ORIENTED MATROID PROGRAMMING

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July 1982

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By

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A thesis
presented to the University of Waterloo
in partial fulfilment of the
requirements for the degree of
Doctor of Philosophy
in
Combinatorics and Optimization

Waterloo, Ontario, 1982

ABSTRACT

Let E be a finite set and let A be an $m \times |E|$ real matrix having A^e as the column of A indexed by $e \in E$. For each vector $x \in \mathbb{R}^E$, $S(x)$ denotes the sign vector of x, that is, $S(x) \in \{+,0,-\}^E$ and $S(x)_e$ is the sign of x_e. The set $\mathcal{C}_A$ of signed vectors of vectors in the row space $\mathcal{R}(A)$ of A represents the partition of $\mathbb{R}^m$ by polyhedral cones induced by the subspaces $\{\lambda \in \mathbb{R}^m : \lambda A^e = 0\}$ $(e \in E)$, and also represents the facial incidence relations of the polyhedral cones.

An oriented matroid (OM) is a set $\mathcal{C}$ of signed vectors satisfying certain axioms that are trivially satisfied by $\mathcal{C}_A$. Hence oriented matroids abstract the incidence properties of the partitioning polyhedral cones induced by the matrix A.

In Part I (Chapter 2 - Chapter 8) we consider an abstraction (Oriented Matroid Programming) of linear programming by oriented matroids. Fundamental results in linear programming are generalized with this setting. From the algorithmic point of view, it is shown that designing a finite pivot method for finding an optimal solution is much more difficult in this general setting than in linear programming, because of the existence of a cycling of nondegenerate pivots.

Chapter 9 considers the geometry of oriented matroids, which generalize well known geometric properties of $C(A)$. Chapter 10 studies an oriented matroid generalization of convex polytopes.

In Chapter 11 we shall develop the shellability of OM polytopes, which abstracts the shellability of convex polytopes. As a consequence,we obtain the Euler's relation for OM polytopes.

Chapter 12 presents a systematic way of transforming an OM to another OM, which is useful for constructing OM's with special properties.

In Chapter 13 a new class (BOM) of OM's is introduced. This class is important in the sense that this class properly contains the class of linear OM's and yet certain theorem proving procedures for linear OM's generalize to this class directly.

ACKNOWLEDGEMENTS

I am deeply grateful to my supervisor Professor Jack Edmonds, who introduced me to oriented matroids, stimulated me constantly with his fertile mind, and gave tireless support which I will never forget.

I would like to thank Professor Robert Bland of Cornell University, whose work motivated me to undertake the present study, for several illuminating and fruitful discussions. Also I wish to thank Professor Bill Pulleyblank for reading this thesis and having made helpful suggestions which improved the presentation of this work considerably.

Appreciative thanks to Arnaldo Mandel, with whom I had many valuable discussions on oriented matroids.

It was fortunate to have the friendship of my colleagues Mustafa Akgül, Kathie Cameron and Ephraim Korach, and with their families Nurcan Akgül and Malka Korach.

My warmest thanks to my wife Vera, who helped and encouraged me through hard times. Without her I would never have completed this work.

I would like to express my gratitude to the Canadian Government for their continued financial support under the Cultural Exchange Programme. Also the financial aid extended by the University of Waterloo is greatly appreciated.

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1. INTRODUCTION

Let E be a finite set and let A be an $m \times |E|$ real matrix having A^e as the column of A indexed by $e \in E$. for each vector $x \in \mathbb{R}^E$, $S(x)$ denotes the signed vector of x, that is, $S(x) \in \{+,0,-\}^E$ and $S(x)_e$ is the sign of x_e. The set $\mathcal{C}_A$ of signed vectors of vectors in the row space $\mathcal{R}(A)$ of A represents the partition of $\mathbb{R}^m$ by polyhedral cones induced by the subspaces $\{\lambda \in \mathbb{R}^m : \lambda A^e = 0\}$ $(e \in E)$, and also represents the facial incidence relations of the polyhedral cones.

An oriented matroid (OM) is a set $\mathcal{C}$ of signed vectors satisfying certain axioms that are trivially satisfied by $\mathcal{C}_A$. Hence oriented matroids abstract the incidence properties of the partitioning polyhedral cones induced by the matrix A.

In a very natural way, the notion of linear programming, convex polytopes or arrangements of hyperplanes has an abstraction by oriented matroids. It has been shown in recent years that many of their fundamental properties are still valid in the oriented matroidal generalization.

On the other hand in Folkman and Lawrence [OM] and in the doctoral dissertation of Mandel [TO] an extremely significant result has been obtained: an oriented matroid is a topological abstraction of the partition of $\mathbb{R}^m$ by the subspaces induced by the matrix A, as well as a combinatorial abstraction. This result has brought in a new insight into the subject, and lead us to understand oriented matroids as more concrete objects which can be visualized.

In this thesis we do not use any topology, however we try to present results in such a way that one can easily interpret the meanings topologically. The first part of the thesis studies the abstraction of linear programming and establishes the fundamental results. The second part mainly considers the geometric properties of oriented matroi

The summary of results will be given in Section B and Section C of this chapter, while Section A furnishes definitions and preliminary concepts of oriented matroids.

The reader is assumed to be familier with matroid theory. Appropriate references are Welsh [MT] , and Crapo and Rota [FC].

A. Axioms and Preliminaries

Let E be a finite set. A signed vector X on E is a vector (X_e: e $\in$ E) such that each component X_e (e $\in$ E) is either +, 0, or - . The negatives of each sign +, 0, - and the negative -X of an signed vector X are defined in the obvious way. We say that an element e $\in$ E separates signed vectors X and X' if $X_e = -X'_e \neq 0$. The composition X $\circ$ X' of signed vectors X and X' is defined to be the signed vector Z on E such that

$$(1.1) \qquad Z_e = \begin{cases} X_e & \text{if } X_e \neq 0 \\ X'_e & \text{otherwise} \end{cases} \qquad (e \in E).$$

For example if E = {1,2,3} and X $\equiv$ (X_1,X_2,X_3) = (0, 0,+), X' $\equiv$ (X'_1, X'_2, X'_3) = (0, -, -), then

(a) -X = (0, 0, -), -X' = (0, + ,+) ;

(b) the only element separates X and X' is 3 ;

(c) X $\circ$ X' = (0, -,+), X' $\circ$ X = (0, -, -) = X'.

Note that the composition is associative but not commutative. We may simply use vectors (on E) for signed vectors (on E), whenever there is no confusion with real vectors. Also we use capital letters X, Y, Z, W etc for signed vectors, and x, y, z, w etc for real vectors. The signed vector of all zero's is denoted by $\underset{\sim}{0}$.

(1.2) An oriented matroid† (abbreviated by OM) on a finite set E is a set $\mathcal{C}$ of signed vectors on E satisfying the following axioms:

(OM-0) $\underset{\sim}{0} \in \mathcal{C}$ (Existence of $\underset{\sim}{0}$) ;

(OM-1) $X \in \mathcal{C} \Rightarrow -X \in \mathcal{C}$ (Symmetry) ;

(OM-2) $X^1, X^2 \in \mathcal{C} \Rightarrow X^1 \circ X^2 \in \mathcal{C}$ (Closedness under Composition*) ;

(OM-3) $\left\langle X^1, X^2 \in \mathcal{C}\right.$ and an element $f \in E$ separates X^1 and X^2 . $\left.\right\rangle$

$$\Longrightarrow \left\langle \begin{array}{ll} \exists X^3 \in \mathcal{C} & \text{such that} \\ X^3_f = 0 & \text{and} \\ X^3_e = (X^1 \circ X^2)_e & \text{for all } e \in E \text{ not separating } X^1 \text{ and } X^2. \end{array} \right\rangle$$

(Elimination Property) .

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† An oriented matroid we define here is what Bland [AL], and Bland and Las Vergnas [OR] call the span of cocircuits of an oriented matroid.

* The notation of composition was first introduced by Bland [AL] [OT] and Bland and Las Vergnas [OR] .

(1.3) An OM can be considered as an abstract structure undirlying a vector subspace V of $\mathbb{R}^E$ as follows: For each $x = (x_e : e \in E) \in \mathbb{R}^E$, let S(x) denote the <u>signed vector of x</u>, that is, the signed vector $(S(x)_e : e \in E)$ on E with $S(x)_e$ being the sign of x_e for each $e \in E$. Then the set S(V) of signed vectors of vectors in V is easily shown to be an OM on E. The properties of V corresponding to the axioms (OM-0) ~(OM-3) for $\mathcal{C} = S(V)$ are just the properties of being closed under taking linear combinations of very special types.

An oriented matroid arising this way is said to be <u>linear</u>. Let A be an m × E matrix with the row space $\mathcal{R}(A) = \{\lambda A : \lambda \in \mathbb{R}^m\}$ being V then the matrix A is said to be a <u>matrix representation</u> of S(V), and conversely S(V) is the <u>OM obtained from the row space</u> V of A.

It should be clear that one can take any ordered field F instead of $\mathbb{R}$ above. An oriented matroid arising from a vector subspace of F^E is often called <u>representable over the ordered field F</u>. However, Mandel pointed out that any representable OM over an ordered field is linear, i.e., representable over $\mathbb{R}$, by using the similar argument as in Lindström [RC] .

There are non-linear oriented matroids and some such examples will be constructed later.

(1.4) Our axioms (1.2)(OM-1) (OM-3) of oriented matroids are far from standard. In fact several authors, Bland and Las Vergnas [OR], Bland [AL], Folkman and Lawrence [OM] gave different axiomatizations of oriented matroids based on quite different points of view.

Bland and Las Vergnas [OR] used axioms defining a dual pair of oriented matroids. Two signed vectors X and Y are said to be orthogonal (denoted by $X * Y$) if either

(i) $X_e = 0$ or $Y_e = 0$ for all $e \in E$; or

(ii) $\exists\, e, f \in E$ s.t. $X_e = Y_e \neq 0$ and

$X_f = -Y_f \neq 0$.

The dual C^* of an OM C is defined by

$$C^* = \{Y : X * Y \text{ for all } X \in C\}.$$

It can be shown that C^* is an oriented matroid on E and the dual C^{**} of C^* is C (see Chapter 8). Bland [AL] used primal-dual axioms of oriented matroids to emphasize the mutual relationship between a primal and a dual OM's and to prove theorems concerning a dual pair of OM's, in particular a generalization of the strong duality theorem of linear programming forseen by Rockafeller [EV].

Bland and Las Vergnas [OR] also gave an axoimatization of oriented matroids which defines the "minimal" members of an OM's in the following sence. The support $\underline{X}$ of a signed vector X on E is the set $\{e \in E : X_e \neq 0\}$ of indices of nonzero components of X. For an OM $\mathcal{C}$ on E, let $\mathcal{V} = \mathcal{V}(\mathcal{C})$ be the set of vectors in $\mathcal{C}$ having minimal nonzero supports, i.e.,

$$\mathcal{V} = \{X \in \mathcal{C} \setminus \{\underset{\sim}{0}\} : X' \in \mathcal{C} \setminus \{\underset{\sim}{0}\} \text{ and } \underline{X}' \subseteq \underline{X} \text{ implies } \underline{X} = \underline{X}\} .$$

The set $\mathcal{V}$ will be called the vertices (or cocircuits) of $\mathcal{C}$. Bland and Las Vergnas gave axioms defining the vertices of an oriented matroid and showed that an oriented matroid is characterized by its vertices. (Actually they gave axioms defining the vertices of $\mathcal{C}^*$ (or the circuits of $\mathcal{C}$). However this axiomatization is equivalent to vertex axioms by the duality.) We shall show the equivalence between our axiomatization and that of Bland and Las Vergnas in Section D.

The approach originally taken by the late Folkman and later continued by Lawrence [OM] differs considerably from Bland and Las Vergnas [OR]. However it is easy to see the equivalence between the two different axiomatizations.

A variety of works which initiated the study of oriented matroids can be seen in Camion [MU], Fulkerson [NT], Minty [AF] and Rockafellar [EV].

(1.5) One of the most significant results on oriented matroid theory is the topological representation of oriented matroid studied by Lawrence [OM] and by Edmonds and Mandel (see Mandel [TO]). The idea is to represent an oriented matroid $\mathcal{C}$ on E by a unit sphere S in $\mathbb{R}^m$ together with the collection $\mathcal{S} = \{(S_e^+, S_e^o, S_e^-) : e \in E\}$ of partitions of S such that for each $e \in E$, either

(a) $S_e^o = S$ and $S_e^+ = S_e^- = \phi$; or

(b) (S_e^+, S_e^o, S_e^-) is a homeomorphic image of a linear partition (S^+, S^o, S^-) of S where

$$S^o = \{\lambda \in S : \lambda_1 = 0\}$$

$$S^+ = \{\lambda \in S : \lambda_1 > 0\}$$

$$S^- = \{\lambda \in S : \lambda_1 < 0\} \quad .$$

For example if $\mathcal{C}$ is a linear OM on E having a matrix representation A as in (1.3) then we can represent $\mathcal{C}$ by a system $(S, E, \mathcal{S})$ where

$$S_e^o = \{\lambda \in S : \lambda A^e = 0\}$$

$$S_e^+ = \{\lambda \in S : \lambda A^e > 0\}$$

$$S_e^- = \{\lambda \in S : \lambda A^e < 0\} \quad .$$

A system $(S, E, \mathcal{S})$ associates the set $\mathcal{C}$ of signed vectors on E as

$$\mathcal{C} = \{T(\lambda) : \lambda \in S\} \cup \{\underset{\sim}{0}\}$$

where $T : S \to \{+,0,-\}^E$ defined by

$$T(\lambda)_e = \begin{cases} + & \text{if} \quad \lambda \in S_e^+ \\ 0 & \text{if} \quad \lambda \in S_e^o \\ - & \text{if} \quad \lambda \in S_e^- \end{cases} \qquad \text{for} \quad e \in E .$$

Mandel [TO] gave simple conditions on $(S, E, \mathcal{S})$ such that a resulting restricted system is equivalent to an OM. Such a system is called a sphere system (see Mandel [TO]).

It should be stressed that our discussion on oriented matroids can always have the corresponding topological counterpart which is very useful in the development of our theory.

(1.6) Before we give a brief introduction to the results presented in this thesis, we shall explain how the matroid structure is enbodied in oriented matroids.

Let $\mathcal{C}$ be an OM on E and let

$M(\mathcal{C}) = \{E \setminus \underline{X} \ : \ X \in \mathcal{C}\}$.

The set $M(\mathcal{C})$ is a matroid on E, or more precisely, the set of flats of a matroid for $M = M(\mathcal{C})$ satisfies the flat axioms (M-0)~(M-2) of matroids:

(M-0) $E \in M$;

(M-1) $F_1, F_2 \in M \Rightarrow F_1 \cap F_2 \in M$;

(M-2) $\langle F_1, F_2 \in M,\ a \notin F_1 \cup F_2 ,\ b \in F_2 \setminus F_1 \rangle$

$\Rightarrow \langle \exists F_3 \in M \text{ s.t. } a \in F_3 \not\ni b \text{ and } F_1 \cap F_2 \subseteq F_3 \rangle$.

Given subset S of E, the <u>closure</u> cl(S) in M (in $\mathcal{C}$) is the smallest flat containing S i.e.,

$$cl(S) = \cap \{F : S \subseteq F \in M\} .$$

A subset S of E is said to be <u>independent</u> in M (or $\mathcal{C}$) if there is no proper subset S' of S with cl(S') = cl(S). A maximal independent subset S of $T \subseteq E$ is said to be a <u>basis of T</u> in M (or $\mathcal{C}$). It is well-known that

> Given $T \subseteq E$, every basis of T has the same cardinality, called the <u>rank</u> r(T) of T.

A basis of E is also called a <u>basis</u> of M (or $\mathcal{C}$). The <u>rank</u> r(M) (or r($\mathcal{C}$)) of M (or $\mathcal{C}$) is the rank r(E) of E. It is also well-known that the poset L(M) of flats of M ordered by inclusion is a geometric lattice, and the height function of L(M) is the rank function r of M restricted to the flats of M. The greatest flat of M is $1_{L(M)} = E$ and the smallest flat of M is $0_{L(M)} = cl(\phi)$.

(1.7) Any partial order $\leq\cdot$ on {+,0,-} induces a partial order $\leq\cdot$ on signed vectors on E in the obvious way:

$$< X \leq\cdot X' > \Longleftrightarrow < X_e \leq\cdot X'_e \quad \text{for all} \quad e \in E > .$$

In this thesis we are interested in two kinds of partial ordering on {+,0,-} . One is the <u>natural linear order</u> $\leq$ defined by $- < 0 < +$. The second one is the <u>conformal relation</u> $\preceq$ defined by $0 \prec -$ and $0 \prec +$ (+ and - are incomparable). Roughly speaking, the thesis is devided into two parts, Part I concerning properties of oriented matroids associated with the linear order $\leq$ and Part II mostly concerning the geometric properties of oriented matroids associated with the conformal relation. We present here the major results in each part separately.

B. Summary of Results (Part I)

Since the simplex method for linear programming was introduced by G.B. Dantzig, the theory of linear programming has been studies in many different directions. One of the most significant and interesting direction is the study of combinatorial theory underlying linear programming initiated by A.W. Tucker. In particular the schema (or basis) form of fundamental theorems in linear programming and the constructive proofs using elegant combinatorial pivot methods gave a new insight into the subject (see Tucker [CL], Balinski [CA]). Later, R.T. Rockafellar [EV] suggested that many properties concerning orthogonal subspaces of $\mathbb{R}^n$, including the strong duality theorem of linear programming, can be stated in purely combinatorial fashion and one should be able to axiomatize a broader structure (namely a system of oriented matroids) in which those properties generalize. Bland [OT], Las Vergnas [MO], Folkman and Lawrence [OM] and Lawrence [OM] independently gave different but equivalent axiomatizations of oriented matroids. In the thesis [OT] of Bland, he succeeded in proving all the results suggested by Rockafellar except for the oriented matroid generalization of the strong duality theorem. The first proof of the theorem, a non-constructive one, was given by Lawrence [OM]. Immediately after this Bland [AL] gave a finite pivot algorithm which abstracts Dantzig's simplex method of linear programming and provided a constructive proof of the "schema" form of the theorem.

Although all the results suggested by Rockafellar were settled, we do not think that our understanding of the results is sufficient, or at least not as deep as our understanding of linear programming theory. One of the reasons is that in both Bland [AL] and Lawrence [OM] the primal-dual mechanism of oriented matroids was heavily emphasized and the proof of the results are very hard to understand geometrically even for the special setting of linear programming, while any statement on oriented matroids using duality can be stated without duality. Secondly, the geometric terms used in linear programming such as polyhedra, cones, feasible directions are completely lost in Lawrence [OM] or implicit in Bland [AL] .

Hence we shall develop the theory of oriented matroid programming using our elementary axioms and without duality. Our purely primal treatment of the subject will be particularly interesting when statements are interpreted in terms of the topological representation of oriented matroids in (1.5). A brief description of our approach and results is in order.

An <u>affine oriented matroid</u> $(\mathcal{C};g)$ is an oriented matroid $\mathcal{C}$ on E together with a fixed element $g \in E$ called the <u>infinite element</u>. Two subsets of $\mathcal{C}$:

$$A = \{X \in \mathcal{C} \; ; \; X_g > 0\}$$
$$A^{\infty} = \{X \in \mathcal{C} \; ; \; X_g = 0\}$$

are called the affine space and the infinite space of $(C;g)$, respectively. A vector $X \in C$ is called a solution if $X \in A$, and a direction if $X \in A^{\infty}$. For a subset F of $E \setminus \{g\}$, the subset of A :

$$P(F) = \{X \in A \ : \ X_F \geq \underset{\sim}{0}\}$$

is called a polyhedron. For a vector X in a polyhedron P, a direction $Z \in A^{\infty}$ is said to be a feasible direction at X in P if $X \circ Z \in P$. For $f \in E \setminus \{g\}$, a solution $X \in A$ is said to maximize f over a polyhedron P if $X \in P$ and there is no feasible direction Z at X in P with $Z_f > 0$.

For two elements g, f in E, an oriented matroid program (OP) $P = (C;\ g,\ f)$ is to find $X \in A$ which maximizes f over the polyhedron (the feasible region) $P(E_1)$, where E_1 is the set $E \setminus \{g,f\}$ of constraint elements. An OP $P = (C;\ g,\ f)$ is feasible if $P(E_1) \neq \phi$, and unbounded if in addition there exists a direction $Z \in A^{\infty}$ such that $Z_{E_1} \geq 0$ and $Z_f > 0$.

We shall give a simple inductive proof of the following theorem in Chapter 3:

Theorem (3.4) (Fundamental Theorem) [Lawrence [OM] and Bland [AL]]

Every feasible OP is either unbounded or has an optimal solution.

Using (3.4) we shall show:

<u>Theorem</u> (3.15) (<u>Feasibility Theorem</u>)[cf. Bland [AL], Corollary 3.4.

For a subset $F \subseteq E \setminus \{g\}$, the polyhedron $\mathcal{P}(F) = \phi$ iff either $\mathcal{A} = \phi$ or there exists a subset F_1 of F, an element $h \in F \setminus F_1$ and $X \in \mathcal{A}$ such that X maximizes h over $\mathcal{P}(F_1)$ and $X_h < 0$.

Let $\mathcal{B}$ be the set of bases of $\mathcal{C}$. The set $\mathcal{B}_1$ of bases of an OP $P = (\mathcal{C}; g, f)$ is defined to be the set $\{B \setminus g : g \in B \in \mathcal{B}$ and $f \notin B\}$. It can be shown that for every basis B of P,

(a) there is a unique solution $X = X(B)$ such that $X_B = \underset{\sim}{0}$, called a <u>basic solution</u>; and

(b) for each $j \in B$, there exists a unique feasible direction $Z = Z^j(B)$ at $X(B)$ in $\mathcal{P}(B)$ such that $Z_{B\setminus\{j\}} = 0$ and $Z_j > 0$, called a <u>basic feasible direction</u>.

<u>Proposition</u> (5.9): For any basis $B \in \mathcal{B}_1$ and any element $i \in E \setminus (B \cup \{g\})$ the following statements are equivalent:

(a) $X(B)$ maximizes i over $\mathcal{P}(B)$;

(b) $Z^j(B)_i \leq 0$ for all $j \in B$.

The following definitions follows naturally from Theorem (3.15) and (5.9).

We say that a basis $B \in \mathcal{B}_1$ is <u>feasible</u> if $X(B) \in \mathcal{P}(E_1)$, <u>optimal</u> if it is feasible and $Z^j(B)_f \leq 0$ for all $j \in B$, <u>unbounded</u> if it is feasible and $Z^j(B)_{E_1} \geq 0$ and $Z^j(B)_f > 0$ for some $j \in B$, and <u>inconsistent</u> if $\exists\, i \in E_1 \setminus B$ s.t. $X(B)_i < 0$ and $Z^j(B)_i \leq 0$ for all $j \in B$.

An OP P is said to be <u>standard</u> if $\mathcal{B}_1 \neq \phi$.

<u>Theorem</u> (5.16) (<u>Basis Form of the Fundamental Theorem</u>)

Every standard OP has either an optimal, unbounded or inconsistent basis.

This theorem can be viewed as two statements:

(a) Every standard OP has either a feasible or inconsistent basis;

(b) Every standard OP having a feasible basis has either an unbounded or optimal basis.

For linear programming the statement (b) can be proved constructively by using the simplex method with a finite pivot rule, and the statement (a) follows immediately by the idea of Phase I of the simplex method. It turns out that this proof for the linear case cannot be extended to the broader setting of oriented matroids, although pivot methods generalize very naturally. The main reason is that the natural abstraction of the simplex method can produce a <u>nondegenerate cycling</u> , that is, a sequence of feasible basis $B^o, B^1, B^2, \ldots, B^k$ such that $B^o = B^k$ and $X(B^{i-1}) \neq X(B^i)$ for at least one $1 \leq i \leq k$, which cannot be detected by any known pivot methods for the simplex method. This peculiar phenomenon explains the necessity of a new approach to the subject. In fact Bland [AL] gave a pivot method for OP's whose finiteness is guranteed by purely combinatorial argument.

In Chapter 6 a new finite pivot method for OP's is given. The statement (b) is a straightforward consequence of the algorithm. The statement (a) follows easily from (b) (This is shown in Section D of Chapter 5). Some examples of OP's for which the simplex produces nondegenerate cycling will be constructed in Chapter 7 and it will be shown that Bland's pivot method produces an infeasible basis for one of the examples. This answers the two open problems raised by Bland [AL] . The discovery of the examples is due to our purely primal treatment of oriented matroid and the topological representation of oriented matroids.

