \) denote the number of lines containing precisely \( k \) points of \( P \) for \( k \geq 2 \). We want to show that \( N_2 \geq 1 \).

We consider the dual configuration \( P^* = \{ p^* | p \in P \} \) of \( n \) lines in the projective plane. These lines determine a graph \( \Gamma_P \). Let \( V \) be the number of vertices (intersections of any of those \( n \) lines or equivalently, points \( l^* \), where \( l \) is a line joining two points of \( P \)), \( E \) the number of edges (line segments connecting two vertices without inner vertices) and \( F \) the number of faces in \( \Gamma_P \). By Euler’s Formula in the projective plane, we have

\[
V - E + F = 1 \tag{1}
\]

Because of the duality, it’s easy to see that

\[
V = \sum_{k=2}^{n} N_k \tag{2}
\]
Let $d(l_i^*)$ be the degree of the vertex $l_i^*$ in $\Gamma_P$. We get the following:

$$2E = \sum_{i=1}^{V} d(l_i^*) = 2 \sum_{i=1}^{V} \text{number of points on } l_i \text{ in } P = 2 \sum_{k=2}^{n} kN_k$$  \hspace{1cm} (3)

For $s \geq 3$, let $M_s$ be the number of faces in $\Gamma_P$ with precisely $s$ edges. It’s easy to see that these equations hold:

$$F = \sum_{s=3}^{n} M_s$$  \hspace{1cm} (4)

$$2E = \sum_{s=3}^{n} sM_s$$  \hspace{1cm} (5)

Now we put everything together and build the equation $3(1) - 3(2) + \frac{1}{2}(3) - 3(4) + (5)$, which gives us:

$$0 = 3 - 3 \sum_{k=2}^{n} N_k + \sum_{k=2}^{n} kN_k - 3 \sum_{s=3}^{n} M_s + \sum_{s=3}^{n} sM_s$$

This is equivalent to:

$$N_2 = 3 + \sum_{k=4}^{n} (k-3)N_k + \sum_{s=4}^{n} (s-3)M_s$$  \hspace{1cm} (6)

Now we’re done, since that implies $N_2 \geq 3$.

To move on, we need the following proposition:

**Proposition 3.1** (Few bad edges). Suppose that $P$ is a set of $n$ points in the projective plane, not all on a line, and suppose that $P$ has at most $Kn$ ordinary lines. Consider the planar graph $\Gamma_P$ obtained by dualising $P$ as described above.
Then $\Gamma_P$ is an “almost triangulation” in the following sense. Say that an edge of $\Gamma_P$ is good if both of its vertices have degree 6, and if both faces it adjoins are triangles. Say that it is bad otherwise. Then the number of bad edges in $\Gamma_P$ is at most $16Kn$.

Proof. For $s \geq 4$, we have $s \leq 4(s - 3)$, and together with (6) it follows:

$$\sum_{s=4}^{n} sM_s \leq 4 \sum_{s=4}^{n} (s - 3)M_s \leq 4N_2 \leq 4Kn$$

Further, by using $d(l^*) = 2|P \cap l|$ and $k \leq 4(k - 3)$ for $k \geq 4$, we get:

$$\sum_{l: d(l^*) > 6} d(l^*) = \sum_{l: |P \cap l| > 3} 2|P \cap l| = n \sum_{k=4}^{n} 2kN_k \leq 8 \sum_{k=4}^{n} (k - 3)N_k \leq 8N_2 \leq 8Kn$$

Also, we have:

$$\sum_{l: d(l^*) = 4} d(l^*) = \sum_{l: |P \cap l| = 2} 2|P \cap l| = 4N_2 \leq 4Kn$$

Now we’re ready to place an upper bound on the number of bad edges. Since each face with $s > 3$ sides gives $s$ bad edges and each vertex $l^*$ with $d(l^*) \neq 6$ gives $d(l^*)$ bad edges, and these are the only sources of bad edges, we get:

$$\text{number of bad edges} \leq \sum_{s=4}^{n} sM_s + \sum_{l: d(l^*) \neq 6} d(l^*) \leq 16Kn$$

\[\square\]

4. Triangular structure in the dual and cubic curves

Again, we have a set of points in the plane called $P$. In the following section, we try to extract some structure out of this point arrangement. In the paper, they achieved that by looking at cubic curves, using a proposition already proven by Chasles. This proposition allows to place a huge amount of points on not so many cubic curves if there are not that many ordinary lines. It even is possible to place a huge amount of the points on one single cubic curve, which in the following chapters then allows us to analyze the structure.

