Polyhedral Computation, Spring 2016
Solutions to Assignment 3

April 13, 2016

Problem 1 (Hilbert Basis): Let $P = \{x : Ax \leq 0\}$ be any rational cone which is not pointed, that is, there is at least one non-zero z satisfying $Az = 0$. This implies that there exists such integral z.

Given any Hilbert basis B, it contains a vector b_i with $Ab_i = 0$. For a short proof, consider any integral z satisfying $Az = 0$. That z can be expressed as a non-negative integer combination $z = \sum_{b_j \in B} \gamma_j b_j$. We conclude that $\sum_{b_j \in B} \gamma_j A b_j = 0$. Since there is a strictly positive γ_j and $Ab_j \leq 0$ for all b_j's, there exists at least one b_i satisfying $Ab_i = 0$. It also follows that $-b_i$ is in P as well.

We are able to find non-negative integer coefficients λ_j so that

$$-b_i = \sum_{b_j \in B} \lambda_j b_j.$$

We claim that $B' = B \backslash \{b_i\} \cup \{b'_i\}$ where $b'_i = (\lambda_i + 2) b_i$ is another Hilbert basis. For a proof, it is enough to express b_i as a non-negative integer combination of vectors in B'. Observe that

$$b_i = b'_i - (\lambda_i + 1) b_i = b'_i + \sum_{b_j \in B' \backslash \{b'_i\}} \lambda_j b_j.$$

Finally, if B is a minimal Hilbert basis, then B' is also minimal. For the sake of a contradiction, suppose B' is not minimal, that is, there is a basis vector b_k in B' such that $B' \backslash \{b_k\}$ is still a Hilbert basis. If so, then b_k can be expressed as a non-negative integer combination

$$b_k = \sum_{b_j \in B' \backslash \{b_k\}} \delta_j b_j.$$

We distinguish two cases. If $k = i$ where i is the index from above, then

$$b_i = \sum_{b_j \in B \backslash \{b_i\}} \delta_j b_j - (\lambda_i + 1) b_i = \sum_{b_j \in B \backslash \{b_i\}} (\delta_j + \lambda_j) b_j$$

by using $b_k = b_i' = (\lambda_i + 2)b_i$, the identity $B'\backslash\{b_k\} = B\backslash\{b_i\}$ and $-b_i = \sum_{b_j \in B} \lambda_j b_j$. The basis B would not be minimal, and the contradiction is established. Working out the details for the case where $k \neq i$ is left to the reader.

Problem 2 (Lower Bound on the Size of a Hilbert Basis): Since C =cone($\{a_1, a_2, \ldots, a_n\}$) is generated by n linearly independent vectors in $\mathbb{Z}^n$, the zonotope Z, as defined in the lecture notes, is full dimensional. Furthermore, it is a combinatorial cube with volume $|\det([a_1, a_2, \ldots, a_n])|$, which tiles the whole space with translations by integer combinations of the basis vectors.

Let V consist of the extreme points of the zonotope Z. Each extreme point belongs to the Hilbert basis B, except the origin, and therefore $|B| \geq |V| - 1 = 2^n - 1$.

For any point $x \in \mathbb{R}^n$, a unit-cube is defined as follows

$$U_x = \{y : x_i - 1/2 \leq y_i \leq x_i + 1/2 \text{ for every } i \in [n]\}.$$

Let us consider the unit-cubes of the integer points in the zonotope Z. The interiors of those cubes do not intersect and their union has volume at least as large as Z. This is because the union tiles the whole space with possible interior overlaps with translations by integer combinations of the basis vectors.

Note that the volume of

$$Z \cap \left(\bigcup_{x \in V} U_x \right)$$

is exactly 1. To cover the remaining volume of Z we need at least $|\det([a_1, a_2, \ldots, a_n])| - 1$ unit-cubes centered at integer points. Thus, the size of the Hilbert basis B is at least $2^n + |\det([a_1, a_2, \ldots, a_n])| - 2$.

The bound is tight. Suppose the cone C is defined by the orthogonal unit-vectors $\{e_1, e_2, \ldots, e_n\}$. Then Z has one unit of volume and the size of B is trivially $2^n - 1$.

Problem 3 (Convexity Checking): For a matrix A^i, let m^i be its number of rows and A^i_j denote row j. Let $P := P_1 \cup P_2 \cup \ldots \cup P_k$ be the union of all convex polytopes and $V := V^1 \cup V^2 \cup \ldots \cup V^k$ be the union of extreme points. Define a constant M as

$$M := \max\{||v - u|| : u, v \in V \text{ and } u \neq v\}.$$

We formulate

$$
\text{(mixed IP)} \qquad \begin{array}{llll}
\max & f & & \\
\text{s.t.} & A^i_j x & \geq b_j - (M+1)s^{ij} + f, & \forall P_i, 1 \leq j \leq m^i \\
& \sum_{j=1}^{m^i} s^{ij} & \leq m^i - 1, & \forall P_i \qquad (1) \\
& s^{ij} & \in \{0,1\}, & \forall P_i, 1 \leq j \leq m^i \\
& f & \geq 0, & \\
& x & = \sum_{v_i \in V} \lambda_i v_i, & \qquad (2) \\
& \sum_{v_i \in V} \lambda_i & = 1, & \qquad (3) \\
& \lambda_i & \geq 0, & \forall v_i \in V \qquad (4)
\end{array}
$$

and claim that this mixed IP has a positive optimal value f^* if and only if the union P is not convex. We prove the "only if" direction. Suppose $f^* > 0$. Because of constraint (1), for every i, there is at least one j such that $s^{ij*} = 0$. But then $A^i_j x^* \geq b_j + f^*$, that is, x^* is not in P_i. It follows that x^* is not in P. At the same time the constraints (2), (3) and (4) imply that x^* is in $\text{conv}(P)$. We conclude that $\text{conv}(P) \neq P$.

For the "if" direction assume that P is not convex. Then there exists x^* such that $x \in \text{conv}(P) \setminus P$. We show that this solution can be extended to a solution (x^*, f^*, s^*) where $s^* = (s^{ij*})_{i=1,\ldots,k; j=1,\ldots,m^i}$, such that $f^* > 0$. The constraints (2), (3) and (4) are satisfied by choice of x^* and do not depend on f^* and s^*. Since $x^* \notin P$ for every i there exists a $\ell(i)$ such that the $\ell(i)$-th constraint of A^i is violated, i.e.,

$$A^i_{\ell(i)} x > b_k.$$

For all $j \neq \ell(i)$ set $s^{ij*} = 1$ and set $s^{i\ell(i)*} = 0$ for all i. Hence the second and third inequality of (1) are satisfied. Then by choice of M we have that for all $j \neq \ell(i)$

$$A^i_j x \geq b_j - M > b_j - (M+1)s^{ij}.$$

It follows that we can choose f^* as

$$f^* = \min_i \{\min\{A^i_{\ell(i)} x - b_k; A^i_j x - b_j - (M+1)s^{ij}, j \neq \ell(i)\}\} > 0.$$

With this choice of (x^*, f^*, s^*) we constructed a solution to the LP with $f^* > 0$.