The duality of oriented matroids and programming will be studied in Chapter 8. The generalization of the LP strong duality theorem is immediate from the fundamental theorem, once the basic properties of duality are proved.

C. Summary of Results (Part II)

For a subset F of E let

$$C(F) = \{X \in C : X_F = \underset{\sim}{0}\} \quad .$$

Clearly $C(F) = C(cl(F))$ for any subset F of E. Consider the poset $L(C)$ of flats (or subspaces) of C :

$$L(C) = \{C(F) \; : F \subseteq E\}$$

ordered by reversed inclusion. We have

$$L(M(C)) \simeq L(C).$$

The greatest flat of C is $1_{L(C)} = \{\underset{\sim}{0}\}$ and the smallest flat of C is $0_{L(C)} = C$. The dimension $d(t)$ of a flat $t \in L(C)$ is defined to be the coheight of t in $L(C)$ minus one. It is easy to verify

$$d(C(F)) = n - r(F) - 1 \quad \text{for} \quad F \subseteq E,$$

where $n = r(C) = r(M(C))$.

The flats of dimension 0 are called the points of C , the flats of dimension 1 are the lines of C and the flats of dimension n-2 are the hyperplanes of C . Unique flats of dimension (-1) and (n-1) are the maximal flat $\{\underset{\sim}{0}\}$ and the smallest flat C , respectively. Some of the basic properties are

(a) The points of C are the flats of the form $\{V, -V, \underset{\sim}{0}\}$ for some vertex V of C ;

(b) The hyperplanes of $\mathcal{C}$ are the flats of the form $\mathcal{C}(\{e\})$ for some $e \in E \setminus cl(\phi)$;

(c) If ℓ is a line, h is a hyperplane and $n \geq 3$ then either $\ell \subseteq h$ or ℓ intersects with h at a point.

Let $\mathcal{C}$ be an OM on E. For two vectors X and X' of $\mathcal{C}$, we say that X is a <u>face</u> of X' if $X \precsim X'$ i.e., X conforms to X'. Any subset t of $\mathcal{C}$ ordered by the conformal relation $\precsim$ is a poset, denoted by L[t]. For $X \in \mathcal{C}$, let

$$\mathcal{C}[X] = \{X' \in \mathcal{C} : X' \precsim X\} .$$

A subset t of $\mathcal{C}$ is said to be a <u>cell</u> (or <u>polytope</u>) if $t = \mathcal{C}[X]$ for some $X \in \mathcal{C}$. Members of a cell are called <u>faces</u> of t. For a cell t in $\mathcal{C}$, L[t] is a lattice, called the <u>face lattice</u> of t.

It is clear that the face lattice of a convex polytope is isomorphic to the face lattice of a cell in a linear OM.

For each $X \in \mathcal{C}$, we define the <u>dimension</u> d(X) of X to be the dimension of the minimal subspace $\mathcal{C}(E\setminus\underline{X})$ containing X, or equivalently the number $n-r(E\setminus\underline{X}) - 1$. The <u>dimension</u> d(t) of a cell t is the dimension $d(1_{L[t]})$ of the greatest face. It will be shown in Chapter 10 that the face lattice L[t] of a cell t satisfies:

(a) The height function of L[t] is the dimension function d restricted to the faces of t minus 1.

(b) Jordan-Dedekind Property: for ordered $X, X' \in t$,

$< X'$ covers X in $t > \iff < d(X') = d(X) + 1 >$

(c) PM property :

$< X, X' \in t, \; X \lneqq X'$ and $d(X') = d(X) + 2 >$

$\Rightarrow < \exists$ exactly two faces of t between X and $X' >$.

Las Vergnas [CV] previously studied various properties of the polar of face lattices of cells. It is well-known that the polar of the face lattice of a convex polytope is isomorphic to the face lattice of some convex polytope. This polarity does not extend to OM cells, since it has been recently shown by Munson and Billera (see Munson [FL]) that there exists an OM cell such that the polar of its face lattice cannot be realized by the face lattice of any OM cell. Hence the properties of the face lattice of an OM cell cannot be directly extended for the polar. One of the properties known to be satisfied for the face lattices of OM cells and not known for the polar is "shellability", which will be studied in Chapter 11.

Bruggesser and Mani [SD] first proved the shellability of a convex polytope: the facets of a convex polytope can be arranged in a sequence $F_1, F_2, \ldots, F_r$ (r = number of facets) in such a way that for each i with $2 \leq i < r$, $F_i \cap (\bigcup_{j=1}^{i-1} F_j)$ is a topological ball of dimension 2 less than the polytope. This result completed the Schläfli's calculation of the Euler characteristic for convex polytopes.

It turns out that we can abstract the notion of shellability for cell complexes combinatorially to posets, using two kinds of shellable posets namely a shellable d-ball and a shellable d-sphere. We shall prove:

Theorem (11.15.a): The face lattice L[t] of every d-dimensional OM cell t is a shellable d-ball and its boundary $\partial L[t] \equiv L[t] \setminus \{1_{L[t]}\}$ is a shellable (d-1)-sphere.

Using the above theorem, we can show:

Corollary Every OM cell t of dimension d satisfies Euler's relation:

$$\sum_{i=0}^{d} (-1)^i f_i(t) = 1$$

where $f_i(t)$ denotes the number of i-dimensional faces of t.

In Chapter 12 we introduce some operations (called perturbations) by which one can transform an OM with certain properties to another OM on the same elements. This operation is very useful to systematically construct nonlinear OM's from linear OM's. It can be easily shown that many of well-known nonlinear OM's such as the non-Pappus, non-Desarques can be constructed from appropriate linear OM's using perturbations. Also, the examples of nonlinear OM's used in Chapter 7 in order to show the possibility of nondegenerate cycling in the simplex method are constructed in a similar way.

In Chapter 13 we define and investigate a new class of OM's, called the <u>BOM's</u>. This class includes all linear OM's, all OM's with rank ≤ 3 and excludes all OM's C for which the simplex method generates no nondegenerate cycling for the OP (C; g, f) for any pair g, f of elements. It will be shown that this class is closed under taking dual i.e., the dual of a BOM is a BOM.

D. Conformal Elimination Property and Vertex Axioms

It is pointed out in Section A that there are several different axiomatizations of oriented matroids used in the literature. Here we shall derive axioms defining the vertices of an oriented matroid , and show that an oriented matroid is characterized by its vertices. This axiomatization are equivalent by duality to the circuit axioms used by Bland and Las Vergnas [OR] .

First of all, we shall obtain a very powerful property of oriented matroids, which will be often used thoughout this thesis. This property is in fact equivalent to the elimination property (1.2 OM-3), and will be called the <u>conformal elimination property</u>.

(1.8) <u>Proposition</u> Let $\mathcal{C}$ be a set of signed vectors on E. Then $\mathcal{C}$ satisfies the elimination property (1.2 OM-3) iff $\mathcal{C}$ satisfies:

(OM-3')
$$\left\langle \begin{array}{l} X^1, X^2 \in \mathcal{C} \text{ and } I \text{ is a nonempty subset of } E \\ \text{such that every } i \in I \text{ separates } X^1 \text{ and } X^2 \end{array} \right\rangle$$

$$\Longrightarrow \left\langle \begin{array}{l} \exists\ j \in I \text{ and } \exists\ X^3 \in \mathcal{C} \text{ such that} \\ \quad X^3_j = 0 , \\ \quad X^3_i = X^1_i \text{ or } 0 \text{ for all } i \in I \text{ (i.e., } X^1_I \succeq X^3_I) \\ \quad X^3_e = (X^1 \circ X^2)_e \text{ for all } e \in E \text{ not separating} \\ \qquad\qquad X^1 \text{ and } X^2 . \end{array} \right\rangle$$

<u>Proof</u> Clearly (OM-3') implies (OM-3).

We shall prove the implication (OM-3) => (OM-3') by induction. Let $(OM\text{-}3')_k$ be the statements obtained from (OM-3') by imposing the condition that a set I is chosen with $|I| \leq k$. Thus $(OM\text{-}3')_1$ is equivalent to (OM-3).

Assume by induction that (OM-3) implies $(OM\text{-}3')_{k-1}$ and we shall show that (OM-3) implies $(OM\text{-}3')_k$. This will prove the result. Let $X^1, X^2 \in \mathcal{C}$, let I be a subset of E such that $|I| = k$ and every $i \in I$ separates X^1 and X^2. Take any $j \in I$. By (OM-3), there exists $X^3 \in \mathcal{C}$ such that $X^3_j = 0$ and $X^3_e = (X^1 \circ X^2)_e$ for all $e \in E$ not separating X^1 and X^2. If $X^3_i = X^1_i$ or 0 for all $i \in I$ we are done. Otherwise let I' be the set $\{i \in I : i$ separates X^1 and $X^3\}$. Since $j \notin I'$, $0 < |I'| < |I| = k$. By the inductive hypothesis $(OM\text{-}3')_{k-1}$ applies to X^1, X^3 and I'. It follows that there exists $\ell \in I' \subset I$ and $X^4 \in \mathcal{C}$ such that $X^4_\ell = 0$, $X^4_i = X^1_i$ or 0 for all $i \in I'$ and $X^4_e = (X^1 \circ X^3)_e$ for all $e \in E$ not separating X^1 and X^3. It is clear that $X^4_i = X^1_i$ or 0 for all $i \in I \setminus I'$, and that $(X^1 \circ X^3)_e = (X^1 \circ X^2)_e$ for all $e \in E$ not separating X^1 and X^2, and further every element $e \in E$ that separates X^1 and X^3 separates X^1 and X^2. This implies that $X^4_e = 0$, $X^4_i = X^1_i$ or 0 for all $i \in I$, and $X^4_e = (X^1 \circ X^2)_e$ for all $e \in E$ not separating X^1 and X^2. Thus $(OM\text{-}3')_k$ follows and the proof is complete □

Recall that the set $\mathcal{V} = \mathcal{V}(\mathcal{C})$ of vertices (or cocircuits) of an OM $\mathcal{C}$ is the set of nonzero vectors in $\mathcal{C}$ which have minimal supports over $\mathcal{C} \setminus \{\underset{\sim}{0}\}$ i.e.,

(1.9) $\mathcal{V} = \{V \in \mathcal{C}\setminus\{\underset{\sim}{0}\} : X \in \mathcal{C} \setminus \{\underset{\sim}{0}\}$ and $\underline{X} \subseteq \underline{V}$ implies $\underline{X} = \underline{V}\}$.

(1.10) <u>Proposition</u> If X is a nonzero vector in an OM $\mathcal{C}$ then for each $j \in \underline{X}$ there exists a vertex $V \in \mathcal{V}$ such that $V_j = X_j$ and V conforms to X.

<u>Proof</u> If $X \in \mathcal{V}$ there is nothing to prove. Let $X \in \mathcal{C} \setminus \mathcal{V}$ and $X \neq \underset{\sim}{0}$. Then there exists $Z \in \mathcal{V}$ with $\phi \neq \underline{Z} \subset \underline{X}$. Let $j \in \underline{X}$. It is enough to show that there exists $W \in \mathcal{C}$ such that $j \in \underline{W} \subset \underline{X}$ and W comforms to X. If $j \in \underline{Z} \subset \underline{X}$ then we are done. If not there are two cases to consider.

<u>Case 1:</u> $j \notin \underline{Z}$.

Let Z' be Z if $Z \not\leq X$, and $-Z$ otherwise. Let $I = \{i \in E$: i separates X and Z'$\}$. Clearly $I \neq \phi$, and using (OM-3') with $X^1 = X$ and $X^2 = Z'$ we obtain $W(= X^3$ in (OM-3')) s.t. $j \in \underline{W} \subset \underline{X}$ and $W \prec X$.

<u>Case 2:</u> $j \in \underline{Z}$ and $Z \not\preceq X$.

If $-Z \preceq X$, then there is nothing to prove. Suppose $-Z \not\preceq X$. Let Z' be Z if $Z_j = X_j$ and $-Z_j = -X_j$. The proof continues similarly to Case 1. □

(1.11) Theorem If X is a nonzero vector in an OM $\mathcal{C}$ then there exist vertices $V^1, V^2, \ldots, V^k$ of $\mathcal{C}$ such that $V^i \preceq X$ for $i = 1,2,\ldots,k$ and $X = V^1 \circ V^2 \circ \cdots \circ V^k$.

(1.12) Corollary An oriented matroid is determined by its vertices.

Using (1.11) one can easily show:

(1.13) Suppose $\mathcal{V}$ is the set of vertices of an OM $\mathcal{C}$. Then the following two conditions hold:

(OV-1) For all $V \in \mathcal{V}$, $V \neq \underset{\sim}{0}$ and $-V \in \mathcal{V}$; and for all $V^1, V^2 \in \mathcal{V}$, $\underline{V}^1 \subseteq \underline{V}^2$ implies $V^1 = V^2$ or $-V^2$;

(OV-2)
$$\left\langle \begin{array}{l} V^1, V^2 \in \mathcal{V},\ f \in E \text{ separates } V^1 \text{ and } V^2 \\ \text{and } g \in E \text{ does not separate } V^1 \text{ and } V^2 \end{array} \right\rangle$$

$$\Longrightarrow \left\langle \begin{array}{l} \exists V^3 \in \mathcal{V} \text{ such that} \\ V^3_f = 0,\quad V^3_g = (V^1 \circ V^2)_g, \text{ and} \\ V^3_e = V^1_e,\ V^2_e, \text{ or } 0 \text{ for all } e \in E \end{array} \right\rangle$$

In fact it is not difficult to show the following:

(1.14) Theorem A set $\mathcal{V}$ of signed vectors on E is the set of vertices of an OM iff $\mathcal{V}$ satisfies the conditions (OV-1) and (OV-2).

We call the conditions (OV-1) and (OV-2) the vertex (or cocircuit) axioms for OM's. Bland and Las Vergnas [OR] used the vertex axioms to define an OM, and they characterized the set $\mathcal{C}$ (what they call the span of OM) in terms of conformal unions of vertices.

E. Deletion, Contraction and Sign Reversal

Before we go to the next chapter we introduce some standard useful operations in oriented matroids.

Let E be a finite set and let $\mathcal{C}$ be an OM on E.

If X is a vector on E and if S is a subset of E, ${}_{S}X$ denotes the vector on E obtained from X by reversing signs of the components X_S i.e.,

$$({}_{S}X)_e = \begin{cases} -X_e & \text{if} \quad e \in S \\ X_e & \text{if} \quad e \in E \setminus S \end{cases}$$

Let

$${}_{S}\mathcal{C} = \{{}_{S}X \ : \ X \in \mathcal{C}\} \quad .$$

Then it is easy to verify :

(1.15) ${}_{S}\mathcal{C}$ is an OM on E for each subset S of E, called <u>the OM obtained from $\mathcal{C}$ by reversing signes on S</u>.

For subsets R and S of E, let

(1.16) $\mathcal{C} \setminus R = \{X_{E \setminus R} \ : \ X \in \mathcal{C}\}$

(1.17) $\mathcal{C} / S = \{X_{E \setminus S} \ : \ X \in \mathcal{C} \text{ and } X_S = 0\}$,

where X_F denotes the <u>subvector</u> $(X_e : e \in F)$ of X for $F \subseteq E$. It is easily seen that both $\mathcal{C} \setminus R$ and $\mathcal{C}/S$ are OM's on $E \setminus R$ and $E \setminus S$, called respectively <u>the minor of $\mathcal{C}$ obtained by deleting R</u> and <u>the minor of $\mathcal{C}$ obtained by contracting S</u>. For disjoint subsets R and S of E, clearly

(1.18) $\mathcal{C} \setminus R / S = \mathcal{C} / S \setminus R$.

PART I

ORIENTED MATROID PROGRAMMING

2. AFFINE ORIENTED MATROIDS, POLYHEDRA, AND OPTIMIZATION

It will be convenient to have the following definitions. The set {+,0,-} of signs is ordered by the binary relation ≤ in the obvious manner : $- < 0 < +$. This relation ≤ can be extended for signed vectors on a finite set E as follows: for signed vectors X^1 and X^2 on E, $X^1 \leq X^2$ (X^1 is less than or equal to X^2) iff $X^1_e \leq X^2_e$ for each $e \in E$. The relation $X^1 \leq X^2$ may be also written $X^2 \geq X^1$ and read X^2 is greater than or equal to X^1. In particular if $X \geq \underset{\sim}{0}$ ($X \leq \underset{\sim}{0}$; respectively) holds then X is said to be nonnegative (nonpositi

These definitions are obviously motivated by the standard usage of these terms for real vectors. For example (0,+,+) > (0,-,+), (0,+,+) is a nonnegative vector, and a vector (0,-,+) is neither nonnegative nor nonpositive.

A. Affine Oriented Matroid

Let $\mathcal{C}$ be an OM on a finite set E. By an affine OM (associated with $\mathcal{C}$) we mean a pair ($\mathcal{C}$; g) where g is a specified element in E called the infinite element. For an affine OM ($\mathcal{C}$; g) consider the following two subsets of $\mathcal{C}$, the affine space $\mathcal{A}$ defined by

(2.1) $\mathcal{A} = \{X \in \mathcal{C} : X_g > 0\}$

and the infinite space $\mathcal{A}^\infty$ defined by

(2.2) $\mathcal{A}^\infty = \{X \in \mathcal{C} : X_g = 0\}$.

In order to distinguish a vector in $\mathcal{A}^\infty$ from a vector in $\mathcal{A}^\infty$, we may call a vector in $\mathcal{A}^\infty$ a direction.

Let $E_o = E \setminus \{g\}$. For subsets F^1 and F^2 of E_o define

(2.3) $\mathcal{P}(F^1,F^2) = \{X \in \mathcal{A} : X_{F^1} \geq 0 \text{ and } X_{F^2} \leq 0\}$

where X_F denotes the subvector $(X_e : e \in F)$ of X for $F \subseteq E$. A subset $\mathcal{P}$ of $\mathcal{A}$ such that $\mathcal{P} = \mathcal{P}(F^1,F^2)$ for some subsets F^1 and F^2 of E_o is called a polyhedron. A pair (F^1,F^2) is a representation of $\mathcal{P}$.

Let $\mathcal{P}$ be a polyhedron. We say that an element $e \in E_o$ supports $\mathcal{P}$ if either

(2.4) $X_e \geq 0$ for all $X \in \mathcal{P}$, or

(2.5) $X_e \leq 0$ for all $X \in \mathcal{P}$.

For any subset F of E_o, let $\mathcal{P}_F$ be the polyhedron defined by

(2.6) $\mathcal{P}_F = \{X \in \mathcal{P} : X_F = \underset{\sim}{0}\}$.

A <u>face</u> of P is a nonempty subset t of P such that $t = P_F$ for some set $F \subseteq E_o$ of elements supporting P .

If $P = P(F^1, F^2)$ for some subsets F^1, F^2 of E_o, then clearly

(2.7) Every element $e \in F^1 \cup F^2$ supports P ;

(2.8) if $F \subseteq F^1 \cup F^2$ and $P_F \neq \phi$, then P_F is a face of P .

By using the fact that a polyhedron is closed under the composition one can show the converse of (2.8):

(2.9) If t is a face of $P(F^1, F^2)$ then there exists a subset F of $F^1 \cup F^2$ such that $t = P_F$.

It should be remarked that any nonempty intersection of two faces of a polyhedron P is again a face of P and hence the set FL(P) of all faces of P together with the empty set ϕ ordered by inclusion is a lattice. It can be shown that the lattice FL(P) abstracts various properties of the face lattice of a real polyhedron such as the <u>Jordan-Dedekind chain property</u> or the <u>PM-ness</u> (see Chapter 9,10). However we will not develop this theory here but later in Part II, simply because there are more important notions to develop in order to study the main subject in Part I. One such notion is <u>optimization</u> which is to be introduced.

Let P be a polyhedron. Given a vector $X \in P$, a feasible direction at X in P is a direction $Z \in A^{\infty}$ such that

(2.10) $X \circ Z \in P$.

The set P^{∞} of directions which are feasible at every vector X in P is called the infinite face of P provided that $P \neq \phi$, i.e.,

(2.11) $P^{\infty} = \{Z \in A \ : \ X \circ Z \in P \ \text{ for all } X \in P\}$.

(2.12) Proposition Let P be a polyhedron with a representation (F^1,F^2) and let $X \in P$. Then a vector z is a feasible direction at X in P iff $z \in A^{\infty}$,

(2.13) $z_i \geq 0$ for all $i \in F^1 \setminus \underline{X}$, and
$z_j \leq 0$ for all $j \in F^2 \setminus \underline{X}$.

Proof Easy.

The following is less trivial.

(2.14) Proposition Let P be a nonempty polyhedron with a representation (F^1,F^2). Then the infinite face P^{∞} is determined by

(2.15) $P^{\infty} = \{Z \in A^{\infty} : \ Z_{F^1} \geq \underset{\sim}{0} \ \text{ and } \ Z_{F^2} \leq \underset{\sim}{0} \ \}$.

Proof Let P_0^{∞} denote the right hand side of (2.15). By the definition

$P_0^{\infty} \subseteq P^{\infty}$. Suppose there exists $Z \in P^{\infty} \setminus P_0^{\infty}$. Let $I = \{e \in F^1 : Z_e < 0 \} \cup \{e \in F^2 : Z_e > 0\}$. Since $Z \in P_0^{\infty}$, $I \neq \phi$.

Let X be any vector in P . Since $X \circ Z \in P$, it follows that $X_i > 0$ for all $i \in I \cap F^1$ and $X_j < 0$ for all $j \in I \cap F^2$. By (1.8 OM-3'), setting $X^1 = X$ and $X^2 = Z$, we obtain $X^3 \in P$ and $\ell \in I$ such that $X^3_\ell = 0$. Then $X^3 \circ Z \notin P$ contradicting $Z \in P^\infty$. This completes the proof. □

B. Optimization

For $f \in E_o$, we say that a vector X maximizes (minimizes respectively) f over a polyhedron P if

(2.16.a) $X \in P$ and

(2.16.b) there is no feasible direction Z at X in P with $Z_f > 0$ $(Z_f < 0)$.

We may use the word optimize in the place of "maximize" or "minimize" whenever the distinction is unnecessary. It should noted that using the language of Bland [AL] the statement (2. says there is no augmenting vector Z with respect to X. The following properties are immediate.

(2.17) If P' is a polyhedron contained in P and if $X \in P'$ optimizes f over P then X optimizes f over P'.

(2.18) If the infinite face P^∞ contains a direction Z with $Z_f > 0$ (respectively, $Z_f < 0$) then there is no maximizer (minimizer) of f in P .

A cone is a polyhedron having a unique minimal (nonempty) face. It follows immediately that if X is a vector in a polyhedron P with a representation (F^1, F^2) then for $\hat{F}^i = F^i \setminus \underline{X}$ $(i = 1,2)$

(2.19) $P(\hat{F}^1, \hat{F}^2)$ is a cone containing P .

By (2.12) we obtain:

(2.20) Suppose P has a representation (F^1, F^2). Then a vector X optimizes f over P iff it optimizes f over the cone $P(\hat{F}^1, \hat{F}^2)$ defined above.

It is clear that the definition of optimality is merely an analogue of the same terminology used in the linear case: a real vector x maximizes a linear function f over a real convex polyhedron iff there is no direction z such that $f(x + \varepsilon z) > f(x)$ for $\varepsilon > 0$. In the similar way it is very useful to abstract the notion of comparing the objective values evaluated at two points. For example if X^1 and X^2 are two vector in a polyhedron P and if $X_f^1 < X_f^2$ then it is easy to show that X^1 is not a maximizer of f in P (Corollary (2.24)). However we are able to say much more than this. For this purpose we introduce the notion of directions between two vectors in an affine space.

For any two distinct vectors X^1, X^2 in the affine space A, we define the set $\mathcal{D}(X^1, X^2)$ of directions from X^1 to X^2 by

(2.21) $$\mathcal{D}(X^1, X^2) = \{Z \in A^\infty : X^1 \circ Z = X^1 \circ X^2 \text{ and } X^2 \circ (-Z) = X^2 \circ X^1\} .$$

Hence, for $X^1, X^2 \in A$

(2.21) $\langle Z \in \mathcal{D}(X^1, X^2) \rangle$

$$\Longleftrightarrow \left\langle \begin{array}{lll} Z \in A^{\infty} & & \\ Z_i = X_i^2 & \text{for all} & i \in E_o \setminus \underline{X}^1 \\ Z_j = X_j^1 & \text{for all} & j \in E_o \setminus \underline{X}^2 \end{array} \right\rangle .$$

The following proposition is straightforward from the definitions and the elimination property (OM-3).