First, they defined a cubic curve: $\gamma$ is called a cubic curve if

$$\gamma = \{[X,Y,Z] : a_1X^3 + a_2X^2Y + a_3XY^2 + a_4Y^3 + a_5X^2Z + a_6XYZ + a_7Y^2Z + a_8XZ^2 + a_9YZ^2 + a_{10}Z^3 = 0\}$$

for $a_1, \ldots, a_{10} \in \mathbb{R}$ not all zero. That is, $\gamma$ is the locus of some homogeneous polynomial in 3 variables of degree 3. Remark that the coordinates $[X,Y,Z]$ denote coordinates in the projective space, i.e. the equivalence class of the vector $(X,Y,Z)$. To visualize those cubic curves, remember that we are allowed to
scale an element of the projective space arbitrarily without changing its equivalence class, and we see that it after scaling remains in the locus. Thus, for a better understanding how these cubic curves look like, scale the points such that $Z = 1$. You get the usual locus of a cubic polynomial in two variables on the plane given by $Z = 1$, of which the cubic curve in the projective space (roughly) is the projection.

![Figure 7: A cubic curve in $\mathbb{RP}^2$ as projection of a usual cubic curve.](image)

As mentioned, the goal is to use those cubic curves to extract some sort of structure out of our point arrangement. This will be achieved with the following proposition of Chasles:

**Proposition 4.1 (Chasles).** Suppose that two sets of three lines define nine distinct points of intersection in $\mathbb{RP}^2$. Then any cubic curve passing through eight of these points also passes through the ninth.

![Figure 8: Sketch of the situation in Chasles proposition](image)
Since the previous results concern the dual of our point set, it is not a farfetched idea to look at the dual of the configuration we have in the proposition. For our example above, the dual looks as follows:

![Figure 9: Dual of Figure 8](image)

We see that this example contains some sort of triangular structure. Indeed, using Proposition 3.1 for bounding the number of bad edges, they found that a huge part of the dual consists of some triangular structure if we have not so many ordinary lines. To analyze this more formally, they defined a triangular grid as follows:

**Definition 4.1 (Triangular grid).** Let $I, J, K$ be three discrete intervals in $\mathbb{Z}$, i.e. they have the form $\{a_-, a_- + 1, \ldots, a_+\}$. A collection of lines $(p^*_i)_{i \in I}, (q^*_j)_{j \in J}, (r^*_k)_{k \in K}$ in $\mathbb{RP}^2$ is called triangular grid with dimensions $I, J, K$ if it fulfills

(i) For $i \in I, j \in J, k \in K$ with $i + j + k = 0$, the lines $p^*_i, q^*_j, r^*_k$ are distinct and meet at a point $P_{ijk}$, which is not incident to any other line of this collection that is not identical to $p^*_i, q^*_j, r^*_k$.

(ii) If $i \in I, j, j' \in J, k, k' \in K$ with $i + j + k = 0 = i + j' + k'$ and $0 < |j - j'| \leq 2$, then $P_{ijk} \neq P_{ij'k'}$. Similarly for all permutations of $i, j, k$ and $j', k'$.

