Cyclic Polytope

The convex hull of n distinct points on the moment curve $\{m(t) = (t^1, t^2, \ldots, t^d)^T : t \in R\}$ in R^d is known as a *cyclic polytope*. It is known that its combinatorial structure (i.e. its face lattice) is uniquely determined by n and d. Thus we often write $c(d, n)$ to denote any polytope combinatorially equivalent to a cyclic d-polytope with n vertices.

Let $f_k(P)$ denote the number of k-faces in a polytope P. McMullen's Upper Bound Theorem shows that the maximum of $f_k(P)$ over all d-polytopes P with n vertices is attained by the cyclic polytopes for all values of $k = 1, 2, \ldots, d-1$.

Theorem 1 (Upper Bound Theorem) *For any d-polytope with n vertices,*

$$f_k(P) \le f_k(c(d, n)), \ \forall k = 1, \ldots, d-1,$$

holds.

The number of k-faces of a cyclic polytope $c(d, n)$ can be explicitely given and thus one can evaluate the order of the upper bound in terms of n and d.

Theorem 2 *For $d \ge 2$ and $0 \le k \le d-1$,*

$$f_k(c(d,n)) = \sum_{r=0}^{\lfloor d/2 \rfloor} \binom{r}{d-k-1} \binom{n-d+r-1}{r} + \sum_{r=\lfloor d/2 \rfloor + 1}^{d} \binom{r}{d-k-1} \binom{n-r-1}{d-r}.$$

In particular, by using the binomial identity

$$\binom{p+q+1}{q} = \binom{p+q}{q} + \binom{p+q-1}{q-1} + \cdots + \binom{p+1}{1} + \binom{p}{0},$$

we have

$$\begin{aligned} f_{d-1}(c(d,n)) &= \binom{n - \lceil \frac{d}{2} \rceil}{n-d} + \binom{n - \lfloor \frac{d}{2} \rfloor - 1}{n-d} \\ &= \binom{n - \lceil \frac{d}{2} \rceil}{\lfloor \frac{d}{2} \rfloor} + \binom{n - \lfloor \frac{d}{2} \rfloor - 1}{\lceil \frac{d}{2} \rceil - 1} \\ &= O(n^{\lfloor \frac{d}{2} \rfloor}) \quad \textit{for any fixed } d. \end{aligned}$$

For example,

P	f_0	f_1	f_2	f_3	f_4
$c(5, 10)$	10	45	100	105	42
$c(5, 20)$	20	190	580	680	272
$c(5, 30)$	30	435	1460	1755	702

The upper bound theorem can be written in dual form which gives, for example, the maximum number of vertices in a d-polytope with m facets.

Theorem 3 (Upper Bound Theorem in Dual Form) *For any d-polytope with m facets,*

$$f_k(P) \le f_{d-k-1}(c(d, m)), \ \forall k = 0, 1, \ldots, d-2,$$

holds.

The original proof of the Upper Bound Theorem is in [2, 3]. There are different variations, see [1, 4, 5].

References

[1] G. Kalai. Linear programming, the simplex algorithm and simple polytopes. *Math. Programming*, 79(1-3, Ser. B):217–233, 1997. Lectures on mathematical programming (ismp97) (Lausanne, 1997), ps file available from http://www.ma.huji.ac.il/~kalai/papers.html.

[2] P. McMullen. The maximum number of faces of a convex polytope. *Mathematika*, XVII:179–184, 1970.

[3] P. McMullen and G.C. Shephard. *Convex polytopes and the upper bound conjecture*. Cambridge University Press, 1971.

[4] K. Mulmuley. *Computational Geometry, An Introduction Through Randamized Algorithms*. Prentice-Hall, 1994.

[5] G.M. Ziegler. *Lectures on polytopes*. Graduate Texts in Mathematics 152. Springer-Verlag, 1994.