(2.22) Proposition Let $X^1, X^2 \in A$ with $X^1 \neq X^2$.
Then the following properties hold:

(a) $\mathcal{D}(X^1, X^2) = -\mathcal{D}(X^2, X^1)$ i.e.
Z is a direction from X^1 to X^2 iff $-Z$ is a direction from X^2 to X^1 ;

(b) $\mathcal{D}(X^1, X^2) \neq \phi$ and $\mathcal{D}(X^1, X^2) \not\ni \underset{\sim}{0}$;

(c) If X^1 and X^2 are vectors in a polyhedron P then every direction from X^1 to X^2 is a feasible direction at X^1 in P ;

(d) If $e \in E_o$ and $X_e^1 < X_e^2$ (respectively, $X_e^1 > X_e^2$) then there exists $Z \in D(X^1, X^2)$ with $Z_e > 0$ ($Z_e < 0$)

(2.23) Proposition If a vector X in a polyhedron P maximizes (minimizes, respectively) f over P then for each vector X' in P different from X, there is no direction Z from X to X' such that $Z_f > 0$ ($Z_f < 0$).

Proof This follows immediately from (2.22c). □

(2.24) <u>Proposition</u> If a vector X maximizes (minimizes) f over a polyhedron P then $X'_f \leq X_f$ ($X' \geq X_f$) for all $X' \in P$ with $X' \neq X$.

<u>Proof</u> Suppose X maximizes f over a polyhedron P. Let $X' \in P$, $X' \neq X$ and suppose $X'_f > X_f$. By (2.22.d), there exists $Z \in D(X,X')$ with $Z_f > 0$. By (2.23), this contradicts the optimality of X. Therefore $X'_f \leq X_f$. □

Proposition (2.24) implies:

(2.25) <u>Corollary</u> If vectors X^1 and X^2 in a polyhedron P optimizes f over P then $X^1_f = X^2_f$.

(2.26) <u>Theorem</u> If X^1 and X^2 optimize f over a polyhedron P then $X^1 \circ X^2$ also optimize f over P.

<u>Proof</u> Suppose X^1 and X^2 maximize f over P. Clearly $X^1 \circ X^2 \in P$. Suppose that $X^1 \circ X^2$ does not maximize f over P. Then there exists $Z^1 \in A$ such that $X^1 \circ X^2 \circ Z^1 \in P$ and $Z^1_f > 0$. Let Z^2 be any direction from X^1 to X^2. By (2.23), $Z^2_f = 0$. Let $Z = Z^2 \circ Z^1$. Hence $Z_f > 0$. Since $Z^2 \in \mathcal{D}(X^1,X^2)$, $X^1 \circ Z^2 = X^1 \circ X^2$ and thus $X^1 \circ Z = X^1 \circ Z^2 \circ Z^1 = X^1 \circ X^2 \circ Z^1 \in P$. This together with the fact $Z_f > 0$ contradicts the optimality of X^1. This completes the proof. □

(2.27) <u>Corollary</u> If X^1 and X^2 optimize f over a polyhedron P then every vector in the minimal face of P containing both X^1 and X^2 optimizes f over P.

3. ORIENTED MATROID PROGRAMMING

In the previous chapter we have introduced the notion of optimization , maximization or minimization over a polyhedron, as a natural abstraction of the same notion in real vector case and we have shown that various geometrical properties of linear optimization are still valid in the oriented matroidal abstraction.

We shall study in this chapter further results on optimization, which specialize to well-known results in linear programming when the associated oriented matroid is restricted to be linear.

We assume in the chapter that $\mathcal{C}$ is an OM on a finite set E , g is a fixed element of E, $\mathcal{A}$ and $\mathcal{A}^{\infty}$ are the affine space $\{X \in \mathcal{C} : X_g > 0\}$ and the infinite space $\{X \in \mathcal{C} ; X_g = 0\}$ of an affine OM $(\mathcal{C}:g)$. Let $E_o = E \setminus \{g\}$. For a subset F of E_o, the polyhedron in $(\mathcal{C}:g)$ having a representation (F,ϕ) is denoted by $\mathcal{P}(F)$ instead of $\mathcal{P}(F,\phi)$.

A. Definitions

Given an element $f \in E_o$, the OM program (abbreviated by OP) $(C;g,f)$ is to

(3.1) maximize f

subject to $X \in P(E_1)$

where f is the objective element, $E_1 = E \setminus \{g,f\} = E_o \setminus \{f\}$ is the set of constraint elements, and $P(E_1)$ is the feasible region. A vector X in A is called a solution. A solution X is feasible if $X \in P(E_1)$, optimal if it solves (3.1) i.e., X is feasible and maximizes f over $P(E_1)$. An OP (3.1) is said to be feasible if $P(E_1) \neq \phi$, and infeasible otherwise. A vector Z in A^∞ is said to be infinite - optimal if $Z_{E_1} \geq 0$ and $Z_f > 0$. We say that an OP (3.1) is bounded if it is infeasible or it has no infinite - optimal vector, and unbounded otherwise (i.e. it is feasible and has an infinite-optimal vector).

B. Fundamental Theorem

The following properties are obvious:

(3.2) An infeasible OP has no optimal solution.

(3.3) An unbounded OP has no optimal solution.

The main theorem of this chapter is a generalization of the Fundamental Theorem of LP:

(3.4) Theorem (Fundamental Theorem)

Every feasible OP is either unbounded or has an optimal solution.

This theorem was proved by Bland [AL, Section 5] in the language of primal dual pairs of oriented matroids. His method of proof was a natural extension of the classical way of proving (3.4) with the linearity restriction on OM using the simplex method. (Start with a vertex of the feasible region and generate a finite sequence of vertices until either an optimal solution or the evidence of unboundedness is obtained.) The equivalence of (3.4) and the Bland theorem depends on the equivalence of the notions of unbounded (primal) solutions and coinconsistency (or dual inconsistency) as described at the end of Section 3.E.

However, just as Theorem 3.4 is very simply stated in its present all primal form, so too we can give a "primal" proof of this theorem which does not required the machinery of primal dual pairs of OM's and pivot operations. In fact all we require is the notion of minors introduced in Section C of Chapter 1.

C. Inductive Proof

Let $P = (C;g,f)$ and let $E_1 = E \setminus \{g,f\}$. For subsets R and S of E_1 we denote by $P \setminus R$ and P / S the <u>subproblems</u> (or <u>minors</u>) of the OP P defined by

(3.6) $P \setminus R = (C \setminus R;\ g,f)$

(3.7) $P / S = (C / S;\ g,f)$.

The following properties are straightforward:

(3.8) If $h \in E_1$ and $P \setminus \{h\}$ is infeasible then P is infeasible ;

(3.9) If $h \in E_1$ and $P / \{h\}$ is unbounded then P is unbounded.

It is not too difficult to find that the Fundamental Theorem (3.4) follows from the following lemma and the above remarks (3.8) and (3.9). (A proof of (3.4) is given in (3.12).)

(3.10) <u>Lemma</u> (c f. Bland [AL], Theorem 5.1) For an OP $P=(C;g,f)$ and for any $h \in E_1$, the following statements ho:

(a) If both $P \setminus \{h\}$ and $P / \{h\}$ have an optimal solution then P has an optimal solution;

(b) If $P \setminus \{h\}$ has an optimal solution and $P / \{h\}$ is infeasible then either P has an optimal solution or P is infeasible;

(c) If $P \setminus \{h\}$ is unbounded and $P / \{h\}$ has an optimal solution then either P is unbounded or P has an optimal solution;

(d) If $P \setminus \{h\}$ is unbounded and $P / \{h\}$ is infeasible then P is either unbounded or infeasible.

In order to prove the above lemma it is somewhat easier to show the following stronger lemma.

(3.11) Lemma (c f. Bland [AL], Theorem 5.1) Consider an OP $P = (C;g,f)$ and $h \in E_1$ and for the following four statements (i), (ii), (i)* and (ii)*:

(i) $X^1 \in A$ and $X^1_{E\setminus\{h\}}$ is an optimal solution for $P \setminus \{h\}$;

(ii) $Z^1 \in A^\infty$ and $Z^1_{E\setminus\{h\}}$ is an infinite-optimal vector for $P \setminus \{h\}$;

(i)* $X^2 \in A$, $X^2_h = 0$ and $X^2_{E\setminus\{h\}}$ is an optimal solution for $P / \{h\}$;

(ii)* $P / \{h\}$ is infeasible ;

where $A = \{X \in C : X_g > 0\}$ and $A^\infty = \{Z \in C : Z_g = 0\}$ are the affine and the infinite spaces of $(C;g)$. The following properties (a) ~ (d) hold:

(a) If (i) and (i)* are true then either X^1 or X^2 is an optimal solution for P ;

(b) If (i) and (ii)* are true then either X^1 is an optimal solution for P or P is infeasible;

(c) If (ii) and (i)* are true then either Z^1 is an infinite-optimal vector or X^2 is an optimal solution for P ;

(d) If (ii) and (ii)* are true then either Z^1 is an infinite-optimal vector or P is infeasible.

<u>Proof</u> (a) Assume (i) and (i)* are true and suppose neither X^1 nor X^2 is an optimal solution for P. Since $X^1_{E\setminus\{h\}}$ is an optimal solution for $P\setminus\{h\}$ but X^1 is not an optimal solution for P, we know that $X^1_h < 0$. Let P' be the polyhedron $P(E_1\setminus\{h\}, \{h\}) = \{X \in A : X_{E_1\setminus\{h\}} \geq \underset{\sim}{0}$ and $X_h \leq 0\}$. Clearly $X^i \in P'$ for $i = 1,2$ and X^1 maximizes f over P'. Let Z' be any direction from X^2 to X^1. Then by (2.21) and (2.23), we have $Z' \in A^\infty$ and

$$Z'_f \geq 0 \quad , \quad Z'_{F\setminus\{h\}} \geq \underset{\sim}{0} \quad \text{and} \quad Z'_h < 0$$

where $F = E_1\setminus \underline{X}^2$. On the other hand since $X^2_{E\setminus\{h\}}$ is optimal for $P / \{h\}$ but X^2 is not optimal for P, there exists $Z \in A^\infty$ s.t.

$$Z_f > 0 \quad , \quad Z_{F\setminus\{h\}} \geq \underset{\sim}{0} \quad \text{and} \quad Z_h > 0 \ .$$

Using the elimination property (1.2 OM-3) for Z and Z' eliminating h, there exists $Z'' \in A^\infty$ such that $Z''_f > 0$, $Z''_h = 0$ and $Z''_F \geq \underset{\sim}{0}$. This contradicts the optimality of $X^2_{E\setminus\{h\}}$ in $P / \{h\}$. Therefore the result holds.

(b) Assume that (i) and (ii)* are true, and suppose the conclusion is false. For the same reason as in the proof of (a), $X^1_h < 0$. Since $P / \{h\}$ is infeasible and P is feasible, there exists $X^o \in A$ s.t $X^o_{E_1} \geq \underset{\sim}{0}$ and $X^o_h > 0$. Using the

$x^1_{E_1\setminus\{h\}} \geq \underset{\sim}{0}$ and the elimination property for x^1 and x^o, we obtain $x^3 \in A$ such that $x^3_{E_1} \geq \underset{\sim}{0}$ and $x^3_h = 0$, contradicting (ii)* . Therefore (b) follows.

(c) Assume (ii) and (i)* are true, and suppose the conclusion is false. This implies that $z^1 \in A^\infty$, $z^1_f > 0$, $z^1_{E_1\setminus\{h\}} \geq \underset{\sim}{0}$ and $z^1_h < 0$, and that there exists $z \in A^\infty$ as in the proof of (a). Using the elimination property for z^1 and z eliminating h, we obtain a contradiction to (i)*. Thus (c) follows.

(d) Assume (ii) and (ii)* are true, and suppose the contrary conclusion. This implies that $z^1_h < 0$ $z^1_f > 0$, $z^1_{E_1\setminus\{h\}} \geq \underset{\sim}{0}$, and that there exists $x^o \in A$ such that $x^o_{E_1} \geq \underset{\sim}{0}$ and $x^o_h > 0$. By using the elimination property for z^1 and x^o eliminating h, we obtain a contradiction to (ii)* . This completes the proof. □

(3.12) <u>Proof of the Fundamental Theorem:</u> Let $P = (C;g,f)$ be any OP and suppose it is feasible and bounded. We must show it has an optimal solution. We use induction on $|E_1|$. If $|E_1| = 0$, any feasible solution is optimal and hence the result is true. Suppose $|E_1| > 0$ and the theorem is true for smaller values of $|E_1|$. Take any element $h \in E_1$. Since P is feasible and bounded, and by the induction and by (3.8) and (3.9), either

(a-1) $P \setminus \{h\}$ is unbounded ; or

(a-2) $P \setminus \{h\}$ has an optimal solution, and either

(b-1) $P / \{h\}$ is infeasible ; or

(b-2) $P / \{h\}$ has an optimal solution.

Thus we have four cases to consider, which fall into the cases (a), (b), (c) and (d) of (3.10). By the lemma and the assumption P must have an optimal solution. This completes the proof. □

In the rest of this chapter we shall see some interesting consequences of this theorem.

D. <u>Feasibility Theorem</u>

An element $g \in E$ is said to be a <u>loop</u> of an OM C if $X_g = 0$ for all $X \in C$. For any subset $F \subseteq E_o \equiv E \setminus \{g\}$ remember that $P(F)$ denotes the polyhedron

$$P(F) = \{X \in A \;:\; X_F \geq \underset{\sim}{0}\}$$

in the affine space $A = \{X \in C \;:\; X_g > 0\}$ of $(C;g)$. We shall derive a necessary and sufficient condition for a polyhedron $P(F)$ to be nonempty. Let F be any fixed subset of E_o. It is obvious that $P(F) = \phi$ if

(3.13) g is a loop of C .

It is less trivial but follows from (2.24) that the following condition implies $P(F) = \phi$:

(3.14) There exists a proper subset F_1 of F, an element $h \in F \setminus F_1$ and a vector $X \in P(F_1)$ such that $X_h < 0$ and X maximizes h over $P(F_1)$.

The following theorem says that if $P(F) = \phi$, at least one of (3.13) or (3.14) holds,

(3.15) Theorem (Feasibility Theorem) [cf. Bland [AL], Cor. 3.4.

Let F be any nonempty subset of E_o. Then the polyhedrc $P(F) \neq \phi$ iff neither (3.13) nor (3.14) holds.

The sufficiency of this theorem has already been observe For the necessity we need the following lemma:

(3.16) Lemma Let F be any subset of E_o and let $h \in F$. If $P(F) = \phi$ and $P(F \setminus \{h\}) \neq \phi$ then there exists a vector $X \in P(F \setminus \{h\})$ maximizing h over $P(F \setminus \{h\})$.

Proof Assume $P(F) = \phi$ and $P(F \setminus \{h\}) \neq \phi$. Suppose the contrary. Consider an OP $\overline{P} = (C \setminus S; g,h)$ where $S = E_o \setminus F$. Since $P(F \setminus \{h\}) \neq \phi$, the OP $\overline{P}$ is feasible. By the Fundamental Theorem (3.4), $\overline{P}$ is either unbounded or has an optimal solution. Observing that there exists no $X \in A$ maximizing h over $P(F \setminus \{h\})$, $\overline{P}$ has no optimal solution and hence it is unbounded. This implies

$$\exists Z \in A^\infty \quad \text{s.t.} \quad Z_h > 0 \quad \text{and} \quad Z_{F\setminus\{h\}} \geq 0 .$$

It follows from the assumption $P(F) = \phi$ and $P(F \setminus \{h\}) \neq \phi$ that

$$\exists X \in A \quad \text{s.t.} \quad X_h < 0 \quad \text{and} \quad X_{F\setminus\{h\}} \geq 0 .$$

Using the elimination property for X and Z, we obtain $X' \in \mathcal{A}$ with $X'_h = 0$ and $X'_{F\setminus\{h\}} \geq 0$. Then $X' \in \mathcal{P}(F)$, contradicting $\mathcal{P}(F) = \phi$. Thus the result follows. □

(3.17) <u>Proof of (3.15) :</u> It is left to show the necessity. It is sufficient to prove

(3.18) $\langle\ \mathcal{P}(F) = \phi$ and g is not a loop $\rangle$

$\Rightarrow \langle$ (3.14) holds $\rangle$.

We shall prove (3.18) by induction on $|F|$.

If $|F| = 0$, then the assumption of (3.18) cannot be satisfied and hence (3.18) holds.

Suppose $|F| \geq 1$ and assume that the statement (3.18) holds for smaller F. Take any element $h \in F$. There are two cases to consider (i) $\mathcal{P}(F\setminus\{h\}) = \phi$ and (ii) $\mathcal{P}(F\setminus\{h\}) \neq \phi$. If $\mathcal{P}(F\setminus\{h\}) = \phi$, by the inductive hypothesis the property (3.14) follows. Suppose $\mathcal{P}(F\setminus\{h\}) \neq \phi$. Then by Lemma (3.16) (3.14) follows immediately. This completes the proof. □

E. <u>The Strong Duality Theorem</u> (<u>Primal Statement</u>)

A subset F of E_1 is said to be <u>cofeasible</u> for (3.1) if

(3.19) $\nexists\ Z \in \mathcal{A}^\infty$ such that $Z_f > 0$ and $Z_F \geq \underline{0}$.

This condition (3.11) is equivalent to

(3.20) The infinite face $\mathcal{P}^\infty(F)$ of the polyhedron $\mathcal{P}(F)$ contains no vector Z with $Z_f > 0$.

A vector $X \in \mathcal{A}$ and a subset $F \subseteq E_1$ are said to be <u>complementary</u> if

(3.21) $X_F = \underset{\sim}{0}$.

(A similar notion of complementarity was introduced by Bland [AL], where the equivalence was established between the two notions of optimality in terms of the existence of a complementary co-feasible solution and the nonexistence of a primal augmentation.)

The following is clear.

(3.22) Proposition A feasible solution X for an OP (3.1) is optimal iff there exists a cofeasible subset F of E_1 such that X and F are complementary.

An OP (3.1) is said to be co-infeasible if the OP has an infinite-optimal vector. Thus the unboundedness of an OP is simply the feasibility together with co-infeasibily, and clearly a co-infeasible OP has no optimal solution. The following theorem which will be shown in Chapter 8 to be equivalent to the OP strong duality theorem (first proved by Lawrence [OM] and later algorithmically by Bland [AL]) is a consequence of the Fundamental Theorem (3.4) and (3.22).

(3.23) Theorem For an OP P = (C; g,f), exactly one of the following statements holds:

(a) P is infeasible or co-infeasible;

(b) There exists a feasible solution X for P and a cofeasible subset F of E_1, and furthermore they can be chosen complementary.

F. The Optimal Face and Full Complementality

In this section we shall study a generalization of the Full Complementality Theorem of linear programming, which makes the Strong Duality Theorem (3.23) slightly stronger.

First we recall a geometrical property of the set of all optimal solutions for an OP.

Consider an OP $P = (C; g, f)$. As we have shown in (2.28)

(3.24) The set P^* of all optimal solutions of P is a face of the feasible region $P(E_1)$, which is called the <u>optimal face</u> of P.

It follows from (2.25) that

(3.25) for the optimal face P^* of P

$$X_f = X'_f \quad \text{for all} \quad X, X' \in P^* \quad .$$

For a solution $X \in A$, a constraint element $e \in E_1$ is said to be <u>active at X</u> if $X_e = 0$. For each subset F of E_1, let P_F denote a face of the feasible region $P = P(E_1)$ defined by

(3.27) $P_F = \{X \in P \ : \ X_F = \underset{\sim}{0}\}$.

It is clear that if an OP P has an optimal solution then the optimal face P^* of P is determined by

(3.28) $\mathcal{P}^{*} = \mathcal{P}_{F^{*}}$

where F^{*} is the set of constraint elements which are active at every optimal solution for P. Since a polyhedron is closed under the composition and $\mathcal{P}^{*}$ is clearly a polyhedron ,

(3.29) there exists an optimal solution X^{*} such that

$X^{*}_{E_1 \setminus F^{*}} > \underset{\sim}{0}$.

It follows from (3.22) that

(3.30) F^{*} is cofeasible.

In fact one can prove a stronger statement on F^{*} than (3.30). First we remark that

(3.31) <u>Proposition</u> If $F \subseteq E_1$ is cofeasible for P and if $e \in F$ and $\exists Z \in A^{\infty}$ s.t. $Z_e > 0$, $Z_f \geq 0$ and $Z_{F \setminus \{e\}} \geq 0$ then $F \setminus \{e\}$ is cofeasible.

<u>Proof</u> Assume $F \subseteq E_1$ is cofeasible and there exists $e \in F$ and $Z \in A^{\infty}$ s.t. $Z_e > 0$, $Z_f \geq 0$ and $Z_{F \setminus \{e\}} \geq \underset{\sim}{0}$. Suppose $F \setminus \{e\}$ is not cofeasible. Then $\exists Z' \in A^{\infty}$ s.t. $Z'_f > 0$ and $Z'_{F \setminus \{e\}} \geq \underset{\sim}{0}$. Since F is cofeasible $Z'_e < 0$. Using the elimination property for Z and Z' we obtain $Z'' \in A^{\infty}$ s.t. $Z''_f > 0$, $Z''_e = 0$ and $Z''_f \geq 0$. This contradicts the cofeasibility of F. Hence $F \setminus \{e\}$ is cofeasible. □

We say that a subset F of E_1 is <u>strongly cofeasible</u> for an OP P if it is cofeasible and $\nexists$ $Z \in A^\infty$ s.t. $Z_e > 0$, $Z_f \geq 0$ and $Z_{F\setminus\{e\}} \geq \underset{\sim}{0}$, for each $e \in F$. Proposition (3.31) implies:

(3.32) If $F \subsetneq E_1$ is cofeasible, then there exists a strongly cofeasible subset F' of F.

Now we can show that the set F^* is strongly cofeasible.

(3.33) <u>Theorem</u> Suppose that an OP P has an optimal solution. Let F^* be the set of constraint elements which are active at every optimal solution. Then F^* is strongly cofeasible.

<u>Proof</u> If $F^* = \phi$, there is nothing to prove. Assume that $F^* \neq \phi$. Suppose that F^* is not strongly cofeasible. Since F^* is cofeasible by (3.30), there exists $e \in F^*$ and $Z^o \in A^\infty$ s.t. $Z^o_e > 0$, $Z^o_f \geq 0$, and $Z^o_{F^*\setminus\{e\}} \geq \underset{\sim}{0}$. Let R, S be the partition of F^* s.t. $Z^o_R > \underset{\sim}{0}$ and $Z^o_S = \underset{\sim}{0}$. By the remark (3.29) we know that there exists an optimal solution X^* s.t. $X^*_{F^*} = \underset{\sim}{0}$, $X_{E_1\setminus F^*} > \underset{\sim}{0}$. Let $X^1 = X^* \circ Z^o$. Clearly X^1 is a feasible solution, however, not an optimal solution for $X^1_{F^*} \neq \underset{\sim}{0}$. It follows that there exists $Z^1 \in A^\infty$ s.t. $Z^1_f > 0$, $Z^1_S \geq \underset{\sim}{0}$. Let $Z^2 = Z^o \circ Z^1$. Then $Z^2 \in A^\infty$, $Z^2_f > 0$, $Z^2_F \geq \underset{\sim}{0}$, which contradicts the cofeasibility of F^*. Therefore F^* is strongly cofeasible. □

Using Theorem (3.33) one can make Theorem (3.15) stronger by replacing (b) of (3.15) by

(3.34) There exists a feasible solution X^* and a strongly cofeasible subset F^* of E_1 such that they are full complementary:

$$X^*_{F^*} = \underset{\sim}{0} \quad \text{and} \quad X^*_{E_1 \setminus F^*} > \underset{\sim}{0} \quad .$$

For a vector $X \in A$ and $F \subseteq E_1$, a pair (X,F) is said to be a complementary pair if X and F are complementary, and an optimal pair if in addition X is feasible and F is cofeasible.

Bland (private communication) pointed out that (3.34) can be proved from (3.15) and the generalization of Tucker's complementary Theorem (see Bland [AL], Corollary 3.2.1) in the context of dual pairs of OM's.