Clearly, this definition is fulfilled for Figure 9 with intervals $I = J = K = \{-1, 0, 1\}$, at least if we reverse the ordering of $r_j$ in their picture. More generally, even greater triangular grids may fulfill those conditions, as seen in Figure 10, if we multiply the indices of $p_i$ by $-1$. 
Now we come to mention why they looked at such triangular grids. The most important Lemma is the following transcription of Chasles Proposition:

**Lemma 4.3.** Let \(i_0, j_0, k_0\) be integers with \(i_0 + j_0 + k_0 = 0\), and define the intervals \(I = \{i_0 - 1, i_0, i_0 + 1\}\), \(J = \{j_0 - 1, j_0, j_0 + 1\}\) and \(K = \{k_0 - 1, k_0, k_0 + 1\}\). If \((p_i)_{i \in I}, (q_j)_{j \in J}\) and \((r_k)_{k \in K}\) are triples of points such that their dual forms a triangular grid of dimension \(I, J, K\), then any cubic curve which passes through eight of them passes through the ninth.

**Proof.** First, we can assume \(i_0 = 0, j_0 = 0, k_0 = 0\), because we can relabel the points by just substracting \(i_0\) from every index of every point \(p_i, q_j\) and \(r_k\). (and the sum of our new indices clearly remains 0).

Now look at the dual; as the dual is a triangular grid, for any points \(p_i, q_j, r_k\) such that \(i+j+k=0\), we know that the dual lines \(p_i^*, q_j^*, r_k^*\) intersect in one point \(P_{i,j,k}\). Back in the original space, this point corresponds to a line \(l_{i,j,k}\) through the points \(p_i, q_j, r_k\).

This means that we have six lines \(l_{0,1,-1}, l_{0,-1,1}, l_{1,0,-1}, l_{1,-1,0}, l_{-1,0,1}, l_{-1,1,0}\), and each of those 9 points is intersection point of two of those lines, which we see regarding the indices. For example \(p_0\) is intersection point of \(l_{0,1,-1}\) and \(l_{0,-1,1}\). Hence, if those nine points all are different, then by Chasles, every line through eight of those points also go through the ninth, as we can divide our set of six lines into two set of three lines defining those nine points.

Thus, we have to prove that those points are different. If we look at \(i = 0\), we see that for any \(j \in J, k \in K\) such that \(j + k = 0\), we get a point \(P_{i,j,k}\). For no two \(j, j' \in \{-1, 0, 1\}\) those points are the same because of the second condition for being a triangular grid. Thus, \(p_i^*\) crosses \(q_j^*\) at another place than \(q_{j'}^*\), i.e. \(q_j \neq q_{j'}\). Hence \(q_{-1}, q_0, q_1\) are different. The same can be done for \(r_{-1}, r_0, r_1\) and \(p_{-1}, p_0, p_1\).

Also, for any \(i + j \leq 1\), we can find a \(k\) such that \(i + j + k = 0\). Then the first part of the definition of triangular grid implies that \(p_i, q_j, r_k\) are distinct.
points, i.e. \( p_i \neq q_j \). Looking at all permutations of variables, the last case that is left to show is \( p_{-1} \neq q_1 \). Clearly, if they were equal, then \( P_{1,-1,0} \) and \( P_{-1,1,0} \) would both be intersection point of \( r_0^* \) and \( p_{-1} = q_1 \), hence \( P_{1,-1,0} = P_{-1,1,0} \), contradicting the second condition for being a triangular grid.

Clearly, one point more or less on a cubic curve does not give the result we are looking for. To strengthen their result, they saw that even the whole triangular grid can be covered by one cubic curve:

**Lemma 4.4.** Suppose that \( m \geq 4 \) is an integer and \( i_-, i_+ \in \mathbb{Z} \) such that \( 2 \leq i_+ \leq m - 2 \) and \( 2 - m \leq i_- \leq -2 \). If we have a collection of points \((p_i)_{i_- \leq i \leq i_+}, (q_j)_{-m \leq j \leq -1}, (r_k)_{1 \leq k \leq m}\) whose dual forms a triangular grid, then all those points lie on a single cubic curve.

**Proof.** For the proof of this, they started with a cubic curve that passes through \( p_{-1}, p_0, p_1, p_2, q_{-3}, q_{-2}, q_{-1}, r_1, r_2 \). Clearly, such a curve exist as the space of homogenous cubic polynomials has dimension 10. (Remember: a homogeneous cubic polynomial has form \( a_1 X^3 + a_2 X^2 Y + a_3 X Y^2 + a_4 Y^3 + a_5 X^2 Z + a_6 X Y Z + a_7 Y^2 Z + a_8 X Z^2 + a_9 Y Z^2 + a_{10} Z^3 \).)

Now, we iteratively apply Lemma 4.3. In the first step, we look at the points \( p_{-1}, p_0, p_1, q_{-3}, q_{-2}, q_{-1}, r_1, r_2, r_3 \). As our cubic curve passes through each point except \( r_3 \), that is through eight of those points, by Lemma 4.3 it passes also through \( r_3 \).

Hence, for \( k = 3 \), we know that \( q_{-j}, r_j \) lie on the cubic curve for \( 1 \leq j \leq k \). For each \( k \geq 3 \), we inductively look at the points \( p_0, p_1, p_2, q_{-(k+1)}, q_{-(k-1)}, r_{k-2}, r_{k-1}, r_k \) to see that \( q_{k+1} \) lies on the cubic curve by Lemma 4.3. Similarly, looking at \( p_0, p_1, p_2, q_{-k}, q_{-(k+1)}, q_{-(k-2)}, r_{k-1}, r_k, r_{k+1} \) gives us the same for \( r_{k+1} \).

Hence, all \( r_k, q_j \) lie on this cubic curve as wanted. For the \( p_i \), one can similarly look at \( p_{-1}, p_i, p_{i+1}, q_{-i-3}, q_{-i-2}, q_{-i-1}, r_1, r_2, r_3 \) for \( i \in \{2, \ldots, i_+ - 1\} \) and \( P_{-(i-1)}, P_{-(i+1)}, q_{-3}, q_{-2}, q_{-1}, r_{i-1}, r_{i-2}, r_{i-3} \) for \( i \in \{1, \ldots, i_- - 1\} \).

This then finishes the proof.

5. Almost triangular structure and covering by cubics

With those main statements about triangular grids, we are now ready to look out for triangular grids. Remember that they called an edge good if both of its vertices has degree 6 and both faces adjoining it are triangles. They now called an edge really good if all paths of length two starting in one of its endpoints consist entirely of good edges, and somewhat bad if it is not really good.

Remark that a really good edge gives us our triangular structure: on both sides of the edge, we have a triangle, and as all edges of the triangle are in paths of length 2, they have to be good as well, which then extends our triangular structure. To visualize this, just look at the case where we only require all paths of length 1 to be good in Figure 11:
Figure 11: An edge \((a, b)\) such that each edge starting in \(a\) or \(b\) is good.

We see that this definition of really good edge gives us the structure in Figure 9, where the points were all on one cubic curve. It is even true that if we have a segment of \(l\) consecutive really good edges, we have a triangular grid of dimensions \([-2, \ldots, 2]; \{-l - 4, \ldots, -1\}; \{1, \ldots, l + 4\}\), which then is contained in one cubic curve by Lemma 4.4.

On the other hand, we know that the number of bad edges is really small if we have few ordinary lines, thus we expect a huge amount of really good edges, which then allows us to place many points on few cubic curves. This then gives next Proposition:

**Proposition 5.1** (Cheap structure theorem). Suppose that \(P\) is a finite set of \(n\) points in the plane. Suppose that \(P\) spans at most \(Kn\) ordinary lines for some \(K \geq 1\). Then \(P\) lies on the union of \(500K\) cubic curves.

**Proof.** The proof starts with disposing the degenerate case: Assume we have a line in the dual that meets fewer than \(500K\) intersection points. We know that each other line intersects it in one of these points, hence one of those intersection points is contained in every line in the dual. In the original space, this corresponds to less than \(500K\) lines that contain all points of \(P\). Since every line is also a cubic curve, we are done in this case.