(3.35) Proposition If (X,F) is an optimal pair, and if F is strongly cofeasible, then $F \subseteq F^*$, where F^* is defined in (3.33).

Proof Let (X, F) be an optimal pair and let F be strongly cofeasible. Suppose that there exists $e \in F \setminus F^*$. By the definition of F^*, there exists an optimal solution X^o s.t. $X^o_e > 0$. Since (X, F) is an optimal pair, $X_F = \underset{\sim}{0}$. Thus $X \neq X^o$ and there exists a direction $Z \in A^\infty$ from X to X^o such that $Z_e > 0$ and $Z_{F\setminus\{e\}} \geq \underset{\sim}{0}$. By the optimality of X and X^o, and by (2.25), $Z_f = 0$. This contradicts the strong cofeasibility of F. □

(3.35) implies:

(3.36) Corollary (Bland [AL]) If (X^i, F^i) is an optimal pair and F^i is strongly cofeasible for $i = 1,2$, then (X^1, F^2) is an optimal pair.

4. BASES OF ORIENTED MATROIDS

A. Elementary Properties

Let E be a finite set and let $\mathcal{C}$ be an o.m. on E.

Let

(4.1) $\quad M(\mathcal{C}) = \{E \setminus \underline{X} : X \in \mathcal{C}\}$.

It can be easily shown that the set $M(\mathcal{C})$ is the set of flats of a matroid, since $M = M(\mathcal{C})$ satisfies the flat axioms of a matroid:

(M-0) $\quad E \in M$;

(M-1) $\quad F_1, F_2 \in M \Rightarrow F_1 \cap F_2 \in M$;

(M-2) $\quad F_1, F_2 \in M, \quad a \notin F_1 \cup F_2, \quad b \in F_2 \setminus F_1$

$\quad \Rightarrow \exists F_3 \in M$ s.t. $a \in F_3 \not\ni b$ and $F_1 \cap F_2 \subseteq F_3$.

Given subset S of E, the closure cl(S) of S in $\mathcal{C}$ is the smallest flat of $M(\mathcal{C})$ containing S i.e. ,

(4.2) $\quad cl(S) = \cap\{F : S \subseteq F \in M(\mathcal{C})\}$.

Clearly, we have

(4.3) $\quad S' \subseteq S \subseteq E \Rightarrow cl(S') \subseteq cl(S)$.

A subset S of E is said to be independent in $\mathcal{C}$ if there is no proper subset S' of S with cl(S') = cl(S). Given $T \subseteq E$, a maximal independent subset of T is called a basis of T in $\mathcal{C}$. Using matroidal properties (M-0) ~ (M-2), we can prove the following: Let T be a subset of E.

(4.4) If a subset S of E is independent and $cl(T) \setminus cl(S) \neq \phi$, then $T \setminus cl(S) \neq \phi$ and $S \cup \{e\}$ is independent for each $e \in T \setminus cl(S)$.

(4.5) If S is a basis of T then $cl(S) = cl(T)$.

(4.6) If S is a basis of T and $j \in S$, then there exists a unique flat F with $S \setminus \{j\} \subseteq F \subset cl(S)$ (therefore $F = cl(S \setminus \{j\})$.

(4.7) Let S be a basis of T and let $i \in T \setminus S$. Then $S \setminus \{j\} \cup \{i\}$ is a basis of T iff $i \notin cl(S \setminus \{j\})$.

(4.8) If S and S' are bases of T and $j \in S \setminus S'$, then there exists $i \in S' \setminus S$ such that $S \setminus \{j\} \cup \{i\}$ is a basis of T.

(4.9) Every basis of T has the same cardinality, called the <u>rank</u> $r(T)$ of T.

A basis of E in $\mathcal{C}$ is also called a <u>basis of $\mathcal{C}$</u>, and the <u>rank</u> $r(\mathcal{C})$ of $\mathcal{C}$ is $r(E)$. Let $\mathcal{B} = \mathcal{B}(\mathcal{C})$ denote the set of all bases of $\mathcal{C}$. The following properties will be very useful:
Let $\mathcal{C}(F)$ denote the set $\{X \in \mathcal{C} : X_F = \underset{\sim}{0}\}$ for $F \subseteq E$.

(4.10) $B \in \mathcal{B}$ iff B is a minimal subset of E such that $\mathcal{C}(B) = \{\underset{\sim}{0}\}$.

(4.11) For each basis $B \in \mathcal{B}$ and $j \in B$, $\mathcal{C}(B \setminus \{j\}) = \{X, -X, \underset{\sim}{0}\}$ for a unique signed vector $X \in \mathcal{C}$ with $X_j = +$. Such X is called the <u>fundamental cocircuit (or vertex) of j in B</u>, denoted by <u>X(B;j)</u>.

(4.12) <u>(Exchange Property)</u> Let $B \in \mathcal{B}$ and let $j \in B$ and $i \in E \setminus B$. Then the following statements (a) ~ (d) are equivalent:

(a) $B \setminus \{j\} \cup \{i\} \in \mathcal{B}$;

(b) $cl(B \setminus \{j\}) \not\ni i$;

(c) $X(B;\ j)_i \neq 0$;

(d) $C(B \setminus \{j\} \cup \{i\}) = \{0\}$.

An element $e \in E$ is said to be a <u>loop</u> of C if $X_e = 0$ for all $X \in C$, and a <u>coloop</u> of C if $\exists\, X \in C$ s.t. $\underline{X} = \{e\}$. The following properties are easy to verify:

(4.13) The following statements are equivalent:

(a) e is a loop of C ;

(b) $\{e\}$is not independent in C ;

(c) $e \in F$ for all $F \in M(C)$;

(d) $e \notin B$ for all $B \in \mathcal{B}(C)$.

(4.14) The following statements are equivalent:

(a) e is a coloop of C ;

(b) $S \cup \{e\}$ is independent for all independent set S;

(c) $E \setminus \{e\} \in M(C)$;

(d) $e \in B$ for all $B \in \mathcal{B}(C)$.

A less obvious property is:

(4.15) <u>Proposition</u> Assume that neither $g \in E$ is a loop nor $f \in E$ is a coloop of $\mathcal{C}$. Then there exists a basis B of $\mathcal{C}$ with $g \in B \not\ni f$.

<u>Proof</u> It follows from the assumption and from (4.13), (4.14) that there exist bases B^1 and B^2 with $g \in B^1$, $f \notin B^2$. If $f \notin B^1$ or $g \in B^2$, we are done. Suppose that $f \in B^1$ and $g \notin B^2$. Since $f \in B^1 \setminus B^2$, using (4.8) there exists $i \in B^2 \setminus B^1$ such that $B^1 \setminus \{f\} \cup \{i\}$ is a basis of $\mathcal{C}$. □

In Chapter 1 we defined the minors $\mathcal{C} \setminus R$ and $\mathcal{C} / S$ of $\mathcal{C}$ for subsets R and S of E as

$$\mathcal{C} \setminus R = \{X_{E\setminus R} \ : \ X \in \mathcal{C}\}$$
$$\mathcal{C} / S = \{X_{E\setminus S} \ : \ X \in \mathcal{C} \text{ and } X_S = \underset{\sim}{0}\}$$

which are OM's on $E \setminus R$ and $E \setminus S$ respectively. The following properties are immediate.

(4.16) <u>Proposition</u> Let R and S be subsets of E. Then the following properties are satisfied:

(a) A subset B of $E \setminus R$ is a basis of $\mathcal{C}$ iff it is a basis of $\mathcal{C} \setminus R$. Moreover if $B \subseteq E \setminus R$ is a basis of $\mathcal{C}$ and $j \in B$ then $X(B;j)_{E\setminus R}$ is the fundamental cocircuit of j in a basis B of $\mathcal{C} \setminus R$;

(b) A subset B of E containing S is a basis of $\mathcal{C}$ iff B \ S is a basis of $\mathcal{C}$ / S. Moreover if B is a basis of $\mathcal{C}$ with $S \subseteq B$ and if $j \in B \setminus S$ then $X(B;j)_{E\setminus S}$ is the fundamental cocircuit of j in a basis B \ S of $\mathcal{C}$ / S .

B. Sign Properties

Let $\mathcal{C}$ be an OM on E and let $\mathcal{B}$ be the set of bases of $\mathcal{C}$.

It is clear that for $B \in \mathcal{B}$ and $j \in B$,

(4.17) $X(B;j)_{B\setminus\{j\}} = \underset{\sim}{0}$ and $X(B;j)_j > 0$ and hence

(4.18) $X(B;j)_B \geq \underset{\sim}{0}$.

The following proposition will be useful:

(4.19) Proposition Let $B \in \mathcal{B}$ and let $i \in E \setminus B$. Then the following two statements are equivalent:

(a) $< X \in \mathcal{C}$ and $X_B \geq \underset{\sim}{0} > \Rightarrow < X_i \geq 0 >$;

(b) $X(B;\ j)_i \geq 0$ for all $j \in B$.

Proof The implication (a) $\Rightarrow$ (b) follows from (4.18). We shall prove (b) $\Rightarrow$ (a). Assume (b) is satisfied and suppose (a) does not hold. Let X be a vector in $\mathcal{C}$ with $X_B \geq \underset{\sim}{0}$, $X_i < 0$ and the set $B \cap \underline{X}$ being minimal with these properties. By (4.10), $B \cap \underline{X}$ is nonempty. Take any $j \in B \cap \underline{X}$. Since $X_j > 0$ and $X(B;j)_i \geq 0$, by using the elimination property (1.2 OM-3) for X and $-X(B;j)$, we obtain $X' \in \mathcal{C}$ with $X'_B \geq \underset{\sim}{0}$, $X'_i < 0$ and $B \cap \underline{X}' \subset B \cap \underline{X}$. This contradicts the choice of X. Therefore (a) holds. □

A corollary of (4.19) and (4.16.b) is

(4.20) Corollary Let $B \in \mathcal{B}$, $S \subseteq B$, and let $i \in E \setminus B$. Then the following two statements are equivalent:

(a) $< X \in \mathcal{C},\ X_{B\setminus S} \geq \underset{\sim}{0}$ and $X_S = \underset{\sim}{0} > \Longrightarrow < X_i \geq 0 >$;

(b) $X(B;\ j)_i \geq 0$ for all $j \in B \setminus S$.

(4.21) Proposition (c f. Bland [AL], Claim 4.3) Let $B \in \mathcal{B}$ and let g and j be two given elements of B. Then the following statements hold:

(a) If K is a subset of E such that $X(B;\ g)_K \geq \underset{\sim}{0}$ and the set $I \equiv \{i \in K\ :\ X(B;\ j)_i < 0\}$ is nonempty then there exists an element $i \in I$ such that $B' \equiv B \setminus \{j\} \cup \{i\} \in \mathcal{B}$ and $X(B'\ ;\ g)_K \geq \underset{\sim}{0}$;

(b) If K is a subset of E such that $X(B\ ;\ j)_K \geq \underset{\sim}{0}$ and the set $I \equiv \{i \in K : X(B;\ g)_i < 0\}$ is nonempty and $X(B\ ;\ j)_I > \underset{\sim}{0}$ then there exists an element $i \in I$ such that $B' \equiv B \setminus \{j\} \cup \{i\} \in \mathcal{B}$ and $X(B'\ ;\ g)_K \geq \underset{\sim}{0}$.

Proof The results follow from (4.12) and the following claim which is straightforward from (1.4).

Claim : If $X^1, X^2 \in \mathcal{C}$ and K is a subset of E much that $X^1_K > \underset{\sim}{0}$ and the set $I \equiv \{i \in K\ :\ X^2_i < 0\}$ is non-empty then there exists an element $i \in I$ and $X^3 \in \mathcal{C}$ such that $X^3_i = 0$, $X^3_K \geq \underset{\sim}{0}$ and $X^3_\ell = (X^1 \circ X^2)_\ell$ for all $\ell \in E$ not separating X^1 and X^2 . □

5. BASIS FORM OF THE FUNDAMENTAL THEOREM

The Fundamental Theorem (3.4) of Oriented Matroid Programming says that every feasible OP is either unbounded or has an optimal solution. We have shown that there is a simple inductive proof of the theorem.

It is well known that the corresponding LP theorem can be proved constructively by a finite simplex method. Since the simplex method produces feasible solutions of special type, basic feasible solutions, this constructive proof yields a stronger form of the theorem, which is sometimes called the basis form or schematic form of the theorem.

In the next two chapters, we prove the basis form of the OP Fundamental Theorem is true, although the finiteness of the simplex method cannot be guaranteed by any known finite pivot rules in a broader context of OP. The proof of the theorem is devided into two parts. The easier part is proved inductively in this chapter, and the more difficult part will be proved by the algorithm in Chapter 6.

This theorem and the basic idea of the proof is not new. This result is due to Bland [AL] . However his presentation of the subject is quite different from ours, because an oriented matroid is defined together with the "dual" oriented matroid in Bland [AL] .

The point of this chapter is to understand the Bland's results in a simplex setting without using duality. This approach is also an attempt to understand the subject more geometrically.

For this chapter we assume that $\mathcal{C}$ is an OM on a finite set E, g and f are given elements of E, and $E_o = E \setminus \{g\}$, $E_1 = E \setminus \{g,f\}$.

A. Standard OP

We say that an OP $P = (C; g,f)$ is standard if neither g is a loop of C nor f is a coloop of C . First of all we remark that a non-standard OP has obvious properties so that such an OP may be excluded from our consideration.

(5.1) Proposition For an OP $P = (C; g,f)$ the following properties hold:

(a) If g is a loop of C then the OP is infeasible.

(b) If f is a coloop of C then the OP is co-infeasible (, thus the OP is unbounded if it is feasible).

Let $\mathcal{B} = \mathcal{B}(C)$ be the set of all bases of C . Let

(5.2) $$\mathcal{B}_1 \equiv \mathcal{B}_1(C) = \{B \subseteq E_1 : B \cup \{g\} \in \mathcal{B}\} .$$

The set $\mathcal{B}_1$ is the set of bases of an OP $(C; g,f)$. By (4.15) we have :

(5.3) An OP $(C; g,f)$ is standard iff the set $\mathcal{B}_1$ of bases of the OP is nonempty.

Let A and A^∞ be the affine space and the infinite space of an affine OM $(C; g)$. One can easily verify the following properties for any basis $B \in \mathcal{B}_1$:

(5.4) There exists a unique vector X in the affine space A such that $X_B = \underset{\sim}{0}$, called a basic solution X(B). (Using the terminology of (4.11), X(B) is the fundamental cocircuit $X(B \cup \{g\};g)$ of g in the basis $B \cup \{g\}$ of C.)

(5.5) The polyhedron $P(B) = \{X \in A : X_B \geq \underset{\sim}{0}\}$ is a cone having the smallest face {X(B)} , called a basic feasible cone.

(5.6) For each basic element $j \in B$, there exists a unique feasible direction $Z \in A^{\infty}$ at X(B) in P(B) such that $Z_j > 0$ and $Z_{B\setminus\{j\}} = \underset{\sim}{0}$, called a basic feasible direction $Z^j(B)$. (By the language of (4.11), $Z^j(B)$ is the fundamental cocircuit $X(B \cup \{g\}, j)$ of j in the basis $B \cup \{g\}$ of C.)

(5.7) For each basic element $j \in B$, the polyhedron $P(B, B\setminus\{j\}) = \{X \in A : X_{B\setminus\{j\}} = \underset{\sim}{0}, X_j \geq 0\}$ is a cone having the smallest face {X(B)} and the infinite face $\{Z^j(B)\}$, called a basic feasible ray $R^j(B)$. Clearly each basic feasible ray $R^j(B)$ is a face of the basic feasible cone.

(5.8) The infinite face $P^{\infty}(B) = \{Z \in A^{\infty}: Z_B \geq \underset{\sim}{0}\}$ of the basic feasible cone P(B) contains the basic feasible direction $Z^j(B)$ for all $j \in B$.

The above terminologies concerning bases of OP are merely imitating the standard use of those terminologies in linear programming. The importance of these terminologies are that they play exactly same role in OM programming as they played in linear programming.

B. Basic Properties

We start with an elementary property.

(5.9) Proposition For any basis $B \in \mathcal{B}_1$ and any element $i \in E \setminus B$ the following three statements are equivalent:

(a) The basic solution $X(B)$ maximizes i over the basic feasible cone $\mathcal{P}(B)$;

(b) $Z_i \leq 0$ for all $Z \in \mathcal{P}^{\infty}(B)$;

(c) $Z^j(B)_i \leq 0$ for all $j \in B$ i.e., every basic feasible direction $Z^j(B)$ $(j \in B)$ has non-positive i-component.

Proof The first two statements are obviously equivalent from definitions. Also by (5.8), the statement (b) implies (c). It is left to prove the implication (c) => (b). Observing that $Z^j(B)$ is the fundamental cocircuit $X(B \cup \{g\}; j)$ of j in the basis $B \cup \{g\}$ of $\mathcal{C}$, it follows from (4.20) that (c) => (b). □

Remember that a subset F of E_1 is cofeasible (for an OP $(\mathcal{C}; g,f)$) if

$$Z_f \leq 0 \quad \text{for all} \quad Z \in \mathcal{P}^{\infty}(F) .$$

Proposition (5.9) tells us two different characterizations of cofeasible basis:

(5.10) A basis $B \in \mathcal{B}_1$ is cofeasible iff

(a) the basic solution $X(B)$ maximizes f over the basic feasible cone $\mathcal{P}(B)$; iff

(b) $Z^j(B)_f \le 0$ for all $j \in B$ i.e. every feasible direction $Z^j(B)$ $(j \in B)$ has nonpositive f-component.

We say that a basis $B \in \mathcal{B}_1$ is <u>feasible</u> if the basic solution $X(B)$ is feasible i.e., $X(B)_{E_1} \ge 0$, and <u>optimal</u> if it is both feasible and cofeasible. Clearly

(5.11) If $B \in \mathcal{B}_1$ is optimal then the basic solution $X(B)$ is an optimal solution.

We say that a basis $B \in \mathcal{B}_1$ is <u>inconsistent</u> if

(5.12) $\exists\, i \in E_1 \backslash B$ s.t. $X(B)_i < 0$ and $Z_j(B)_i \le 0$ for all $j \in B$.

(5.13) <u>Proposition</u> If there exists an inconsistent basis then the OP is infeasible.

<u>Proof</u> Suppose $B \in \mathcal{B}_1$ is an inconsistent basis. By (5.9) there exists $i \in E_1 \backslash B$ such that $X(B)$ maximizes i over $\mathcal{P}(B)$ and $X(B)_i < 0$. It follows from (2.24) that $\mathcal{P}(B \cup \{i\}) = \phi$ and hence $\mathcal{P}(E_1) = \phi$. This proves the result. □

A basis $B \in \mathcal{B}_1$ is said to be <u>co-inconsistent</u> if

(5.14) $\exists j \in B$ such that $Z^j(B)$ is an infinite-optimal vector i.e., $Z^j(B)_f > 0$ and $Z^j(B)_{E_1} \geq 0$.

It follows immediately from the definition that

(5.15) If there is a co-inconsistent basis then the OP is infeasible or unbounded.

We say that a basis $B \in \mathcal{B}_1$ is <u>unbounded</u> if it is both feasible and co-inconsistent.

C. The Theorem

It should be clear that the inconsistency, the optimality and the unboundedness of a basis B are sign properties on the basic solution X(B) and the basic feasible directions $\{Z^j(B) : j \in B\}$, and that the existence of each inconsistent, optimal or unbounded basis implies the infeasibility, the existence of an optimal solution, or the unboundedness of the OP.

The following is the main theorem:

(5.16) Theorem (Basis Form of the Fundamental Theorem)
Every standard OP has either an optimal, unbounded, or inconsistent basis.

This theorem can be viewed as the following two statements:

(5.17.a) Every standard OP has either a feasible or inconsistent basis;

(5.17.b) Every standard OP having a feasible basis has either an optimal or unbounded basis.

Bland [AL, §5] has stated a result for dual pairs of oriented matroids which implies Theorem (5.16). He has also described (private conversation) how it is possible to extend his pivoting operation in order to obtain this result. The proof presented here (developed independently) based on a purely primal approach can in fact be easily adapted to prove the result stated by Bland.

D. Phase I

In this section we shall prove the easier part of Theorem (5.16). Namely we shall prove Feasibility Theorem (5.17.a), provided that the Optimality Theorem (5.17.b) is true. Since Theorem (5.16) is equivalent to (5.17.a) together with (5.17.b), once (5.17.b) is proved (in the next chapter) the main theorem (5.16) will be proved.

One reason for separating the proof of the feasibility part from the proof for the optimality part is that the first part is considerably simpler than the other one. Also it will be clear that obtaining the Feasibility Theorem from the Optimality is essentially an analogue of the Phase I of the simplex method, although the proof will be by induction.

(5.18) Proof ((5.17.b) => (5.17.a)).

Suppose (5.17.b) is true. Let $P = (\mathcal{C}; g,f)$ be a standard OP and we want to show that P has either a feasible or inconsistent basis. Since P is standard, there is a basis $B \in \mathcal{B}_1$. We assume by induction that for any nonempty subset F of $E_1 \setminus B$ the subproblem $P \setminus F = (\mathcal{C} \setminus F; g,f)$ has either a feasible or inconsistent basis. (Clearly B is a feasible basis of $P \setminus (E_1 \setminus B)$.) Take any $r \in E_1 \setminus B$ and consider the subproblem $P' \equiv P \setminus \{r\}$. The set $\mathcal{B}_1'$ of bases of P' is $\{B' : r \notin B' \in \mathcal{B}_1\}$. Since P' is standard, it follows from the induction that there exists a basis B' of P' which is either feasible or inconsistent for P'. It is easy to see that if B' is an inconsistent basis of P' then it is an inconsistent basis of P . Hence we can assume that it is a feasible basis of P'. Now consider the OP $P'' = (C\setminus\{f\} ; g, r)$. Note that B' is a feasible basis of P". By (5.17.b), there exists a basis B" of P" which is either optimal or unbounded for P". If $X(B'')_r \geq 0$ then B" is a feasible basis and we are done. We can assume

(5.19) $X(B'')_r < 0.$

There are two cases to consider.

<u>Case 1:</u> B" is optimal for P".

In this case, by (4.16) and (5.10) we have

$$z^j(B'')_r \leq 0 \quad \text{for all } j \in B'' .$$

This together with (5.19) implies the inconsistency of B".

<u>Case 2:</u> B" is unbounded.

Again by (4.16) and (5.14) we obtain

(5.20) $\exists\, j \in B''$ such that

$$z^j(B'')_r > 0 \quad \text{and}$$

$$z^j(B'')_i \geq 0 \quad \text{for all } i \in E_1\setminus \{r\}$$

Since $z^j(B'')_r \neq 0$, the set $\hat{B} \equiv B'' \setminus \{j\} \cup \{r\}$ is a basis of P. By (5.19) and since

$$X(B'')_i \geq 0 \quad \text{for all } i \in E_1\setminus \{r\} ,$$

we can apply the elimination property (1.2 OM-3) for X(B") and Z^j(B") eliminating r to obtain a feasible vector X satisfying $X_{\hat{B}} = 0$. This implies $X(\hat{B}) = X$ and hence $\hat{B}$ is a feasible basis.

This completes the proof. □

6. PIVOT METHODS

The simplex method for linear programming which was discovered by Dantzig [LP] is one of the most celebrated algorithms in both theory and practice. The main mechanism of the algorithm is the pivot operation which replaces a canonical system of linear equalities by another equivalent canonical system.

In this chapter we study the pivot operation in OM programming which abstracts the same terminology in linear programming. We shall introduce a pivot method as a general algorithm for OM programming. We introduce the simplex method as a feasible pivot method, and we point out that there is no known finite pivot rule for the simplex method in a broader setting of oriented matroid programming. Then a finite pivot method is proposed, from which the Optimality Theorem (5.17.b) follows immediately.

It should be noted that Bland [OT][AL] first pointed out that pivoting operations and the simplex method in linear programming have the natural abstractions in the context of dual pairs of OM's. Here, we shall employ these notions from a different point of view, one that does not concern the duality of OM's.

For this chapter we assume that $\mathcal{C}$ is an OM on a finite set E, g and f are given elements of E, P is the OP $(\mathcal{C};g,f)$, $E_o = E \setminus \{g\}$ and $E_1 = E \setminus \{g,f\}$.