Otherwise, every line meets more than \(500K\) intersection points. In this case, they analyzed the number of somewhat bad edges. Remark that for every somewhat bad edge there is a path of length two or less that contains a bad edge and starts at the endpoint of the edge. This is why they associated to each somewhat bad edge the shortest path starting with this edge that ends in a bad edge. Either the edge itself is bad, or otherwise, those paths start with 1 or 2 good edges and end in a bad one. For each bad edge, we can bound the number of such paths it appears in: there are at most 5 paths of the form \(bad - good\) as the midpoint would have degree 6. Analogously, there are at most 25 paths of the form \(bad - good - good\). As this holds for any bad edge, the amount of such paths altogether is bounded by \((5 + 25) \cdot 16Kn + 16Kn \leq 500Kn\). This then directly bounds the number of somewhat bad edges which were associated to such paths.

Hence, one out of the \(n\) dual lines has to contain less than \(500K\) bad edges by pigeon hole principle. Call it \(p^*\). These edges partition \(p^*\) into at most \(500K\) partitions of consecutive really good edges, where a point between two somewhat bad edges is looked at as a partition of length 0.
As every dual line $q^*$ intersects $p^*$ in one of these partitions, we can for each partition of length 0 choose the line corresponding to the intersection point, while for partitions of length bigger 1, we know that one cubic curve is enough to cover all nodes intersecting there. As lines are a special case of a cubic curves, we achieved a covering of $P$ by 500$K$ cubic curves as we wanted.

This proposition might tell us something about the structure, but in the paper, they even estimated the number of points on each irreducible component of the cubic curves. For each of the presented statements, they also mentioned a stronger version, which finally let them achieve the following, much stronger proposition:

**Proposition 5.3** (Intermediate structure theorem). *Suppose that $P$ is a finite set of $n$ points in the plane. Suppose that $P$ spans at most $Kn$ ordinary lines for some $K \geq 1$. Then one of the following three alternatives holds:

(i) $P$ lies on the union of an irreducible cubic $\gamma$ and an additional $2^{75}K^5$ points.

(ii) $P$ lies on the union of an irreducible conic $\sigma$ and an additional $2^{64}K^4n$ lines. Furthermore, $\sigma$ contains between $\frac{n}{2} - 2^{76}K^5$ and $\frac{n}{2} + 2^{76}K^5$ points of $P$, and $P \setminus \sigma$ spans at most $2^{82}K^4n$ ordinary lines.

(iii) $P$ lies on the union of $2^{16}K$ lines and an additional $2^{87}K^6$ points.

6. Ordinary points in unions of lines

This part of the paper shows how one can reduce the number of lines in the previous structure theorem stated in Proposition 5.3. The proofs in this section are fairly technical, which is why they were not covered in the presentation. However, all that is needed in the later sections is the following result.

**Proposition 6.1.** *Suppose that a set $P \subset \mathbb{P}^2$ of size $n$ lies on a union $l_1 \cup \ldots \cup l_m$ of lines, and that $P$ spans at most $Kn$ ordinary lines. Suppose that $n \geq n_0(m, K)$ is sufficiently large. Then all except at most $3K$ of the points of $P$ lie on a single line.*

*We can take the function $n_0$ to be of the form $n_0(m, K) = K \exp(\exp(Cm))$.*

The proof is separated into different Propositions and Lemmas. In **Proposition 6.2** almost all of the points (namely at least $\frac{2}{3}n$) are on one line. In contrast to this, in **Proposition 6.3** every line contains at least a certain multiple $cn$ of the points. The latter case has to be separated into the situation, when all lines are concurrent (**Proposition 6.4**) and when they are not concurrent (**Proposition 6.6**).

Using Proposition 6.1 we are able to reduce the number of lines in Proposition 5.3 and prove a stronger structure theorem.
Proposition 6.13. Suppose $P \subset \mathbb{P}^2$ is a set of size $n > 100$ spanning at most $Kn$ ordinary lines, where $1 \leq K \leq c(\log(\log(n))^c$ for some sufficiently small absolute constant $c$. Then $P$ differs in at most $O(K^{O(1)})$ points from a subset of a set of one of the following three types:

(i) A line;

(ii) The union of an irreducible conic and a line;

(iii) An irreducible cubic curve.

In the proof we simply use the fact, that in the second and third case of Proposition 5.3 the points lying on lines still span few ordinary lines.

To formulate and understand the next results, we need a short excursion.

Excursion: Group structure on irreducible cubic curves

Let $\gamma = \{p \in \mathbb{P}^2; f(p) = 0\}$ be an irreducible cubic curve (i.e. $f \in \mathbb{R}[x, y, z]$ homogeneous of degree 3, $f$ cannot be decomposed into a product of two polynomials of strictly smaller degree). Let $\gamma^*$ be the smooth points on $\gamma$, that is points where grad($f$) does not vanish. If all the points of $\gamma$ are smooth, it is called a smooth or non-singular curve, otherwise it is called singular. Take 0 to be a point of inflection on $\gamma^*$, that is a point, where the tangent to the curve meets the curve to order 3. We want to define an abelian group structure $\oplus$ on $\gamma^*$ with the following property:

$P, Q, R$ are the intersection points (with multiplicity) of $\gamma^*$ with a line $l$ \iff $P \oplus Q \oplus R = 0$.

This is possible with the following construction: for $P, Q \in \gamma^*$ distinct points, the line $PQ$ through $P$ and $Q$ meets $\gamma^*$ in a third point $R$ (which may coincide with either $P$ or $Q$). As we want $P \oplus Q \oplus R = 0$ we have $R = \ominus P \ominus Q$ Taking the line $OR$ through the neutral element and $R$, we get again a third intersection point $R'$ satisfying $R' = \ominus O \ominus R = P \oplus Q$. The case above is the most general (or non-degenerate) case. If some of the points $P, Q, R, O, R'$ coincide, we get a slightly different picture (for instance the “line through $P$ and $P'$” is the tangent line to $\gamma$ at $P$). All possible cases are listed in the following picture (where 0 is the point at infinity in direction of the $y$-axis):

It is possible to prove that this operation makes $(\gamma^*, \oplus)$ into an abelian group.
In fact it is possible to classify it as a (topological) group:

\[(\gamma^*, \odot) \cong \mathbb{R} \text{ or } \mathbb{R} \times (\mathbb{Z}/2\mathbb{Z}) \iff \gamma \text{ is singular,} \]

\[(\gamma^*, \odot) \cong \mathbb{R}/\mathbb{Z} \text{ or } (\mathbb{R}/\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \iff \gamma \text{ is non-singular.} \]

Finally note that by a projective transformation, every smooth irreducible cubic curve can be transformed into a curve in Weierstrass normal form, that is a curve of the form

\[\gamma = \{[x, y, z] \in \mathbb{P}^2; y^2z = x^3 + axz^2 + bz^3\} \]

with real constants \(a, b \in \mathbb{R}\) satisfying \(4a^3 + 27b^2 \neq 0\).

7. The detailed structure theorem

With these new definitions we are able to state the strongest structure theorem in this presentation.

**Theorem 7.1.** Suppose \(P \subset \mathbb{P}^2\) is a set of size \(n\) spanning at most \(Kn\) ordinary lines for some \(K \geq 1\) and suppose \(n \geq \exp(\exp(CK^C))\) for some sufficiently large absolute constant \(C\). Then, after applying a projective transformation if necessary, \(P\) differs by at most \(O(K^2)\) points from an example of one of the following three types:

(i) \(n - O(K^2)\) point on a line;

(ii) The set \(X_{2m}\) for some \(m = \frac{1}{2}n - O(K^2)\);

(iii) A coset \(H \oplus x, 3x \in H\), of some finite subgroup \(H\) of the real points on an elliptic curve \(E\) in Weierstrass normal form, with \(H\) having cardinality \(n + O(K^2)\).

Note that the polynomial bounds \(O(K^2)\) can be improved to linear bounds \(O(K)\) (see Theorem 1.5). In the following we give an outline of the proof of Theorem 7.1. However the Theorem itself is enough to prove the Dirac-Motzkin conjecture, so anyone only interested in this final step of the proof may continue with section 8.

The proof proceeds by looking closer at the cases 2. and 3. in Proposition 6.13. In both cases we apply the following Lemma from additive combinatorics using either the group structure on an irreducible cubic or a pseudo-group structure on a conic together with a line.