A. Pivot Operations

Let $\mathcal{B}$ be the set of bases of $\mathcal{C}$ and let $\mathcal{B}_1$ be the set of bases of P:

$$\mathcal{B}_1 = \{B \setminus \{g\} \quad : \quad g \in B \in \mathcal{B}\}.$$

It follows from the exchange property of bases (4.12) that

(6.1) Proposition For $B \in \mathcal{B}_1$, $j \in B$ and $i \in E_1 \setminus B$,

$$< B \setminus \{j\} \cup \{i\} \in \mathcal{B}_1 > \iff < Z^j(B)_i \neq 0 > \quad .$$

If $B \in \mathcal{B}_1$, $j \in B$, $i \in E_1 \setminus B$ and if $Z^j(B)_i \neq 0$, the replacement of a basis B by $B \setminus \{j\} \cup \{i\}$ is said to be the pivot (operation) at (i,j) in a basis B. A pivot replacing B by B' is degenerate if $X(B) = X(B')$.

The following is immediate:

(6.2) Proposition Let $B \in \mathcal{B}_1$, $j \in B$, $i \in E_1 \setminus B$ and $Z^j(B)_i \neq 0$. Then the pivot at (i,j) in B is degenerate iff $X(B)_i = 0$.

(6.3) A pivot method is an algorithm which start with a given basis $B^o \in \mathcal{B}_1$ and generates a sequence $B^o, B^1, B^2, \ldots$ of bases using pivot operations.

We assume that

(6.4) The oracle of a pivot method is to give the basic solution X(B) and the basic feasible direction $Z^j(B)$ for any basis $B \in \mathcal{B}_1$ and any $j \in B$.

B. Feasible Pivot Method

A pivot operation replacing $B \in \mathcal{B}_1$ by $B' \in \mathcal{B}_1$ is said to be feasible if both B and B' are feasible bases. A pivot method is said to be feasible if it generates feasible bases only, or equivalently it only uses feasible pivot operations.

Setting $K = E_1$ in Proposition (4.21.a) implies:

(6.4) Proposition (Bland [AL]) Let $B \in \mathcal{B}_1$ be a feasible basis. If $j \in B$ and the set $I \equiv \{i \in E_1 \setminus B: z^j(B)_i < 0\}$ is nonempty then there exists $i \in I$ such that $B \setminus \{j\} \cup \{i\}$ is a feasible basis.

For $B \in \mathcal{B}_1$, $i \in B$ and $j \in E_1 \setminus B$, a pivot at (i,j) in B is said to be simplex if the following three conditions are satisfied:

(6.5.a) it is a feasible pivot ;

(6.5.b) $z^j(B)_i < 0$; and

(6.5.c) $z^j(B)_f > 0$.

Now we have:

(6.6) Proposition (Bland [AL]) If $B \in \mathcal{B}_1$ is feasible then either it is unbounded, optimal, or there exists a simplex pivot in B.

<u>Proof</u> Let B be a feasible basis. Suppose it is neither optimal nor unbounded. Thus it is not cofeasible, and by (5.10) this means

$$\exists\, j \in B \quad \text{such that} \quad Z^j(B)_f > 0.$$

Since it is not co-inconsistent, and by (5.14), we know

$$I \equiv \{i \in E_1 \setminus B \ : \ Z^j(B)_i < 0\} \neq \phi \quad .$$

By Proposition (6.3), we know that there is $i \in I$ such that a pivot at (i,j) in B is a simplex pivot. This completes the proof. □

The simplex method is a feasible pivot method which naturally follows from Proposition (6.6):

(6.7) <u>Simplex Method</u>

<u>Input</u>: An initial feasible basis B^o.

<u>Output:</u> Either an unbounded or optimal basis if it terminates.

<u>Initialization:</u> Set $B = B^o$

<u>Steps</u>

(S1) Select any element $\ell \in B$ such that $Z^\ell(B)_f > 0$. If there is no such ℓ, stop (=> the basis is optimal).

(S2) Test: $I \equiv \{i \in E_1 \setminus B : Z^\ell(B)_i < 0\} = \phi$? If so, stop (=> the basis is unbounded). Otherwise select any $k \in I$ such that the basis $B \setminus \{\ell\} \cup \{k\}$ is feasible

(S3) Perform a simplex pivot at (k,ℓ) in B and repeat the above procedure with the new basis $B' = B \setminus \{\ell\} \cup \{k\}$.

It is wellknown that the simplex method is not finite in general, even if OP's are restricted to be linear (see Beale [C D]). This is because the simplex method may cycle, that is, to generate a sequence

(6.8) $B^o \to B^1 \to B^2 \to \cdots\cdots \to B^k$

of feasible bases such that $B^o = B^k$ (where $B^{i-1} \to B^i$ denotes the pivot operation replacing B^{i-1} by B^i).

Here we should make distinction between two types of cycling.

(6.9) The first one is a degenerate cycling, that is, a cycling of pivots, all of which ($B^{i-1} \to B^i$ for $i = 1,2,\ldots,k$) are degenerate.

(6.10) The second one is a nondegenerate cycling, that is, a cycling of pivots, not all of which are degenerate.

It is easy to see that

(6.11) Proposition If the simplex method produces a cycling (6.8) for a linear OP then it is a degenerate cycling.

Hence if there is a systematic rule (pivot rule) which restricts the selection of elements ℓ and k to leave and to enter the basis in the simplex method, and which excludes the possibility of degenerate cycling then the basis form of the Fundamental Theorem (5.6) with linear restriction follows immediately.

(6.12) In fact Bland [FS] [AL] has shown that with the following rule the simplex method never produces any degenerate cycling.

(6.13) Smallest Subscript Rule (α)

Initialization Give any ordering on the constraint elements: $E_1 = \{e_1, e_2, \ldots, e_n\}$.

Rules

(α1) In Step (S1), if there is more than one element which can be selected, select the element ℓ with the smallest subscript:

(α2) In Step (S2), if there is more than one element which can be selected (i.e. $|I| > 1$) tl select the element k with the smallest subscript

It is natural to conjecture that Proposition (6.11) is true in general without linear restriction and hence that the simplex method with (α) is finite for any OP. Unfortunately this is not true as we show in Chapter 7:

(6.14) Theorem There is an OP for which the simplex method produces a nondegenerate cycling.

(6.15) Theorem There is an OP for which the simplex method with the Smallest Subscript Rule cycles.

Theorem (6.14) answers one of the problems raised by Bland [AL] negative

C. Finite Pivot Method: The Algorithm A

We say that a pivot method is finite if it terminates in finite number of iterations for any given OP.

It has been remarked that there is no known feasible pivot method which is finite and which terminates in either unbounded or optimal basis.

In this section we will present a finite pivot method which either produces an unbounded or an optimal basis. This algorithm is a modification of the finite pivot method of Bland [AL]. Our algorithm is not known to be feasible, while the Bland's algorithm will be shown to be non-feasible in Chapter 7.

Before we describe the algorithm we shall present the main lemma by which the validity of the algorithm will be guaranteed.

Let $h \in E_1$ be any given constraint element. As defined before, $P \setminus \{h\}$ and $P / \{h\}$ denote the subproblems $(C \setminus \{h\} ; g,f)$ and $(C / \{h\} ; g,f)$ of P, respectively. By (4.16), the sets $\mathcal{B}_1(P \setminus \{h\})$ and $\mathcal{B}_1(P / \{h\})$ of bases of the subproblems are defined by

(6.16.a) $\qquad \mathcal{B}_1(P \setminus \{h\}) = \{B : h \notin B \in \mathcal{B}_1\}$

(6.16.b) $\qquad \mathcal{B}_1(P / \{h\}) = \{B \setminus \{h\} : h \in B \in \mathcal{B}_1\}$, where $\mathcal{B}_1$ is the set of bases of P. It is easy to see the following:

(6.17.a) If B is an inconsistent basis of P \ {h} then B is an inconsistent basis of P ;

(6.17.b) If B is an co-inconsistent basis of P / {h} then B ∪ {h} is an co-inconsistent basis of P. (Note that this statement is still valid when "co-inconsister is replaced by unbounded).

The main lemma is the following:

(6.18) <u>Lemma</u> (Bland [AL], Theorem 5.1) Let $h \in E_1$ be a fixed constraint element and let B^1 and B^2 be bases of P with $h \in B^2 \setminus B^1$. Further suppose that $B^2 \setminus \{h\}$ is an optimal basis of $P / \{h\}$ but B^2 is not an optimal basis of P. Then the following properties hold:

(a) If B^1 is an optimal basis of P \ {h} then it is an optimal basis of P as well;

(b) If there exists $j \in B^1$ such that $z^j(B^1)_f > 0$ and $z^j(B^1)_{E_1 \setminus \{h\}} \geq \underset{\sim}{0}$ (i.e. B^1 is an co-inconsistent basis of P \ {h}) then $z^j(B^1)_h > 0$ (thus B^1 is an co-inconsistent basis of P as well).

<u>Proof</u> The proof of this lemma is very similar to that of Lemma (3.11).

(a) Assume that B^1 is an optimum basis of $P\setminus\{h\}$. Suppose the conclusion is false, that is, neither B^1 nor B^2 is optimal. Let $X^i = X(B^i)$ for $i = 1,2$. Since B^1 is optimal for $P\setminus\{h\}$ but not for P, $X^1_h < 0$. Let P' be the polyhedron $\{X \in A : X_{E_1\setminus\{h\}} \geq 0 \text{ and } X_h \leq \underset{\sim}{0}\}$. Clearly $X^i \in P'$ for $i = 1,2$, and X^1 maximizes h over P'. Let Z' be any direction from X^2 to X^1. Then by (2.21) and (2.23),

$$Z'_f \geq 0,\ Z'_{B^2\setminus\{h\}} \geq \underset{\sim}{0} \ \text{ and } \ Z'_h < 0 \ .$$

On the other hand, since B^2 is optimal for $P / \{h\}$ but not for P, we know that

$$Z_f > 0,\ \ Z_{B^2\setminus\{h\}} = \underset{\sim}{0} \ \text{ and } \ Z_h > 0$$

for $Z = Z^h(B^2)$. Using the elimination property (1.2 OM-2) for Z and Z' eliminating h, we obtain $Z'' \in A^\infty$ s.t.

$$Z''_f > 0,\ \ Z''_{B^2\setminus\{h\}} \geq \underset{\sim}{0} \ \text{ and } \ Z''_h = 0 \ .$$

This contradicts the optimality of B^2 in $P / \{h\}$. Therefore the result holds.

(b) $Z^j(B^1)_h \geq 0$ follows from a similar argument as above. $Z^j(B^1)_h \neq 0$ follows from the fact that $X(B^1)$ maximizes f over $\{X \in A : X_h = 0 \text{ and } X_{E_1\setminus\{h\}} \geq \underset{\sim}{0}\}$. □

The following pivot method which makes use of Lemma (6.18) will be shown to be finite and produces either an optimal or co-inconsistent basis provided an initial feasible basis is given.

(6.19) <u>Algorithm A</u>

<u>Input</u> : An feasible basis B^o

<u>Output</u> : Either an optimal or co-inconsistent basis.

<u>Data Structure</u> (to be updated each iteration)

(i) a basis B of the OP

(ii) a subset R of $E_1 \setminus B$

(iii) the set $T = R \cup B$ with a linear order of elements $T = \{t_1, t_2, \ldots, t_{|T|}\}$.

<u>Initialization</u>

Set $B = B^o$, $R = \phi$, $T = B^o$, and give any linear order $T = \{t_1, t_2, \ldots, t_{|T|}\}$.

<u>Steps</u>

(A1) Test: $J(B) \equiv \{j \in B : Z^j(B)_f > 0\} = \phi$? If so, stop ($\Rightarrow$ B is optimal). Otherwise select the largest number s such that $1 \leq s \leq |T|$ and $t_s \in J(B)$, and let $\ell = t_s$.

(A2) Let $R^s = R \cap \{t_1, t_2, \ldots, t_s\}$ and let $I = E_1 \setminus (B \cup R^s)$. Test : $Z^\ell(B)_i \geq 0$ for all $i \in I$. If so, stop ($\Rightarrow$ B is co-inconsistent). Otherwise select any $k \in I$ such that $Z^\ell(B)_k < 0$ and

$$X(B \setminus \{\ell\} \cup \{k\})_I \geq 0 \ .$$

(A3) Repeat the same procedure with the new basis $B' = B \setminus \{\ell\} \cup \{k\}$ and with $R' = R^s \cup \{\ell\}$ and $T' = R' \cup B'$ with any linear order of elements $T' = \{t'_1, t'_2, \ldots, t'_{|T'|}\}$ satisfying $t'_i = t_i$ for all $1 \le i \le s$.

(6.20) Proposition Suppose that B is a basis obtained by the algorithm A, and let the associated data structure be given by (6.19) (i),(ii),(iii). Then for each $t_j \in R$ $(1 \le j \le |T|)$, there exists a basis $\bar{B}$ obtained previously by the algorithm with the associated data structures $\bar{R}, \bar{T} = \{\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_{|\bar{T}|}\}$ $(= \bar{R} \cup \bar{B})$ satisfying

(a) $\bar{t}_i = t_i$ for $1 \le i \le j$;

(b) $\bar{t}_i \in \bar{R}$ iff $t_i \in R$ $(1 \le i \le j - 1)$; and

(c) $\bar{t}_j \in J(\bar{B})$ and $\bar{t}_{j+1}, \ldots, \bar{t}_{|T'|} \notin J(\bar{B})$ (thus $\bar{t}_j$ was chosen to leave the basis $\bar{B}$).

Proof This follows immediately from the algorithm. □

(6.21) Proposition Suppose that (i), (ii), (iii) in (6.19) are the data structures obtained by the algorithm in some iteration. Then the following holds:

(a) $X(B)_i \ge 0$ for all $i \in E_1 \setminus R$;

(b) If $1 \le j \le |T|$ and $t_i \notin J(B)$ for all $j \le i \le |T|$ then $X(B)_i \ge 0$ for all $i \in E_1 \setminus R^{j-1}$ where $R^{j-1} = R \cap \{t_1, \ldots, t_{j-1}\}$.

<u>Proof</u> The proof is tedious but not difficult. Use induction on the number of iterations. The properties (a) and (b) are obvious in the first iteration, and these in a general iteration follows from (6.18) and (6.20). □

Now we state the main theorems:

(6.22) <u>Theorem</u> The Algorithm A is finite.

(6.23) <u>Theorem</u>

(a) If the Algorithm A stops at the step (A1) then the last basis B is optimal.

(b) If the Algorithm A stops at the step (A2) then the last basis B is co-inconsistent, and furthermore, for $\ell = t_s$ the element chosen to leave the basis in (A1), $Z^{\ell}(B)_i > 0$ for all $i \in R^s$ and $X(B)_j \geq 0$ for all $j \in E_1 \setminus R^s$.

<u>Proof of (6.22):</u> Let B and B' be two consecutive bases obtained by the algorithm and let $T = \{t_1, \ldots, t_{|T|}\}$ and $T' = \{t'_1, \ldots, t'_{|T'|}\}$ be the associated ordered sets. Let γ be the integral vector $(\gamma_1, \gamma_2, \ldots, \gamma_m)$ defined by

$$\gamma_1 < \gamma_2 < \cdots < \gamma_m \quad \text{and}$$

$$\{t_{\gamma_1}, t_{\gamma_2}, \ldots, t_{\gamma_m}\} = B$$

where m is the cardinality of a basis (= r(C)-1). Let γ' be the corresponding vector for B' and T'. By the step (B3) we know that there exists $1 \leq s \leq |T|$ such that $t'_i = t_i$ for all $1 \leq i \leq s$ and

$$t_j \in B \iff t'_j \in B' \qquad (1 \in j \in s - 1)$$

and

$$t_s \in B \text{ and } t'_s \notin B' .$$

This implies that γ' is lexicographically greater than γ. Since the cardinality of a set T cannot exceed $|E_1|$, the lexicographically increasing sequence of γ's associated with the algorithm must be finite. This completes the proof. □

Proof of (6.23) The part (a) follows immediately from (6.21). We shall prove the part (b). Suppose that the algorithm stops at (A2). By the selection of pivot element $\ell = t_s$ in (A1), and by (6.21)

(6.24) $$X(B)_i \geq 0 \qquad \text{for all} \quad i \in E_1 \setminus R^s .$$

Also, by the stopping criterion in (A2) we have

(6.25) $$Z^\ell(B)_i \geq 0 \qquad \text{for all} \quad i \in E_1 \setminus R^s .$$

Suppose that $Z^\ell(B)_{t_j} \leq 0$ for some $t_j \in R^s$ and let $\hat{j}$ be the largest j with this property. We shall obtain a contradiction. First we remark that

(6.26) $$Z^\ell(B)_i \geq 0 \qquad \text{for all} \quad i \in E_1 \setminus R^{\hat{j}} .$$

Setting $h = t_{\hat{j}}$, $\hat{R} = R^{\hat{j}-1}$, $\hat{S} = B \cap \{t_1, \ldots, t_{\hat{j}-1}\}$ and $\hat{P} = P \setminus R / S$. It follows from (6.20) that there is a basis $\bar{B}$ produced by the algorithm such that $\hat{S} \subseteq \bar{B}$ and $\bar{B} \setminus \hat{S}$ is an optimal basis of $\hat{P}$ / {h} but not of $\hat{P}$. It follows from (6.26) and the fact $\hat{R} \cap \bar{B} = \phi$ that

(6.27) $\quad z^{\ell}(B)_i \geq 0 \quad$ for all $\quad i \in \hat{E}_1 \setminus \{h\}$

where $\hat{E}_1 = E_1 \setminus (R \cup S)$. By (6.18.b) we have $z^{\ell}(B)_h < 0$, a contradiction. This completes the proof. □

(6.28) Lemma If B is a co-inconsistent basis obtained by the Algorithm A then it takes at most one pivot to obtain an unbounded basis.

Proof If B is unbounded, there is nothing to prove. Otherwise by (6.23.b) we have

$$\phi \neq I \equiv \{ i \in E_1 : X(B)_i < 0 \} \subseteq R^S \quad ,$$

and $z^{\ell}(B)_I > \underset{\sim}{0}$. Since $z^{\ell}(B)_{E_1} \geq 0$ and using (4.21.b) (with $K = E_1$, $j = \ell$), there exists $i \in I$ such that the basis $B' = B \setminus \{\ell\} \cup \{i\}$ is feasible (i.e., $X(B')_{E_1} \geq \underset{\sim}{0}$). Observing that $z^{\ell}(B) = z^{i}(B')$, the basis B' is unbounded. □

The above results (6.22), (6.23) and (6.28) imply the following result which proves the optimality theorem (5.17.b).

(6.29) Theorem There is a finite algorithm for any OP which transforms a feasible basis into either an optimal or unbounded basis.

7. NONDEGENERATE CYCLING OF THE SIMPLEX METHOD

In this chapter, we will construct an example of OP for which the simplex method can produce a cycle of pivots, which contains at least one nondegenerate pivot. The construction will start with a linear OM with 8 elements, and determine an OP on the OM (by choosing two elements, the infinite and the objective elements) such that there is a cycle of 6 pivots, 3 of which are degenerate simplex pivots and the rest are non-degenerate non-simplex pivots. Then we use an operation, called a perturbation by which the OM will be transformed to a slightly different OM on the same elements in such a way that the bases of the old OP remain bases of the new OP, the feasible bases of the old remain feasible, and the old cycle becomes a cycle of all simplex pivots, 3 of which are nondegenerate.

It will be also shown that one can further transform this OP to a new OP for which the simplex method can produces a cycle of pivots all of which are nondegenerate. This example will be used to show that (i) the simplex method with the Smallest Subscript Rule is not finite (ii) the Bland's Finite Algorithm for OP's (which will be described in this chapter) produces an infeasible solution and hence it is not a feasible pivot method.

A. The Starter

Let E be the set $\{g,1,2,3,\ldots,6,f\}$ of 8 elements. Let A be $4 \times E$ matrix given

$$(7.1) \qquad A = \begin{array}{c} \begin{matrix} g & 1 & 2 & 3 & 4 & 5 & 6 & f \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & \frac{1}{2} & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & \frac{1}{2} & -1 \end{bmatrix} \end{array},$$

and let $C(A)$ be the linear OM having a matrix representation A. Thus $C(A)$ is the set of incidence sign vectors of vectors X in the row space $R(A)$ of A:

$$(7.2) \qquad R(A) = \{x \in R^E: x = (\lambda_o, \lambda_1, \lambda_2, \lambda_3)\, A \quad \text{for some } (\lambda_o, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^4\}$$

Consider the linear OP $P = (C(A); g,f)$. This OP corresponds to the following LP:

$$(7.3) \qquad \text{maximize } x_f$$

$$x \in R(A)$$

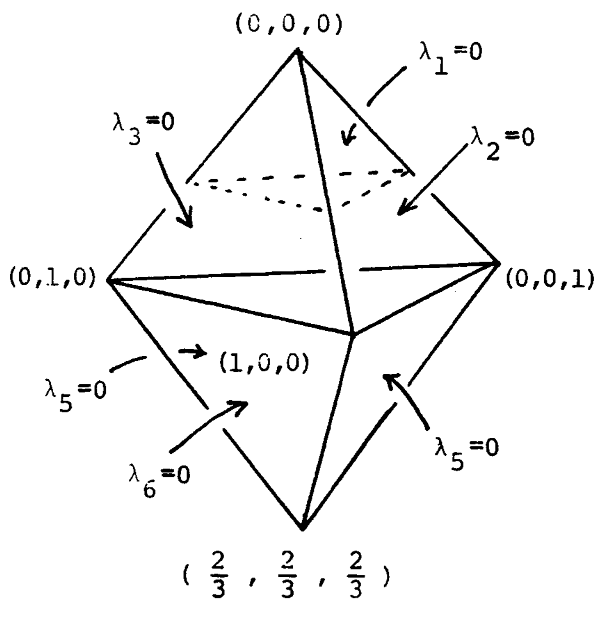
$$x_i \geq 0 \qquad i = 1,2,\ldots,6$$

$$x_g = 1$$

which is reduced to

$$(7.4)\qquad \begin{array}{ll} \text{maximize} & -\lambda_1 - \lambda_2 - \lambda_3 + \frac{1}{2} \\ \text{subject to} & \lambda_1 \geq 0 \\ & \lambda_2 \geq 0 \\ & \lambda_3 \geq 0 \\ & \frac{1}{2}\lambda_1 - \lambda_2 - \lambda_3 + 1 \geq 0 \\ & -\lambda_1 + \frac{1}{2}\lambda_2 - \lambda_3 + 1 \geq 0 \\ & -\lambda_1 - \lambda_2 + \frac{1}{2}\lambda_3 + 1 \geq 0 \end{array}$$

Geometrically the feasible region of (7.4) is a 3-dimensional polytope of 6 facets, 12 edges and 5 vertices. Clearly the optimal solution of (7.4) is $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (0, 0, 0)$. Through the relation: $x = \lambda A$, the i-th constraint in (7.4) corresponds to the non-negative constraint of i-th variable x_i, for $i = 1,2,\ldots,6$. Thus, at the optimal solution x^* of (7.3), $x_1^* = x_2^* = x_3^* = 0$ and x_4^*, x_5^*, x_6^* are positive. This implies that the optimal solution X^* of the OP P^* is $(X_g^*, X_1^*, \ldots, X_6^*, X_f^*) \equiv (+,0,0,0,+,+,+,+)$.



It is clear that

(7.5) $F \subseteq E$ is independent in $\mathcal{C}(A)$ iff the corresponding column vectors $\{A^e : e \in F\}$ are linearly independent.

(7.6) $B \subseteq E$ is a basis of $\mathcal{C}(A)$ iff the corresponding column vectors $\{A^e : e \in B\}$ are linearly independent and $|B| = 4$. (e.g. $\{g,1,2,3\}$ is a basis of $\mathcal{C}(A)$).

(7.7) Let $E_1 = E \setminus \{g,f\} = \{1,2,3,4,5,6\}$. A subset B of E_1 is a basis of the OP P iff the corresponding column vectors $\{A^e = e \in B\}$ with the first row deleted are linearly independent and $|B| = 3$. (e.g. $\{1,2,3\}$ is a basis.)

(7.8) For the basis $B = \{g,1,2,3\}$ of $\mathcal{C}(A)$, and for $j \in B$, the fundamental vertex $X(B;j)$ is the incidence signed vector of the row of A in which j-component is one. (e.g. $X(B;g) = (+,0,0,0,+,+,+,+)$ and $X(B;2) = (0,0,+,0,-,+,-,-)$.)