**Proposition A4.** Suppose that \(A, B, C\) are three subsets of some abelian group \(G\), \(n - K \leq |A|, |B|, |C| \leq n + K\) for some \(n, K \geq 1\). Suppose that \(K = o(n)\). Suppose that there are at most \(Kn\) pairs \((a, b) \in A \times B\) for which \(a + b \notin C\). Then there is a subgroup \(H\) of \(G\) and cosets \(x + H, y + H\) such that

\[|\Delta(x + H)|, |\Delta(y + H)|, |\Delta(x + y + H)| \leq 7K.\]

As the group structure on the cubic curve is canonically given by the operations described in the excursion, we begin with the analysis of the third case in Proposition 6.13. Note that in the proof given in the paper you will need the additional fact that given a set of \(n\) points, adding or deleting \(K\) points changes the number of ordinary lines spanned by the resulting set by at most \(O(Kn + K^2)\). The proof of this fact is a straightforward induction.
Lemma 7.2. Suppose $P \subset \mathbb{P}^2$ is a set of size $n$ spanning at most $Kn$ ordinary lines for some $K \geq 1$. Suppose that all but $K$ of the points of $P$ lie on an irreducible cubic curve $\gamma$, and suppose that $n > CK$ for some suitably large absolute constant $C$. Then $\gamma$ is smooth, thus an elliptic curve. After placing it in Weierstrass normal form by a projective transformation, there is a coset $H \oplus x$ of $\gamma$ with $3x = x \oplus x \oplus x \in H$ such that $|P \Delta (H \oplus x)| = O(K)$.

The proof of this Lemma uses the fact, that in order to bound the number of ordinary lines by $Kn$, for almost every pair of points $(a, b) \in (P \cap \gamma^*)^2$ the line joining $a$ and $b$ must meet $\gamma^*$ in a third point which lies on $P$ (otherwise it is an ordinary line). Group theoretically this gives exactly the condition of Proposition A4. Thus $P \cap \gamma^*$ differs from a coset $H \oplus x$ of a finite subgroup $H$ of $\gamma^*$ by few points. Using the fact that different cosets of the same subgroup are disjoint, we obtain $3x \in H$ and the existence of a large but finite subgroup $H \subset \gamma^*$ implies that $\gamma$ is smooth due to the group theoretic classification of $(\gamma^*, \oplus)$ given at the end of the excursion.

In order to apply the same kind of argument to the second situation in Proposition 6.13, we need a replacement for the group structure on a cubic curve. This is given by the following Proposition.

Proposition 7.3. Suppose that $\sigma$ is an irreducible conic and that $l$ is a line. Then there is an abelian group $G = G_{\sigma,l}$ with operation $\oplus$ and bijective maps $\psi_{x} : G \to \sigma^*$, $\psi_{l} : G \to l^*$ such that $\psi_{x}(x), \psi_{x}(y)$ and $\psi_{l}(z)$ are collinear precisely if $x \oplus y \oplus z = 0$. Furthermore $G_{\sigma,l}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{R}$ if $|\sigma \cap l| = 2$, to $\mathbb{R}$ if $|\sigma \cap l| = 1$ and to $\mathbb{R}/\mathbb{Z}$ if $|\sigma \cap l| = 0$. In the last case, we can apply a projective transformation such that $l =$ line at infinity and $\sigma = $ unit circle and we can take

$$\Psi_{\sigma} : \mathbb{R}/\mathbb{Z} \to \sigma, t \mapsto [\cos(2\pi t), \sin(2\pi t), 1],$$

$$\Psi_{l} : \mathbb{R}/\mathbb{Z} \to l, t \mapsto [\cos(2\pi t), \sin(2\pi t), 0].$$

Thus with basically the same argumentation as in Lemma 7.2, we obtain the following result, which finishes the proof of Theorem 7.1.