(7.9) The pivot (or row) operations in A preserve the OM, e.g., a pivot on the 2nd row 7th column in A results in

$$A_1 = \begin{array}{c} \begin{matrix} g & 1 & 2 & 3 & 4 & 5 & 6 & f \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & \frac{3}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & -1 & 0 & 0 & -\frac{1}{2} & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & -\frac{3}{2} & \frac{3}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 1 & -\frac{3}{4} & -\frac{3}{2} & 0 & -\frac{3}{2} \end{bmatrix} \end{array}$$

where the obvious basis of C(A) is {g,2,3,6} and the fundamental vertices in the basis are shown . It will be convenient to perform one more pivot on the 4-th row 6-th column in A_1 to obtain

$$A_o = \begin{array}{c} \begin{matrix} g & 1 & 2 & 3 & 4 & 5 & 6 & f \end{matrix} \\ \left[\begin{matrix} 1 & 1 & 0 & 0 & \frac{3}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{2}{3} & 0 & \frac{2}{3} & -1 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 & 1 & \frac{9}{4} & 0 & 0 & -\frac{3}{2} \\ 0 & -3 & 0 & -\frac{2}{3} & \frac{1}{2} & 1 & 0 & 1 \end{matrix}\right] \end{array}$$

where the basis {g,2,5,6} is shown.

(7.10) From (7.9) $B^o = \{2,5,6\}$ and $B^1 = \{2,3,6\}$ are bases of the OP and in fact they are feasible, since the corresponding basic solutions

$X(B^o) \equiv X(B^o \cup \{g\}\ ;\ g) = g\ 1\ 4\ \underline{f}$ and

$X(B^1) \equiv X(B^1 \cup \{g\}\ ;\ g) = g\ 1\ 4\ \underline{f}$ are feasible

(where we use a new notation for signed vectors and $g\ 1\ 4\ \underline{f}$ denotes the sign vector X on E such that for the non-underlined elements $X_g = X_1 = X_4 = 0$, for the underlined element $X_f = -$, and the rest of components are zero.) Remarking that the basic feasible direction $Z^5(B^o) \equiv X(B^o \cup \{g\}\ ;\ 5) = \underline{1}\ \underline{3}\ 4\ 5\ f$ from the last row of A_o, $Z^5(B^o)_f = +$ and hence replacement of B^o by B^1 is a simplex pivot (see the circled entry in A_o is the pivot entry), although it is degenerate.

(7.11) Because of the symmetry in the matrix A, the permutation β on E:

$$\beta = \begin{pmatrix} g & 1 & 2 & 3 & 4 & 5 & 6 & f \\ g & 2 & 3 & 1 & 5 & 6 & 4 & f \end{pmatrix}$$

preserves many properties of the OM $C(A)$.

For example, let

$$B^2 = \beta(B^o) = \{3,4,6\}$$
$$B^3 = \beta(B^1) = \{1,3,4\}$$

be bases of P. Then

$$B^4 = \beta(B^2) = \{1,4,5\}$$
$$B^5 = \beta(B^3) = \{1,2,5\}$$

are also bases of P. Clearly $\beta(B^4) = B^o$ and $\beta(B^5) = B^1$.

For the permutation β, and a signed vector X on E, we define $\beta(X)$ as the signed vector on E s.t. for each $e \in E$.

$$\beta(X)_e = X_{\beta^{-1}(e)} \quad .$$

Then we can easily see that

$$X(B^2) = \beta(X(B^o))$$
$$X(B^3) = \beta(X(B^1))$$
$$X(B^4) = \beta(X(B^2))$$
$$X(B^5) = \beta(X(B^3))$$

Since β fixes g and f, it preserves the feasibility and hence all of the bases $B^o, B^1, \ldots, B^5$ are feasible.

Also, by (7.10) and by the symmetry, the replacement of B^2 by B^3, B^4 and B^5 are degenerate simplex pivots. Geometrically, three basic solutions correspond to the three vertices in the middle of the polytope in Fig.1. The degeneracy is caused by four constraint hyperplanes intersecting at each of the three vertices.

(7.12) Consider the bases $B^1 = \{2,3,6\}$ and $B^2 = \{3,4,6\}$. From the matrix A_1, the basic feasible direction $Z^2(B^1) \equiv X(B^1 \cup \{g\}, 2) = \underline{1}\ 2\ \underline{4}\ 5$ has zero f-component. Therefore, the pivot replacing B^1 by B^2 is not a simplex pivot. By the symmetry the replacements of B^3 by B^4, B^5 by B^1 are not simplex pivots.

(7.13) So far we observed that there is a cycle of 6 pivots: $B^o \to B^1 \to \cdots \to B^5 \to B^o$ three of which are degenerate pivots, and the rest are non-simplex nondegenerate pivots.

B. The Vertex Perturbation

For a signed vector X on E, for an element $e \in E$, and for $\alpha = +, 0,$ or $-$, we define $X + e^\alpha$ as the sign vector on E such that

$$(X + e^\alpha)_j = \begin{cases} X_j & \text{if} \quad j \in E \setminus \{e\} \\ \alpha & \text{if} \quad j = e \end{cases} .$$

(7.14) Theorem (Vertex Perturbation): Let C be an OM on a finite set E. Suppose that there is an element f and a vertex V of C satisfying the following conditions (a) and (b):

(a) $V_f = 0$;

(b) $< X \in C \, , \ X_f = 0 \, , \ V \prec X >$
$\Longrightarrow \ < X + f^- \in C >$.

Then $\hat{C} = C \setminus \{V, -V\} \cup \{\hat{V}, -\hat{V}\} \cup N \cup -N$

is an OM on E, where

$$\hat{V} = V + f^+$$
$$N = {}^oN \cup {}^+N$$
$${}^oN = \{X + f^o : V \prec X \in C \ \text{ and } \ X_f = -\}$$
$${}^+N = \{X + f^+ : V \prec X \in C \ \text{ and } \ X_f = -\} .$$

The above theorem was first proved by J. Edmonds, A. Mandel and the author. In Chapter 12, we show that a considerably more general theorem is true.

Let $\mathcal{C}$ be an OM on a set E. For an element $f \in E$, a vertex V of $\mathcal{C}$ is said to be <u>f-flexible (in $\mathcal{C}$)</u> if the conditions (a) and (b) of (7.14) are satisfied. For $f \in E$ and an f-flexible vertex V of $\mathcal{C}$, the OM $\hat{\mathcal{C}}$ in (7.14) is said to be <u>obtained from $\mathcal{C}$ by perturbing f around V</u>.

Suppose that V^o is an f-flexible vertex of $\mathcal{C}$ for a fixed $f \in E$, and suppose that $\hat{\mathcal{C}}$ is obtained from $\mathcal{C}$ by perturbing f arround V^o. The following properties are immediate:

(7.15) If $V + f^- \in \mathcal{C}$ then $\hat{\mathcal{C}} = \mathcal{C}$.

(7.16) If $X' \in \hat{\mathcal{C}}$ then there exists $X \in \mathcal{C}$ such that $\underline{X} \subseteq \underline{X}'$. Therefore, if V is a vertex of $\mathcal{C}$ and $V \in \hat{\mathcal{C}}$ then V is a vertex of $\hat{\mathcal{C}}$ as well.

The next property is less obvious:

(7.17) If B is a basis of $\mathcal{C}$ then it is a basis of $\hat{\mathcal{C}}$ as well.

For a basis B of $\hat{\mathcal{C}}$, and for $j \in B$, $\hat{X}(B,j)$ denotes the fundamental vertex of j in B in $\hat{\mathcal{C}}$.

(7.18) If B is a basis of $\mathcal{C}$ and $j \in B$

(a) $\hat{X}(B;j) = X(B;j)$ if $\hat{X}(B;j) \neq \pm V^o$;

(b) $\hat{X}(B;j) = X(B;j) + f^+$ if $X(B;j) = V^o$;

(c) $\hat{X}(B;j) = X(B;j) + f^-$ if $X(B;j) = -V^o$.

(7.19) If a vertex V (as well as V^o) is f-flexible in $\mathcal{C}$ and $V \neq \pm V^o$ then V is an f-flexible vertex in $\hat{\mathcal{C}}$. Thus the vertex perturbations of f can be done successively around initially f-flexible vertices which are distinct and no two of which are antipodal.

C. The Construction

Consider the linear OM $C(A)$ and the OP $P = (C(A); g,f)$.

In (7.12) we remarked that a pivot replacing $B^1 = \{2,3,6\}$ by $B^2 = \{3,4,6\}$ is not a simplex pivot, since the basic feasible direction $V^1 \equiv Z^2(B^1)$ has zero in f-component. However, we can show that V^1 is an f-flexible vertex in $C(A)$ and thus a new OM can be obtained from $C(A)$ by perturbing f around V^1 so that the corresponding basic feasible direction $\hat{V}^1 \equiv \hat{Z}^2(B^1)$ has + in f-component in the new OP $\hat{P}$ and the pivot replacing B^1 by B^2 becomes a simplex pivot. It is clear from (7.18) that the perturbation around a basic feasible direction, and all the other basic feasible directions. Therefore the simplex method produces a sequence of pivorts: $B^o \to B^1 \to B^2$. By the symmetry of $C(A)$, $V^2 \equiv \alpha(V^1)$ and $V^3 \equiv \alpha(V^2)$ are basic feasible directions $Z^3(B^3)$ and $Z^1(B^5)$, which are f-flexible in $C(A)$. Using (7.19), we can perform the perturbation of f around V^1, V^2 and V^3 successively to obtain a new OM, say $\overline{C}$, so that for an OP $\overline{P}$ $(\overline{C}; g,f)$ the simplex method can produce a cycle of pivots: $B^o \to B^1 \to \cdots \to B^5 \to B^o$, three of which (i.e., $B^o \to B^1$, $B^2 \to B^3$, $B^4 \to B^5$) are degenerate pivots and the rest of which (i.e., $B^1 \to B^2$, $B^3 \to B^4$, $B^5 \to B^o$) are nondegenerate.

It is left to show:

(7.20) Each vertex V^i $(i = 1,2,3)$ is f-flexible in $C(A)$.

<u>Proof</u> It is enough to show for $V^1 = 1\ 2\ \underline{4}\ 5$.
Let A_1^e be the e-column vector of A_1 , for $e \in E$. Suppose $X \in C(A)$ with $X_f = 0$, V^1 X and $X \neq V$. Since $C(A) = C(A_1)$, this implies that $\lambda \in \mathbb{R}^4$ such that

(i) $\lambda A_1^1 > 0$, $\lambda A_1^2 > 0$, $\lambda A_1^4 < 0$, $\lambda A_1^5 > 0$;
(ii) $\lambda A_1^f = 0$,
(iii) at least one of λA_1^g , λA_1^3 , λA_1^6 is nonzero.

By the structure of A_1, we have

(iv) $\lambda A_1^g \neq 0 \iff \lambda_o \neq 0$;
(v) $\lambda A_1^3 \neq 0 \iff \lambda_3 \neq 0$;
(vi) $\lambda A_1^6 \neq 0 \iff \lambda_1 \neq 0$.

Then there exists $\lambda' \in \mathbb{R}^4$ obtained from λ by changing one of the nonzero components in $\{\lambda_o, \lambda_1, \lambda_3\}$ by very small amount ε such that $\lambda' A_1^f < 0$, and $\lambda\ A_1^i$ and $\lambda' A_1^i$ have the same sign for all $i \neq f$. Therefore, the incidence signed vector of $\lambda' A$ is $X + f^+$ which is in $C(A)$. This completes the proof. □

D. Further Results on Cycling

In this section, we construct an example of OP for which

(a) The simplex method produces a cycle of pivots all of which are nondegenerate (as opposed to "some of which" for the OP $\bar{P} = (\bar{\mathcal{C}}\ ; g,f)$ constructed in the Section C);

(b) Bland's Smallest Subscript Rule produces the same cycle of pivots as in (a) and thus it is not finite;

(c) Bland's (finite) Algorithm for OP's produces an infeasible solution, starting from a basic feasible solution.

The results (b) and (c) answer the open questions raised by Bland [AL] negatively.

The construction of the example starts with the OM $\bar{\mathcal{C}}$ and applies some vertex perturbations around 3 feasible vertices of $\bar{P}$, which are degenerate.

Consider the OP $(\mathcal{C}(A);\ g,f)$ and let

$$W^o = X(B^o) = g\ 1\ 4\ \underline{f}$$
$$W^2 = X(B^2) = g\ 2\ 5\ \underline{f}$$
$$W^4 = X(B^4) = g\ 3\ 6\ \underline{f}$$

where B^i's are feasible bases of the OP defined in (7.10) and (7.11). First of all we remark that

(7.21) Each of the vertices W^o, W^2, W^4 are respectively 3-feasible, 1-flexible, 2-flexible in $\mathcal{C}(A)$.

<u>Proof</u> The proof is similar to (7.20) and is omited.

From (7.10), we have $Z^5(B^o) = \underline{1}\ \underline{3}\ 4\ 5\ 2$ and hence $W^o \circ Z^5(B^o) = g\ 1\ \underline{3}\ 4\ 5\ f \in \mathcal{C}(A)$. Let

$$W^1 = W^o \circ Z^5(B^o) = g\ 1\ \underline{3}\ 4\ 5\ \underline{f}$$
$$W^3 = W^2 \circ Z^6(B^2) = g\ \underline{1}\ 2\ 5\ 6\ \underline{f} = \beta(W^1)$$
$$W^5 = W^4 \circ Z^4(B^4) = g\ \underline{2}\ 3\ 4\ 6\ \underline{f} = \beta(W^3) \quad .$$

Hence we have

(7.22) $W^i \in \mathcal{C}(A)$ for $i = 1,2,\ldots,5$.

It follows from (7.19), (7.20) and (7.21) that

(7.23) It is possible to perform vertex perturbations of f around V^1, V^2, V^3 successively and then perform vertex perturbations of an element 3 around W^o, 1 around W^2 and 2 around W^4 to $\mathcal{C}(A)$.

Let $\bar{\bar{\mathcal{C}}}$ be the OM obtained from $\mathcal{C}(A)$ by the operations in (7.23). In other words, $\bar{\bar{\mathcal{C}}}$ is the OM obtained from $\bar{\mathcal{C}}$ (constructed in Section C) by the last three perturbations specified in (7.23). Let $\bar{\bar{P}} = (\bar{\bar{\mathcal{C}}}\ ; g,f)$. The following properties are easily verified from (7.22) and (7.14):

(7.24) Let

$$\bar{\bar{W}}^o = W^o + 3^+ \quad (= g\ 1\ 3\ 4\ \underline{f})$$
$$\bar{\bar{W}}^2 = W^2 + 1^+ \quad (= g\ 1\ 2\ 5\ \underline{f})$$
$$\bar{\bar{W}}^4 = W^4 + 2^+ \quad (= g\ 2\ 3\ 6\ \underline{f}) \quad .$$

Then $\overline{\overline{W}}^i \in \overline{\overline{C}}$ for $i = 0,2,4$.

Observing that

$W^o \prec W^1$, $W^2 \prec W^3$, $W^4 \prec W^5$ and $W^1_3 = W^3_1 = W^5_2 = -$, the construction and (7.14) implies:

(7.25) Let

$\overline{\overline{W}}^1 = W^1 + 3^o$ (= g 1 4 5 $\underline{f}$)

$\overline{\overline{W}}^3 = W^3 + 1^o$ (= g 2 5 6 $\underline{f}$)

$\overline{\overline{W}}^5 = W^5 + 2^o$ (= g 3 4 6 $\underline{f}$)

Then $\overline{\overline{W}}^i \in \overline{\mathcal{C}}$ for $i = 1,3,5$.

By (7.17), each basis $B^i \cup \{g\}$ $(i=0,1,\ldots,5)$ of $\mathcal{C}(A)$ is a basis of $\overline{\overline{\mathcal{C}}}$. Thus, using (7.24) and (7.25) we obtain:

(7.26) $\overline{\overline{X}}(B^i) = \overline{\overline{W}}^i$ for $i = 0,1,\ldots,5$,

where $\overline{\overline{X}}(B^i)$ is the basic solution for the basis B^i $\overline{\overline{P}}$ $(i = 0,1,\ldots,5)$. Thus each B^i is a feasible basis of $\overline{\overline{P}}$ $(i = 0,1,\ldots,5)$.

(7.27) Since $\overline{\overline{\mathcal{C}}}$ can be obtained from $\overline{\mathcal{C}}$, the OM constructed in Section C, by perturbations around vertices W^o, W^2, W^4 with positive g-component and by (7.18), the basic feasible directions for the OP $(\overline{\mathcal{C}}\ ;\ g,f)$ remain basic feasible directions for $(\overline{\overline{\mathcal{C}}}\ ;\ g,f)$. Thus, by (7.26), the simplex method can produce a cycle of pivots: $B^o \to B^1 \to B^2 \to \cdots \to B^5 \to B^o$ for $\overline{\overline{P}}$. Furthermore each pivot in the cycle for $\overline{\overline{P}}$ is nondegenerate since $\overline{\overline{W}}^i \neq \overline{\overline{W}}^j$ for any $0 \le i < j \le 5$.

Let $e_1 = 1$, $e_2 = 2, \ldots, e_6 = 6$. Now we shall show that the simplex method with Bland's Smallest Subscript Rule (α) described in Chapter 5 can produce the same cycle $B^o \to B^1 \to \cdots \to B^5 \to B^o$ of pivots. We remark the following:

(7.28) The pivot $B^o \to B^1$ for $\bar{\bar{P}}$ is uniquely selected by the simplex method. By the symmetry $B^2 \to B^3$, $B^4 \to B^5$ are uniquely selected by the simplex method.

<u>Proof</u> By the construction of C , we have

$$\bar{\bar{Z}}^j(B^o) = \bar{Z}^j(B^o) \qquad \forall\ j \in B^o$$

where $\bar{\bar{Z}}^j(B)$ and $\bar{Z}^j(B)$ denote the basic feasible directions in $\bar{\bar{P}}$ and $\bar{P}$, respectively. From the matrix A_o ,

$$Z^2(B^o) = \underline{1}\ 2\ 3\ 4\ \underline{f}$$

$$Z^5(B^o) = \underline{1}\ \underline{3}\ 4\ 5\ f$$

$$Z^6(B^o) = \underline{1}\ 3\ \underline{4}\ 6$$

Remarking that $B^o = \{2,5,6\}$, $B^5 = \{1,2,5\}$, $Z^6(B^o) = -Z^1(B^5) = -V^3$,

$$\bar{Z}^6(B^o) = Z^6(B^o) + f^- = \underline{1}\ 3\ \underline{4}\ 6\ \underline{f}$$

It follows that

$$\bar{\bar{Z}}^2(B^o) = Z^2(B^o)$$

$$\bar{\bar{Z}}^5(B^o) = Z^5(B^o)$$

$$\bar{\bar{Z}}^6(B^o) = Z^6(B^o) + f^-$$

and $\bar{\bar{Z}}^5(B^o)$ is the only basic feasible direction for B^o in $\bar{\bar{P}}$ whose f-component is positive. Therefore $B^o \to B^1$ is the only simplex pivot in B^o. By the symmetry of $\bar{\bar{C}}$, the result follows. □

(7.29) The pivot $B^1 \to B^2$ is the simplex pivot selected by Bland's Smallest Subscript Rule, and so are the pivots $B^3 \to B^4$, and $B^5 \to B^1$, for the OP $\bar{\bar{P}}$.

<u>Proof</u> We prove the first statement, since the rest will follow similarly. First we observe that $B^1 = \{2,3,6\}$,

$$\bar{\bar{Z}}^2(B^1) = \bar{Z}^2(B^1) = Z^2(B^1) + f^+ = \underline{1}\ 2\ \underline{4}\ 5\ f$$

$$\bar{\bar{Z}}^3(B^1) = \bar{Z}^3(B^1) = Z^3(B^1) = 1\ 3\ \underline{4}\ \underline{5}\ \underline{f}$$

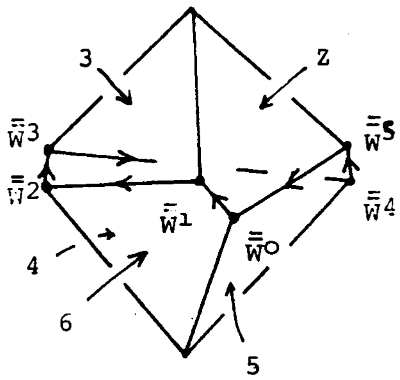
$$\bar{\bar{Z}}^6(B^1) = \bar{Z}^6(B^1) = Z^6(B^1) = \underline{1}\ \underline{4}\ 5\ 6\ f$$

This implies that the simplex method selects either the element 2 or 6 to leave the basis B^1,and hence the Bland's rule (α) selects 2, which results in the pivot $B^1 \to B^2$. This completes the proof. □

It follows from (7.28) and (7.29) that

(7.30) Bland's Smallest Subscript Rule (α) produces a cycle $B^o \to B^1 \to \cdots \to B^5 \to B^o$ of pivots for the OP $\bar{\bar{P}}$.

Geometrically the cycling $B^o \to B^1 \to \cdots \to B^5 \to B^o$ of non-degenerate pivots can be described by the figure :



where $\bar{\bar{W}}^i = \bar{\bar{X}}(B^i)$ for $i = 0,1,\ldots,5$.

The rest of this section will be devoted to a description of Bland's Algorithm and to show that it can produce an infeasible solution.

The original description Bland's Algorithm [AL] was in recursive fashion so that in order to apply the algorithm for an OP one has to consider certain subproblems of the OP. Thus it is rather difficult to analyse the behavior of the algorithm with this description. Here we shall give a nonrecursive description of the algorithm, by which one can observe how the algorithm works without considering "nested" class of subproblems.

(7.30) <u>Bland's Algorithm</u>

<u>Input</u> : An OP $(C; g,f)$ and an initial feasible basis $\bar{B}$.

<u>Output</u> : Either an optimal or co-inconsistent basis.

<u>Data Structure</u> (to be updated each iteration)

(i) a basis B of the OP ;

(ii) an ordered subset $R = \{r_1, r_2, \ldots, r_q\}$ of $E_1 \setminus B$;

(iii) a partition of the basis B into (q + 1) subsets $B_o, B_1, \ldots, B_q$.

<u>Initialization</u>

Set $B = \bar{B}$, $R = \phi$, $q = 0$, $B_o = B$.

<u>Steps</u>

(B1) Select the largest number p such that $0 \leq p \leq q$ and $B_p^+ \equiv \{\ell \in B_p : Z^\ell(B)_f > 0\} \neq \phi$. If there is no such p, stop (=> the basis is optimal). Otherwise select any element $\ell \in B_p^+$.

(B2) Let $I = E_1 \setminus (B \cup \{r_1, r_2, \ldots, r_p\})$.
Test: $Z^\ell(B)_i \geq 0$ for all $i \in I$?
If so, stop (=> the basis is co-inconsistent).
Otherwise select any $k \in I$ such that $Z^\ell(B)_k < 0$ and $X(B \setminus \{\ell\} \cup \{k\})_I \geq 0$.

(B3) Repeat the above procedure with the new basis $B' = B \setminus \{\ell\} \cup \{k\}$ and with $R' = \{r_1', r_2', \ldots, r_{q'}'\}$ and $B' = B_o' \cup \cdots \cup B'_{q'}$, defined by

$$q' = p + 1$$

$$r_i' = \begin{cases} r_i & \text{if} \quad 1 \leq i \leq p \\ \ell & \text{if} \quad i = p + 1 \end{cases}$$

$$B'_i = \begin{cases} B_i & \text{if} \quad 0 \le i \le p-1 \\ B_p^+ \setminus \{\ell\} & \text{if} \quad i = p \\ (B_p \setminus B_p^+) \cup B_{p+1} \cup \cdots \cup B_q \cup \{k\} & \text{if} \quad i = p+1. \end{cases}$$

Bland's Algorithm and the Algorithm A described in (6.19) are very similar, and in fact the following basic properties of Bland's Algorithm can be proved using a similar argument as one used to prove the corresponding properties (6.22) and (6.23) of the Algorithm A.

(7.31) Theorem (Bland) Bland's Algorithm is finite.

(7.32) Theorem (Bland)

(a) If Bland's Algorithm stops at the Step (B1) the last basis B is optimal.

(b) If Bland's Algorithm stops at the Step (B2) then the last basis B is co-inconsistent.