Lemma 7.4. Suppose $P \subset \mathbb{P}^2$ is a set of size $n$ spanning at most $Kn$ ordinary lines for some $K \geq 1$ and $n > n_0(K)$. Suppose further that all except $K$ of the points lie on the union of an irreducible conic $\sigma$ and a line $l$ such that $n/2 + O(K)$ of the points lie on each of $\sigma, l$. Then, after a projective transformation, $P$ differs from one of the sets $X_n^\sigma$ at most $O(K)$ points.

8. The Dirac-Motzkin conjecture

Now we want to see how Theorem 7.1 implies the Dirac-Motzkin conjecture for large $n$. As in the presentation, we give a proof of a slightly weaker statement, which is easily obtained. For the proof, we need the following Lemma.

Lemma 8.1. Suppose that $P \subset \mathbb{P}^2$ differs in $K$ points from a coset $H \oplus x$ of a subgroup $H$ of some elliptic curve $\gamma$ in Weierstrass normal form, where $3x = x \oplus x \oplus x \in H$. Then $P$ spans at least $n - O(K)$ ordinary lines.

The proof of this Lemma uses the fact, that for a coset $H \oplus x$ as described above, the tangent lines to all $n$ points are ordinary due to group theoretic
arguments and that adding or deleting a bounded number of points can only destroy a bounded number of these ordinary lines (as from a given point there are at most 6 tangent lines to a cubic curve). Now we formulate and prove in detail a slightly weaker version of the Dirac-Motzkin conjecture.

**Theorem.** There exists an absolute constant $n_0 \in \mathbb{N}$ such that the following is true: suppose that $P \subset \mathbb{P}^2$ is a set of $n$ points, not all on one line for some $n > n_0$. Then $P$ spans at least $\frac{n}{2} - O(1)$ ordinary lines.

**Proof.** If $P$ spans more than $n$ ordinary lines, we are already done. Assume this is not the case, then we can apply Theorem 7.1 with $K = 1$ for $n$ large enough. After a projective transformation we are in one of the cases of Theorem 7.1.

In case 1. we have a subset $P' \subset P$ of $n - O(1)$ points on one line. As $P$ is not collinear, we have some point $p \in P \setminus P'$ then $P' \cup \{p\}$ spans $n - O(1)$ ordinary lines and each of the additional $O(1)$ points in $P \setminus P'$ can only destroy one of the lines. Thus we have $n - O(1)$ ordinary lines, which for large $n$ is bigger than $\frac{n}{2} - O(1)$.

In case 2. $P$ differs in at most $O(1)$ points from a set $X_{2m}$ for some $m = \frac{n}{2} - O(1)$. The set $X_{2m}$ itself spans already $\frac{n}{2} - O(1)$ ordinary lines, which are all tangent lines to the unit circle. Removing a point can at most destroy the ordinary line through this point, that is one line. Adding a point can destroy at most 2 lines, as from a given point there are at most 2 tangent lines to the unit circle going through this point. Thus the $O(1)$ points, in which $P$ differs from $X_{2m}$ can only destroy $O(1)$ of the ordinary lines of $X_{2m}$ and thus we have $\frac{n}{2} - O(1)$ ordinary lines left.

For the last case, we immediately get the result by applying Lemma 8.1.

As we have seen above, the only case where the number of ordinary lines could be less than $\frac{n}{2}$ is case 2. In order to get the full strength of the asymptotic Dirac-Motzkin conjecture, we apply the following Proposition.

**Proposition 8.2.** There is an absolute constant $C$ such that the following is true. Suppose that $P \subset \mathbb{P}^2$ differs from $X_{2m}$ in at most $K$ points, and that $P$ spans at most $4m - CK$ ordinary lines. Then $P$ is a B"or"oczky example or a near-B"or"oczky example.

In our situation, we take $K$ to be the implicit bound encoded in $O(1)$ in case 2. of Theorem 7.1. For $m$ large enough we have $4m - CK > 2m$. Thus for $n$ large enough assuming $P$ spans at most $2m = n - O(1)$ ordinary lines, we can apply the above Proposition and see that $P$ is a B"or"oczky example or a near-B"or"oczky example. As both types do not violate the Dirac-Motzkin conjecture (as seen in the first part of the paper), the proof is complete.

**Sources**