It is clear from the description (6.7) of the simplex method that the simplex method stays feasible , i.e., it produces only feasible bases. Bland's Algorithm (B) has many similarities to the simplex method in that both methods start with a feasible basis and update a basis using pivot operations. In fact it is easy to see that if Bland's Algorithm would stay feasible then we could conclude that it is a simplex method. This open question whether Bland's Algorithm stays feasible was asked by Bland [AL]. We shall answer to this question negatively in the rest of this section.

First we explain how the algorithm (B) may produce an infeasible basis. Suppose that the algorithm (B) is applied to some OP $P = (C; g,f)$ and that a feasible basis B with partition $B_o \cup B_1 \cup \cdots \cup B_q$ and a subset $R = \{r_1, r_2, \ldots, r_q\}$ are produced in some iteration. And further suppose that

(7.33) $\quad B_o = \phi$.

This implies that in the step (B1) the index p cannot be chosen to be zero, and hence when the element k is selected in (B2) the nonnegativity of the r_1-component in the new basic solution $X(B \setminus \{\ell\} \cup \{k\})$ will not be guaranteed and can possibly be negative. This is exactly what can happen when Bland's Algorithm is applied to the OP $\overline{\overline{P}} = (\overline{\overline{C}}\ ; g,f)$ constructed earlier.

Suppose that Bland's Algorithm is applied to the OP $\overline{\overline{P}}$ with the initial feasible basis $B^1 = \{2,3,6\}$. For initialization we set $B = B^1$, $R = \phi$, $q = 0$, $B_o = B$. As we remarked in the proof of (7.29), $B_o^+ = \{2,6\}$. We can select $\ell = 2$ in the step (B1). In the step (B2), set $I = E_1 \setminus B^1 = \{1,4,5\}$ and we can select $k = 4$ since $Z^2(B)_4 < 0$ and $B^1 \setminus \{2\} \cup \{4\} = \{3,4,6\} = B^2$ is a feasible basis. Then we update the data structure by setting

$$q \leftarrow 0 + 1 = 1$$

$$r_1 \leftarrow 2$$

$$B_o \leftarrow B_o^+ \setminus \{2\} = \{6\}$$

$$B_1 \leftarrow (B_o \setminus B_o^+) \cup \{4\} = \{3,4\} .$$

$$B \leftarrow \{3,4,6\} \ (= B^2).$$

As we saw in (7.28), $B_1^+ = \phi$ and $B_o = \{6\}$ and hence we must select $p = 0$, $\ell = 6$ and $k = 1$ to obtain $B^2 \setminus \{6\} \cup \{1\} = B^3$. Then for updating, set

$$q = 0 + 1 = 1$$

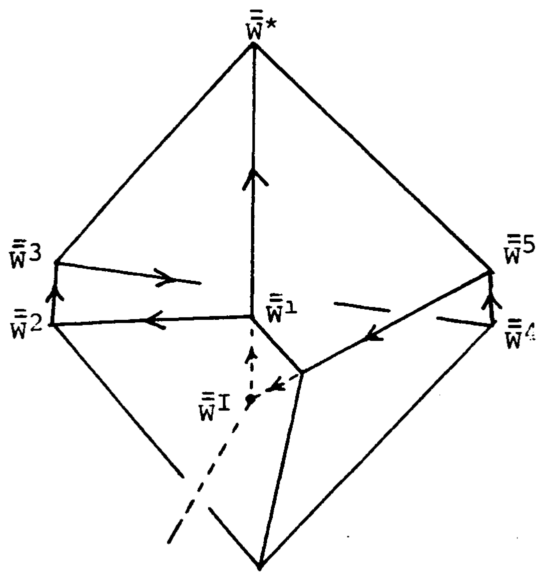
$$r_1 = 6$$

$$B = B^3$$

$$B_o = B_o^+ \setminus \{6\} = \phi$$

$$B_1 = (B_o \setminus B_o^+) \cup B_1 \cup \{1\} = \{1,3,4\} \quad .$$

Now we have the situation (7.33) we discussed above and we will ignore the nonnegativity of 6-component in every basic solutions which will be obtained from now on. The Table.1 indicates the changes of data structure from the first iteration to the last (7-th) iteration when we obtain an optimal basis $B^* = \{1,2,3\}$. Note that the sixth basis B^I will be shown to be infeasible (in(7.34)). It is rather tedious to verify that Bland's Algorithm produces the sequence $B^1 \to B^2 \to \cdots \to B^5 \to B^I \to B^*$. Intuitively the following figure shows how it can produce an infeasible solution $\bar{\bar{X}} \equiv \bar{\bar{X}}(B^I)$ which is in the negative side of the element 6.



We must show:

(7.34) The set $B^I = \{2,3,5\}$ is an infeasible basis of $\bar{\bar{P}}$ and $\bar{\bar{X}}(B^I) = g\ 1\ 4\ \underline{6}\ \underline{f}$.

<u>Proof</u> From the matrix A_o in (7.9), we have

$$X(B^o) = g\ 1\ 4\ \underline{f}$$
$$-Z^6(B^o) = 1\ \underline{3}\ 4\ \underline{6}$$

where $B^o = \{2,5,6\}$. Let $X = X(B^o)\ o\text{-}Z^6(B^o) = g\ 1\ \underline{3}\ 4\ \underline{6}\ \underline{f}$ $\epsilon\ C$. Since $\bar{\bar{C}}$ is obtained from C using the vertex perturbation of the element 3 around $X(B^o)$ (= W^o) and by the fact that X W^o and X_3 = − , $X + 3^o = g\ 1\ 4\ \underline{6}\ \underline{f}\ \epsilon\ \bar{\bar{C}}$. It follows that $X(B^I) = g\ 1\ 4\ \underline{6}\ f$ and hence B^I is infeasible. □

From Table 1 and (7.34), we obtain the claimed result:

(7.35) Bland's Algorithm can produce an infeasible basis and hence it is not a simplex method.

Since the two algorithms, Bland's Algorithm (B) and the algorithm (A) (in (6.8)) are similar, one might suspect that the algorithm (A) is not a feasible method either. We don't know the answer to this question. The only thing we can show is that the algorithm A stays feasible for the OP $\bar{\bar{P}}$ starting with any feasible basis.

Iteration	B	q	B_o	B_1	B_2	B_3	R = $(r_1,\ldots,r_p)$	P	ℓ	I	k	$\bar{\bar{X}}(B)$
1	$B^1=\{2,3,6\}$	0	{2,3,6}	-	-	-	ϕ	0	2	{1,4,5}	4	g 1 4 5 $\underline{f} = \bar{\bar{W}}_1$
2	$B^2=\{3,4,6\}$	1	{6}	{3,4}	-	-	(2)	0	6	{1,2,5}	1	g 1 2 5 $\underline{f} = \bar{\bar{W}}_2$
3	$B^3=\{1,3,4\}$	1	ϕ	{1,3,4}	-	-	(6)	1	3	{2,5}	5	g 2 5 6 $\underline{f} = \bar{\bar{W}}_3$
4	$B^4=\{1,4,5\}$	2	ϕ	{1,4}	{5}	-	(6,3)	1	4	{2,3}	2	g 2 3 6 $\underline{f} = \bar{\bar{W}}_4$
5	$B^5=\{1,2,5\}$	2	ϕ	ϕ	{1,2,5}	-	(6,4)	2	1	{3}	3	g 3 4 6 $\underline{f} = \bar{\bar{W}}_4$
6	$B^I=\{2,3,5\}$	3	ϕ	ϕ	{5}	{2,3}	(6,4,1)	2	5	{1}	1	g 1 4 $\underline{6}$ $\underline{f} = \bar{\bar{W}}^I$
7	$B^*=\{1,2,3\}$	3	ϕ	ϕ	ϕ	{1,2,3}	(6,4,5)	-	-	-	-	g 4 5 6 f $= \bar{\bar{W}}^*$: optimal

Table: 1

8. DUALITY

The duality in oriented matroids is an abstraction of the duality (or the orthogonality) in vector subspaces in $\mathbb{R}^n$. Two signed vectors X and Y on E are said to be orthogonal if the existence of an element separating X and Y implies the existence of an element on which the signs of X and Y agree. The dual C^* of an OM C is the set of all signed vectors orthogonal to every vector in C. Two basic properties are that C^* is also an OM, and that the dual $(C^*)^*$ of C^* is C. This implies that any property of C can be stated equivalently as a property of C^* and vice versa.

Duality in oriented matroids has been studied extensively by Bland [OT][AL], Bland and Las Vergnas [OR], Folkman and Lawrence [OM], Las Vergnas [OM], and Lawrence [OM].

In this chapter we review the fundamental properties of duality in oriented matroids. Those include the two basic properties mentioned above and the so-called "Painting Lemma" by Bland and Las Vergnas. Our main interest is to show how the results of OM programming in the previous chapters 3 and 5 can be translated through the basic properties of duality to OM generalizations of well known results in LP duality. Among others we obtain the OP strong duality theorem (Lawrence [OM] and Bland [AL]) and the OP full complementary theorem.

A. Orthogonality

Let E be a finite set. We say that two signed vectors X and Y on E are orthogonal (denoted by $X * Y$) if either

(8.1.a) $X_e = 0$ or $Y_e = 0$ for all $e \in E$; or

(8.1.b) $\exists\ e, f \in E$ such that

$$X_e = Y_e \neq 0 \quad \text{and} \quad X_f = -Y_f \neq 0 \quad .$$

For any set C of vectors on E, we define the (orthogonal) dual C^* of C as

(8.2) $$C^* = \{Y : X * Y \text{ for all } X \in C\} ,$$

i.e. the set of all vectors on E orthogonal to all vectors in C .

For a set C of vectors on E, and for subsets F, F^1, F^2 of E, let

(8.3) $$C(F) = \{X \in C : X_F = \underset{\sim}{0}\}$$

(8.4) $$C(F^1,F^2) = \{X \in C : X_{F1} \geq \underset{\sim}{0} \text{ and } X_{F2} \leq \underset{\sim}{0}\} .$$

Hence $C(F,F) = C(F)$ for $F \subseteq E$.

For a vector X on E, let

(8.5.a) $$X^+ = \{e \in E : X_e > 0\}$$

(8.5.b) $$X^- = \{e \in E : X_e < 0\} .$$

It is clear that if C is a set of vectors on E and if Y is a vector on E then

(8.6) $\quad C(\underline{Y}) = C(Y^+, Y^-) \cap C(Y^-, Y^+),$

and if in addition C satisfies the symmetry (1.2 OM-1) then

(8.7) $\quad C(Y^+, Y^-) = - C(Y^-, Y^+)$.

It is easy to verify the following:

(8.8) Proposition Let C be a set of vectors on E satisfying the symmetry property (1.2 OM-1). Then for a vector Y on E,

$$< Y \in C^* > \quad <=> \quad < C(Y^+, Y^-) = C(\underline{Y}) > \quad .$$

The following theorem will be useful:

(8.9) Theorem (Generalized Farkas Lemma) Let C be a set of vectors on E satisfying both the symmetry property (1.2 OM-1) and the closedness under the composition (1.2 OM-2). Then for any fixed element $g \in E$ exactly one of the following statements holds:

(a) $\exists X \in C$ s.t. $X_g > 0$ and $X \geq \underset{\sim}{0}$;

(b) $\exists Y \in C^*$ s.t. $Y_g > 0$ and $Y \geq \underset{\sim}{0}$.

Proof Obviously (a) and (b) cannot both hold. Suppose neither (a) nor (b) is satisfied and we shall obtain a contradiction. For each subset F of E, F(+) denotes the vector on E such that for $e \in E$

$$F(+)_e = \begin{cases} + & \text{if} \quad e \in F \\ 0 & \text{if} \quad e \in E \setminus F \end{cases}$$

Let F^o be a maximal subset of E such that $F^o(+) \in C$. Such F^o exists because (b) is not satisfied and $C(E,\phi) \setminus C(E) \neq \phi$ by (8.8). Since (a) does not hold, $g \notin F^o$. Let $\overline{F}^o = E \setminus F^o$. Since $g \in \overline{F}^o$ and $\overline{F}^o(+) \notin C^*$, it follows from (8.8) that

$$\exists Z \in C(\overline{F}^o,\phi) \setminus C(\overline{F}^o).$$

Let $X^1 = F^o(+) \circ Z$. Since C is closed under the composition $X^1 \in C$. Note that $X^1 \geq 0$ and $F^o \subset \underline{X}^1$. This contradicts the choice of F^o. Therefore the result holds. □

A special case of Theorem (8.9), where C is an OM (i.e., C satisfies the elimination property (1.2 OM-3) and (1.2 OM-0) in addition to the other two properties), was proved by Bland and Las Vergnas (see Bland [AL], Corollary 3.1.1). Theorem (8.9) due to Edmonds, Mandel and the author is certainly more general than the Bland-Las Vergnas theorem.

B. The Dual OM

The main theorem in this section is:

(8.10) Theorem (Bland and Las Vergnas [OR]) Let be an OM on a finite set E. Then,

(a) the dual C^* of C is an OM on E; and

(b) the dual C^{**} of C^* is C.

We shall prove (8.10) by induction on $|E|$. For this we need some remarks.

For any set C of vectors on E and for any subset F of E, we define the deletion operation $C \setminus F$ and the contraction operation C / F in the same way as they are defined when C is an OM:

$$C \setminus F = \{X_{E \setminus F} \; : \; E \in C \}$$

$$C / F = \{X_{E \setminus F} \; : \; X \in C \text{ and } X_F = \underset{\sim}{0}\} \quad .$$

(8.11) Proposition ([OR]) Let be an OM on E. Then the following properties hold for any subset F or E:

(a) $(C \setminus F)^* = C^* / F$;

(b) $(C / F)^* = C^* \setminus F$.

Proof It suffices to prove it for the case $|F| = 1$. The proof of (a) is not difficult but a bit tedious. The part (b) is easy and in fact we don't need to use the elimination property (1.2 OM-3). □

(8.12) <u>Proof of (8.10):</u>

(a): The first two axioms (OM-0) and (OM-1) are clearly satisfied by $\mathcal{C}^*$. To verify (OM-2), take any two vectors Y^1 and Y^2 from $\mathcal{C}^*$ and take any vector X from $\mathcal{C}$. Consider three cases (a) $\underline{X} \cap \underline{Y}^1 \neq \phi$, (b) $\underline{X} \cap \underline{Y}^1 = \phi$ and $\underline{X} \cap \underline{Y}^2 \neq \phi$ (c) $\underline{X} \cap (\underline{Y}^1 \cup \underline{Y}^2) = \phi$, and $(\underline{Y}^1 \circ \underline{Y}^2) * X$ follows immediately each case.

The most difficult part is to verify (OM-3). Use induction on $|E|$. Trivial if $|E| = 1$. Suppose $|E| = k$ (≥ 2) and take any two vectors Y^1 and Y^2 from $\mathcal{C}^*$ with $f \in E$ separating Y^1 and Y^2. Let F be the set of elements separating Y^1 and Y^2. If $F = \{f\}$, Y^3 defined by $Y^3_{E\setminus\{f\}} = (Y^1 \circ Y^2)_{E\setminus\{f\}}$ and $Y^3_f = 0$ is a vector in $\mathcal{C}^*$ we want. Otherwise, let $F^1 = F\setminus\{f\}$ and use the OM-ness of $\mathcal{C}^*\setminus F^1$ ($=(\mathcal{C}/F^1)^*$, by (8.11.b)), and the result follows easily.

(b): We use induction on $|E|$. By (8.11.a) and (8.10.a) we know by induction that $\mathcal{C}/F = \mathcal{C}^{**}/F$ for any $\phi \neq F \subseteq E$. Let $\mathcal{V}$ and $\hat{\mathcal{V}}$ be the sets of vertices of $\mathcal{C}$ and $\mathcal{C}^{**}$, respectively. By (1.8) enough to show $\mathcal{V} = \hat{\mathcal{V}}$. Clearly $\mathcal{V}/F = \hat{\mathcal{V}}/F$ for $\phi \neq F \subseteq E$. If $\underline{V} = E$ for some $V \in \mathcal{V}$ then $\mathcal{V} = \{V, -V\}$ and we can easily show $\mathcal{C} = \mathcal{V} = \hat{\mathcal{V}} = \mathcal{C}^{**}$. Otherwise for all $X \in \mathcal{V}$ $\underline{X} \neq E$ and this together with the equality $\mathcal{V}/F = \hat{\mathcal{V}}/F$ ($\phi \neq F \subseteq E$) implies $\mathcal{V} = \hat{\mathcal{V}}$, and hence $\mathcal{C} = \mathcal{C}^{**}$. □

Theorem (8.10) and (8.11) imply a theorem of Bland and Las Vergnas [OR]:

(8.12) Theorem Let $\mathcal{C}$ be an OM on E and let R and S be disjoint subsets of E. Then $\mathcal{C} \setminus R / S$ and $\mathcal{C}^* \setminus S / R$ is a dual pair of OM's (i.e. the dual of each other).

An important corollary of (8.9) and (8.12) is the so-called "Painting Lemma" : (Bland [OT][AL], Las Vergnas [MO], and Bland and Las Vergnas [OR])

(8.13) Corollary (Painting Lemma)

For an OM $\mathcal{C}$ on E, given a partition of E into subsets R, S and T, and a fixed $g \in T$, exactly one of the following statements hold:

(a) $\exists X \in \mathcal{C}$ s.t. $X_g > 0,\ X_T \geq \underset{\sim}{0}$ and $X_S = \underset{\sim}{0}$;

(b) $\exists Y \in \mathcal{C}^*$ s.t. $Y_g > 0.\ X_T \geq \underset{\sim}{0}$ and $Y_R = \underset{\sim}{0}$.

C. Dual Bases

We assume in this section that $\mathcal{C}$ is an OM on E.

Recall that a subset B of E is a basis of an OM $\mathcal{C}$ on E iff it is a minimal subset of E such that

$$\mathcal{C}(B) = \{\underset{\sim}{0}\}$$

(see (4.10)). We will show that

(8.14) Theorem A subset B of E is a basis of $\mathcal{C}$ iff $E \setminus B$ is a basis of $\mathcal{C}^*$.

In order to show (8.14) we need two lemmas:

(8.15) Lemma If B is a basis of an OM $\mathcal{C}$, then $\mathcal{C}^*(E \setminus B) = \{\underset{\sim}{0}\}$.

Proof Suppose B is a basis of $\mathcal{C}$ and suppose that $\underset{\sim}{0} \neq Y \in \mathcal{C}^*(E \setminus B)$. Take any $f \in B$ with $Y_f \neq 0$. Let X be the fundamental cocircuit X(B;f) of f in a basis B of $\mathcal{C}$. Since $\underline{X} \cap \underline{Y} = \{f\}$, X and Y are not orthogonal, a contradiction. □

(8.16) Lemma (Bland [OT] [AL]) For a basis B of an OM $\mathcal{C}$ on E and for an element $f \in E \setminus B$, let Y be a vector on E defined by

$$Y_e = \begin{cases} -X(B;\, e)_f & \text{if} \quad e \in B \\ + & \text{if} \quad e = f \\ 0 & \text{otherwise} \end{cases}$$

(where X(B;e) is the fundamental cocircuit of e in B) then $Y \in \mathcal{C}^*$.

<u>Proof</u> Suppose that $Y \notin \mathcal{C}^*$. Then there exists $X \in \mathcal{C}$ s.t. X and Y are not orthogonal and no element $e \in E$ separates X and Y, and $|\underline{X} \cap B|$ is minimal with these properties. Thus $X_f \geq 0$, and since B is a basis, $|\underline{X} \cap B| \geq 1$. If $X_f = 0$, then there exists $g \in B$ such that $X_g = Y_g = -X(B;g)_f \neq 0$. Otherwise let g be any element of $\underline{X} \cap B$. By the choice of X, $Y_g = -X(B;g)_f = X_g$ or 0. Let X^o be $X(B;g)$ if $X_g = -$ and $-X(B;g)$ if $X_g = +$. Using elimination property (OM-3) for X and X^o we obtain $X^1 \in \mathcal{C}$ such that $X^1_g = 0$, $X^1_f > 0$, and $X^1_{B\setminus\{g\}} = X_{B\setminus\{g\}}$. This implies that X^1 and Y are not orthogonal and $|\underline{X}^1 \cap B| = |\underline{X} \cap B| - 1$, contradicting the choice of X. This completes the proof. □

(8.17) <u>Proof of (8.14):</u> By the duality (8.10) it is sufficient to prove one direction. Let B be a basis of $\mathcal{C}$. Then it follows from (8.15) and (8.16) that $E \setminus B$ is a minimal subset of E such that $\mathcal{C}^*(E \setminus B) = \{\underset{\sim}{0}\}$. Thus $E \setminus B$ is a basis of $\mathcal{C}^*$. □

(8.18) For a basis D of $\mathcal{C}^*$ and for $i \in D$, we denote by $Y(D;i)$ the fundamental cocircuit of i in D. It is easy to see from (8.14) and (8.16) that for any basis B of $\mathcal{C}$,

(8.19) $Y(E \setminus B;\, i)_j = -X(B;\, j)_i$

for $i \in E \setminus B$ and $j \in B$.

D. Duality in Oriented Matroid Programming

For this section we assume that C is an OM on E, g and f are two fixed elements of E.

For a (primal) OP $P = (C;\ g,f)$ the dual OP P^* of P is defined as the OP

(8.20) $P^* = (C^*;\ f,g)$,

that is, the OP obtained from P by replacing C by its dual and interchanging the roles of the infinite element g and the objective element f. Obviously by the duality (8.10.b),

(8.21) $P^{**} = P$.

Let E_1 be the set $E \setminus \{g,f\}$ of constraint elements of P (and P^* as well).

For a subset F of E_1, let

(a) $P(F) = \{X \in C :\ X_g > 0 \text{ and } X_F \geq 0\}$,

(b) $P^\infty(F) = \{X \in C :\ X_g = 0 \text{ and } X_F \geq 0\}$,

(c) $Q(F) = \{Y \in C^*:\ Y_f > 0 \text{ and } Y_F \geq 0\}$,

(d) $Q^\infty(F) = \{Y \in C^*:\ Y_f = 0 \text{ and } Y_F \geq 0\}$.

Then, for a subset F of E_1, $P(F)$ is a polyhedron in an affine OM $(C;g)$ whose infinite face is $P^\infty(F)$, and $Q(F)$ is a polyhedron in an affine OM $(C^*;f)$ whose infinite face is $Q^\infty(F)$. In particular, $P = P(E_1)$ is the feasible region of a primal OP

P with the infinite face $P^{\infty} = P^{\infty}(E_1)$, and $Q = Q(E_1)$ is the feasible region of the dual OP P^* with the infinite face $Q^{\infty} = Q^{\infty}(E_1)$.

Recall that a subset F of E_1 is cofeasible for a primal OP P if

(8.22) $\nexists\, Z \in C$ s.t. $Z_g = 0$, $Z_f > 0$ and $Z_F \geq 0$,

and strongly cofeasible if it is cofeasible and

(8.23) $\nexists\, Z \in C$ s.t. $Z_g = 0$, $Z_e > 0$, $Z_f \geq 0$, $Z_{F\setminus\{e\}} \geq 0$, for all $e \in F$.

Using the Painting Lemma (8.13) and the axiom (1.2 OM-2) for C^*, we can obtain the following:

(8.24) Proposition Consider a primal OP $P = (C; g,f)$.

(a) A subset F of E_1 is cofeasible for P iff there exists a feasible solution Y to the dual OP such that $Y_{E_1\setminus F} = \underset{\sim}{0}$.

(b) A subset F of E_1 is strongly cofeasible for P iff there exists a feasible solution Y to the dual OP such that $Y_{E_1\setminus F} = \underset{\sim}{0}$ and $Y_F > \underset{\sim}{0}$.

Bland [AL] defined that a vector $X \in C$ and a vector $Y \in C^*$ are complementary (for an OP P) if

(8.25) $X_e = 0$ or $Y_e = 0$ for all $e \in E_1$, and in this case a pair (X,Y) is called a <u>complementary pair</u>. Using (8.24), Proposition (3.22) implies:

(8.26) <u>Proposition</u> (Bland [AL]) A feasible solution X to be primal OP P is optimal iff there exists a feasible solutio Y to the dual OP such that X and Y are complimentary.

A <u>optimal pair</u> is a complementary pair (X,Y) such that X is feasible for a primal P and Y is feasible for the dual P^* . Using (8.24.b), (3.36) implies:

(8.27) <u>Theorem</u> (Bland [AL]) If (X^1, Y^1) and (X^2,Y^2) are optimal pairs, then (X^1,Y^2) is an optimal pair.

The following theorem is a generalization of the full complementary slackness theorem of linear programming:

(8.28) <u>Theorem</u> If a primal OP $(C;\ g,f)$ has an optimal solution then there exists a full complementary pair (X, Y), that is,a complementary pair satisfying:

$$\text{either} \quad X_e > 0 \quad \text{or} \quad Y_e > 0 \quad \text{for all } e \in E_1 .$$

<u>Proof</u> As in Theorem (3.33), let F^* be the set of all constraint elements which are active at every optimal solution. Let X be a composition of all optimal solutions. Then, X is also an optimal solution, and clearly $X_F = \underset{\sim}{0}$ and $X_{E_1\setminus F^*} > \underset{\sim}{0}$. By the strong cofeasibility of F^* (Theorem (3.33)) and (8.24.b), we have a feasible solution Y to the dual such that $Y_{F^*} > \underset{\sim}{0}$ and $Y_{E_1\setminus F^*} = \underset{\sim}{0}$. Clearly, (X,Y) is a full complementary paiı. □

Recall that a primal OP is cofeasible if the set E_1 of constraints is cofeasible, and co-infeasible otherwise. It is straightforward from (8.24.a) that:

(8.29) <u>Proposition</u>

(a) A primal OP is cofeasible iff the dual OP is feasible;

(b) A primal OP is co-infeasible iff the dual OP is infeasible.

By (8.29), the primal statement (3.23) of the strong duality theorem is equivalent to a primal-dual statement:

(8.30) <u>Theorem</u> (Strong Duality Theorem, Lawrence [OM]): For a primal OP P and the dual OP P^*, exactly one of the following statements hold:

(a) Either the primal or the dual is infeasible;

(b) Both the primal and the dual have feasible solutions X and Y which are complementary.

Let $\mathcal{B}$ be the set of bases of $\mathcal{C}$ and $\mathcal{D}$ be the set of bases of $\mathcal{C}^*$, and let

(8.31) $\mathcal{B}_1 = \{B \subseteq E_1 \ : \ B \cup \{g\} \in \mathcal{B}\}$,

(8.32) $\mathcal{D}_1 = \{D \subseteq E_1 \ : \ D \cup \{f\} \in \mathcal{D}\}$,

which are the set of bases of a primal OP P and the set of bases of the dual OP P^*. By (8.14), we have

(8.33) $\mathcal{D} = \{E \setminus B \ : \ B \in \mathcal{B}\}$, and

(8.34) $\mathcal{D}_1 = \{E_1 \setminus B \ : \ B \in \mathcal{B}_1\}$.

The terminologies developed in Section B of Chapter 5 on bases of OP have the duality through the duality (8.19) of fundamental cocircuits:

(8.35) Proposition Let B be any subset of E_1. Then the following properties hold:

(a) B is a feasible (co-feasible, respectively) basis of P iff $E_1 \setminus B$ is a co-feasible (feasible) basis of P^* ;

(b) B is an inconsistent (co-inconsistent, respectively) basis of P iff $E_1 \setminus B$ is a co-inconsistent (inconsiste basis of P^* ;

(c) B is an optimal basis of P iff $E_1 \setminus B$ is an optimal basis of P^* .

The basis form (8.36) of the Strong Duality Theorem is straightforward from (5.16) and (8.35).

(8.36) Theorem (Bland [AL]): For every standard OP, exactly one of the following statements hold:

(a) there exists either an inconsistent or co-inconsistent basis;

(b) there exists an optimal basis.

PART II

GEOMETRY OF ORIENTED MATROIDS

9. FLATS

A. Posets

A partially ordered set (or poset) , here, is a finite set L together with a partial ordering $\leq$ on P. The join and the meet of a subset X of P are denoted by $\vee X$ and $\wedge X$, respectively, if they exist. If P is a poset a, b ϵ P, we say that a covers b if $b < a$ and there is no element $c \in P$ with $b < c < a$. A chain c in a poset P is any totally ordered subset $\{a_o, a_1, \ldots, a_k\}$ of P, where $a_o < a_1 < \cdots < a_k$ and the length of a chain c is k. For any comparable elements a, b ($a \leq b$) of a poset P the interval [a, b] is the poset of all elements between a and b i.e. $\{c \in P : a \leq c \leq b\}$.

Let P be a poset containing a least element $0 = 0_P$. The height h(x) of an element $x \in P$ is defined to be the maximum of length of a chain: $0 = a_o < a_1 < \ldots < a_k = x$, between 0 and x. The maximum height of any element of P is the height h(P) of P. The closure of an element $a \in P$ is the interval [o, a] , the set of all elements below a.

A graded poset is a poset P with 0 and a function $g : P \to Z$ such that

(G1) $a > b \Rightarrow g(a) > g(b)$; and

(G2) a covers b $\Rightarrow g(a) = g(b) + 1$.

We say that a poset satisfies the Jordan-Dedekind (Chain) Property if

(J-D) Every maximal chain between the same elements have the same length.

Clearly a poset with 0 satisfies the Jordan-Dedekind property iff it is graded by the height function.

A lattice is a poset L in which any two elements have a join and a meet. A lattice is complete if each of its subsets has a join and a meet. A nonempty complete lattice has the least element $0 = \wedge L$ and the largest element $1 = \vee L$. An element a of a lattice is called an atom if it covers 0. An atomic lattice is a lattice in which every element except 0 is a join of atoms. A lattice L is called semimodular if for all $x, y \in L$, x and y cover $x \wedge y$ then $x \vee y$ covers both.

The polar (or dual) L of a lattice L is the lattice defined by the converse ordering relation on the same elements.

B. Matroids

Let M be an matroid on a finite set E, i.e., M is a set of subsets of E satisfying the axioms.

(M-0) $E \in M$;

(M-1) $F_1, F_2 \in M \Rightarrow F_1 \cap F_2 \in M$;

(M-2) $< F_1, F_2 \in M\ ,\ a \in F_1 \cup F_2\ ,\ b \in F_2 \setminus F_1 >$

$\Rightarrow < \exists F_3 \in M$ s.t. $a \in F_3 \not\ni b$ and $F_1 \cap F_2 \subseteq F_3 >$.

Members of M are called flats of M. Given subset S of E, the closure cl(S) of S in M is the smallest flat of $M(C)$

(9.1) $cl(S) = \cap\{F : S \subseteq F \in M(C)\}$.

Clearly we have

(9.2) $S' \subseteq S \subseteq E \Rightarrow cl(S') \subseteq cl(S)$.

A subset S of E is said to be independent in M if there is no proper subset S' of S with $cl(S') = cl(S)$. Given $T \subseteq E$, a maximal independent subset of T is called a basis of T in M.

The following properties are well-known and can be easily proved from the axioms (M-0) ~ (M-2):

Let T be a subset of E.

(9.3) If a subset S of E is independent and $cl(T) \setminus cl(S) \neq \phi$ then $T \setminus cl(S) \neq \phi$ and $S \cup \{e\}$ is independent for every $e \in T \setminus cl(S)$;

(9.4) If S is a basis of T then $cl(S) = cl(T)$;

(9.5) If S is a basis of T and $j \in S$ then there exists a unique flat F with $S \setminus \{j\} \subseteq F \subset cl(S)$ (hence $F = cl(S \setminus \{j\})$;

(9.6) If S is a basis of T and if $i \in T \setminus S$ then $S \setminus \{j\} \cup \{i\})$ is a basis of T iff $i \notin cl(S \setminus \{j\})$;

(9.7) If S and S' are bases of T and if $j \in S \setminus S'$ then there exists $i \in S' \setminus S$ such that $S \setminus \{j\} \cup \{i\}$ is a basis of T;

(9.8) Every basis has the same cardinality, called the rank $r(T)$ of T.

A basis of E in M is also called a <u>basis of M</u>, and the <u>rank $r(M)$ of M</u> is defined to be rank $r(E)$ of E.

The rank function r of M satisfies the following properties:

(9.9) $r(\phi) = 0$;

(9.10) $0 \leq r(S) \leq |S| \quad (S \subseteq E)$;

(9.11) $S \subseteq T \subseteq E \Rightarrow r(S) \leq r(T)$;

(9.12) $r(S) + r(T) \geq r(S \cup T) + r(S \cap T) \quad (S, T \subseteq E)$.

Let $L(M)$ be the poset of flats of an matroid M ordered by inclusion. From the axioms (M-0) and (M-1), it follows that

(9.13) $L(M)$ is a lattice where the join $F_1 \vee F_2$ and the meet $F_1 \wedge F_2$ are defined by

$$F_1 \wedge F_2 = F_1 \cap F_2$$

$$F_1 \vee F_2 = \cap\{F : F_1 \cup F_2 \subseteq F \in M\} ,$$

for $F_1, F_2 \in L(M)$.

(9.14) $L(M)$ has the largest flat $1_{L(M)} \equiv E$ and the smallest flat $0_{L(M)} \equiv cl(\phi)$.

One can easily show from (9.3) that

(9.15) If $F_1, F_2 \in M$ and $F_1 \subset F_2$, then $r(F_1) < r(F_2)$ and furthermore for each $e \in F_2 \setminus F_1$ there exists a flat $F_3 \in M$ such that $F_1 \subset F_3 \subseteq F_2$, $e \in F_3$ and $r(F_3) = r(F_1)$

This implies that

(9.16) If $F_1, F_2 \in M$ and $F_1 \subset F_2$, then the length of every maximal chain from F_1 to F_2 is $r(F_2) - r(F_1)$.

By the fact that $r(0_{L(M)}) = 0$,

(9.17) The height function of $L(M)$ is the rank function r of M restricted to the flats of M ;

(9.18) $L(M)$ is graded by its height function and hence $L(M)$ satisfies the Jordan-Dedekind property.

Remarking that the atoms of $L(M)$ are the flats of rank 1, the property (9.15) with setting $F_1 = 0_{L(M)}$ implies

(9.19) $L(M)$ is an atomic lattice.

It follows from (9.12), (9.17) and (9.18) that

(9.20) $L(M)$ is semimodular.

Let $\mathcal{C}$ be an OM on E. It is easily shown that the set

(9.21) $M(\mathcal{C}) = \{E \setminus \underline{X} : X \in \mathcal{C}\}$

is a matroid on E, called the underlying matroid of $\mathcal{C}$. The rank $r(\mathcal{C})$ of $\mathcal{C}$ is $r(E)$, the rank of the underlying matroid. A subset S of E is called independent in $\mathcal{C}$ if it is independent in $M(\mathcal{C})$.

C. Flats of an Oriented Matroid

Let $\mathcal{C}$ be an OM. For a subset T of E, let

(9.22) $\mathcal{C}(T) = \{X \in \mathcal{C} : X_T = 0\}$.

A subset t of $\mathcal{C}$ is said to be a flat of $\mathcal{C}$ if there exists a subset T of E such that $t = \mathcal{C}(T)$. Let $L(\mathcal{C})$ be the poset of flats of $\mathcal{C}$ ordered by reverse inclusion. The following properties are obvious:

(9.23) $\mathcal{C}$ is the minimum flat in $L(\mathcal{C})$ and $\{\underset{\sim}{0}\}$ is the maximum flat in $L(\mathcal{C})$;

(9.24) $t_1, t_2 \in L(\mathcal{C}) \Rightarrow t_1 \cap t_2 \in L(\mathcal{C})$.

Hence,

(9.25) $L(\mathcal{C})$ is a lattice, in which the meet $t_1 \wedge t_2$ and the join $t_1 \vee t_2$ are defined by

$$t_1 \vee t_2 = t_1 \cap t_2$$
$$t_1 \wedge t_2 = \cap\{t : t_1 \cup t_2 \subseteq t \in L(\mathcal{C})\} ,$$

for $t_1, t_2 \in L(\mathcal{C})$.

Let $M(\mathcal{C})$ be the underlying matroid of $\mathcal{C}$, and let cl(T) be the closure of a set $T \subseteq E$ in $M(\mathcal{C})$. Clearly,

(9.26) $\mathcal{C}(T) = \mathcal{C}(\mathrm{cl}(T))$ for any $T \subseteq E$;

(9.27) $F_1, F_2 \in M(\mathcal{C})$ and $F_1 \neq F_2 \Rightarrow \mathcal{C}(F_1) \neq \mathcal{C}(F_2)$.

Thus, for each flat t of $\mathcal{C}$ there exists a unique flat F(t) of $M(\mathcal{C})$ such that

$$t = \mathcal{C}(F(t)) .$$

The following properties hold:

(9.28) $F(t) = \cap\{E \setminus \underline{X} : X \in t\}$ $(t \in L(C))$;

(9.29) The lattices $L(C)$ and $L(M(C))$ are isomorphic for there are order-preserving bijections:

$$F \in L(M(C)) \longrightarrow C(F) \in L(C)$$
$$t \in L(C) \longrightarrow F(t) \in L(M(C)) .$$

The properties (9.17) ~ (9.20), (9.29) imply:

(9.30) The height $\rho = \rho_{L(C)}$ of $L(C)$ is defined by $\rho(t) = r(F(t))$ $(t \in L(C))$;

(9.31) $L(C)$ is graded by its height function ρ , and hence it satisfies the J-D property;

(9.32) $L(C)$ is a atomic lattice;

(9.33) $L(C)$ is a semimodular lattice.

The <u>dimension</u> $d(t)$ of a flat $t \in L(C)$ is defined to be $r(C) - \rho(t) - 1$. The <u>k-flats</u> of C are the flats of C having dimension k, for $-1 \le k \le r(C) - 1$.

It follows from (9.26) and (9.30) that

(9.34) A flat t of C is a k-flat iff $t = C(S)$ for an independent set S of $M(C)$ with $|S| = r(C) - k - 1$;

(9.35) The only (-1)-flat is $\{\underset{\sim}{0}\} = 1_{L(C)}$;

(9.36) The 0-flats are the flats of the form $\{\underset{\sim}{0} , V, -V\}$ for some vertex V of C , called the <u>points</u> of C;

(9.37) The $(r(C)-2)$-flats are the flats of the form $C(\{e\})$ for some $e \in E \setminus cl(\phi)$, called the hyperplanes of C ;

(9.38) If t is a k-flat $(-1 \leq k \leq r(C)-2)$ and h is a hyperplane then either $t \subseteq h$ or $t \cap h$ is a $(k-1)$-flat;

(9.39) The only $(r(C)-1)$-flat is $C = 0_{L(C)}$.

The 1-flats of C are called the lines of C . It is straightforward from (9.36) and (9.38) that

(9.40) If ℓ and h are respectively a line and a hyperplane of C and $r(C) \geq 3$ then either $\ell \subseteq h$ or they intersect at a point.

10. CELLS

A natural generalization of convex polytopes by oriented matroids will be considered in this chapter. Our generalization is related by (poset) polarity to another oriented matroidal generalization of convex polytopes due to Las Vergnas [CV]. The basic properties of the face lattice of a convex polytope will be shown to be valid in a more general setting of oriented matroids.

In this chapter we assume that C is an OM on a finite set E. Vectors in C will be called faces.

A. Face Lattice and J-D Property

For a vector X on E, let $C[X]$ denote the set of all faces of C conforming to X:

(10.1) $C[X] = \{X' \in C : X' \preceq X\}$.

A subset t of C is said to be a cell (or polytope) in C

(10.2) $t = C[X]$

for some vector X on E.

(10.3) Proposition Let t be a cell of an OM C . Then the following properties hold:

(a) $< X^1, X^2 \epsilon\ t > \Rightarrow < X^1 \circ X^2 \epsilon\ t >$

(a cell is closed under the composition) ;

(b) $< X^1, X^2 \epsilon\ t > \Rightarrow < X^1 \circ X^2 = X^2 \circ X^1 >$

(the composition in commutative in t).

By (10.3.b), for any subset s of a cell t the <u>composition</u> $\circ \cdot s = \circ\{X: X \epsilon\ s\}$ of vectors in s is uniquely defined. For each cell t, let L[t] denote the poset t ordered by the conformal relation. By the above remarks: for a cell t

(10.4) the poset L[t] is a lattice in which the join and meet operations are defined by

$$X^1 \vee X^2 = X^1 \circ X^2 = X^2 \circ X^1 \quad ,$$

$$X^1 \wedge X^2 = \circ\{X \epsilon\ t : X \preceq X^1 \text{ and } X \preceq X^2\}$$

for $X^1, X^2 \epsilon\ t$;

(10.5) the largest face of L[t] is $\circ\, t$ and the least face of L[t] is $\underset{\sim}{0}$. Since $\circ\, t \epsilon\ C$ and $t = C[\circ t]$ for any cell t,

(10.6) a subset t of C is a cell iff $t = C[W]$ for some face $W \epsilon\ C$.

For a face X in C , let the <u>dimension</u> d(X) of X as the dimension d(t) of the largest flat $t = C(E \setminus \underline{X})$ of C containing X. By (9.34)

(10.7) $d(X) = r(C) - r(E \setminus \underline{X}) - 1$,

where r is the rank function of $M(C)$.

First we observe that

(10.8.a) The unique face of dimension (-1) is $\underset{\sim}{0}$;

(10.8.b) The faces of dimension 0 are the vertices of C .

(10.9) Theorem Let t be a cell of C . Then the following statements hold:

(a) The height $h(X)$ of $X \in t$ in $L[t]$ is $d(X) + 1$.

(b) The lattice $L[t]$ is graded by d, and hence satisfies the J-D property.

Knowing that $d(\underset{\sim}{0}) = -1$, the above theorem follows from the next proposition.

(10.10) Proposition If $X^1, X^2 \in C$ with $X^2 \prec X^1$ then $d(X^1) > d(X^2)$ and furthermore there exists $X^3 \in C$ s.t. $X^2 \prec X^3 \preceq X^1$ and $d(x^3) = d(X^2) + 1$.

Proof: Let $X^1, X^2 \in C$ and $X^1 \prec X^2$. The first statement is clear from (9.15), (9.21) and (10.7). For the second statement we can assume that $d(X^1) > d(X^2) + 1$. It is enough to show that there is $X \in C$ s.t. $X^2 \prec X \prec X^1$. By (10.7) we have $r(E \setminus \underline{X}^2) > r(E \setminus \underline{X}^1) + 1$. Since $L(M(C))$ is graded by r, there exists $X^3 \in C$ s.t. $\underline{X}^2 \subset \underline{X}^3 \subset \underline{X}^1$. Let $X^4 = X^2 \circ X^3$. If $X^4 \preceq X^1$ then we are done. Assume $X^4 \not\preceq X^1$

and set $F = E \setminus \underline{X}^2$. Let I be the set of elements separating X^1 and X^4. Clearly $I \subset F$. Using (1.8) for X^1, X^4 and I, we obtain $X^5 \in C$ and $j \in I$ such that $X_j^5 = 0$, $X^2 \prec X^5 \prec X^1$. This completes the proof. □

B. Facets

For $X, X' \in \mathcal{C}$, X' is said to be a (proper) face of X if $X' \preceq X$ ($X' \prec X$). The following properties are equivalent for $X, X' \in \mathcal{C}$ with $X' \preceq X$:

(10.11.a) $d(X') = d(X) - 1$;

(10.11.b) X' is a maximal proper face of X;

(10.11.c) X covers X' in $L[\mathcal{C}[X]]$.

A face X' of $X \in \mathcal{C}$ is said to be a facet of X (or $\mathcal{C}[X]$) if one of the above conditions holds.

For $X \in \mathcal{C}$, an element $e \in E$ is said to be a facet element of X if $X_e \neq 0$ and there exists a facet X' of X with $X'_e = 0$.

By Proposition (10.10), the following is immediate:

(10.12) Proposition If $X, X' \in \mathcal{C}$ and $X' \prec X$ then there exists a facet element e of X with $e \in \underline{X} \setminus \underline{X}'$.

We shall investigate basic properties of facets and facet elements in this section. For this it will be convenient to have some new notations.

A symbol $\cdot$ is called a dot whose negative $-(\cdot)$ is defined to be itself. A signed-dot vector X on a finite set E is a vector $(X_e : e \in E)$ where $X_e \in \{+,0,-,\cdot\}$ for $e \in E$. A signed-dot vector may be called an sd vector. The negative $-X$ of an sd vector X is defined in the obvious way. The composition $X^1 \circ X^2$ of sd vectors X^1 and X^2 is defined by

$$(X^1 \circ X^2)_e = \begin{cases} X^1_e & \text{if} \quad X^1_e \neq 0 \\ X^2_e & \text{otherwise} \end{cases} \qquad (e \in E). \tag{10.13}$$

We define the binary relation $\preceq$ on $\{+,0,-,\bullet\}$ by $0 \prec \alpha$ and $\alpha \prec \bullet$ for $\alpha \in \{+,0,-,\bullet\}$. This binary relation induces the <u>conformal relation</u> on sd vectors as follows: for sd vectors X^1 and X^2 on E, X^1 <u>conforms to</u> X^2 (denoted by $X^1 \preceq X^2$) if $X^1_e \preceq X^2_e$ for all $e \in E$.

It should be clear that signed vectors are sd vectors, and the operations on sd vectors are simply the same operations on signed vectors when they are restricted for signed vectors.

If X is an sd vector on E, $F \subseteq E$ and $\alpha \in \{+,0,-,\bullet\}$, $X + F^\alpha$ denotes the sd vector defined by

$$(X + F^\alpha)_e = \begin{cases} \alpha & \text{if} \quad e \in F \\ X_e & \text{otherwise.} \end{cases} \tag{10.14}$$

For an sd vector X, C[X] is defined by (10.1). Hence for an sd vector X and a subset F of E,

$$C[X + F^0] \subseteq C[X] \subseteq C[X + F^\bullet] \quad . \tag{10.15}$$

For a signed vector X on E, let $\lceil X \rceil$ denote the maximal face of X comforming to X, or equivalently the maximal face of the cell C[X]. It is easy to see that

(10.16) an element $e \in E$ is a facet element of X iff $d(\lceil X + e^0 \rceil) = d(X) - 1$.

The following proposition says that a cell $C[X]$ is determined by the facet elements of X together with the zero elements $E \setminus \underline{X}$, for each $X \in C$.

(10.17) Proposition Let $X \in C$ and let F be a set of non-facet elements of X in $\underline{X}$. Then

$$C[X] = C[X + F^{\bullet}] .$$

Proof Suppose the statement is false for some $X \in C$ and some F. We can assume that F is a minimal set of non-facet elements in $\underline{X}$ with the property that $C[X + F^{\bullet}] \neq C[X]$. By (10.15), $C[X] \subset C[X + F^{\bullet}]$. Let $W \in C[X + F^{\bullet}] \setminus C[X]$. By the choice of F,

$$W_F = - X_F \quad \text{and}$$
$$W_{E\setminus F} \preceq X_{E\setminus F} \quad .$$

Using (1. 8) setting $X^1 = X$, $X^2 = W$ and $I = F$, we obtain $X' \in C$ and $j \in F$ such that

$$J_j' = 0 \quad , \quad X_F' \prec X_F \quad \text{and}$$
$$X_{E\setminus F}' = X_{E\setminus F} \quad .$$

This implies that $X' \prec X$. By (10.12) there exists a facet element e of X with $e \in \underline{X} \setminus \underline{X}' \subseteq F$, a contradiction. □

(10.18) Corollary Let $X^i \in \mathcal{C}$ for $i = 1,2,3$ and let $X^1 \prec X^2 \preceq X^3$. Then there exist a facet element e of X^3 with $e \in \underline{X}^2 \setminus \underline{X}^1$.

Proof: Let $F = \underline{X}^2 \setminus \underline{X}^1$. Remarking that $X^1 \circ (-X^2) \in \mathcal{C}[X^3 + F^{\cdot}] \setminus \mathcal{C}[X^3]$, by (10.17) F must contain a facet element of X^3. □

(10.19) Theorem Let $X \in \mathcal{C} \setminus \{\underset{\sim}{0}\}$ and let $X' \in \mathcal{C}$ be a k-dimensional face of X. Then there exists at least (d-k) facets of X, of which X' is a face.

Proof By the J-D property (10.9) of $L[\mathcal{C}[X]]$ there exists a chain of faces:

$$X' = X^k \prec X^{k+1} \prec \dots \quad X^{d-1} \prec X^d = X$$

in $L[\mathcal{C}[X]]$ where $d(X^i) = i$ for $i = k,\dots,d$. Since $X^{i-1} \prec X^i \prec X^d$ for $i = k+1,\dots,d-1$, and by (10.18) and (10.12), there exist a facet element $e_i \in \underline{X}^i \setminus \underline{X}^{i-1}$ of X for $i = k+1,\dots,d$. We claim that the facets $W^i \equiv \lceil X + e_i^o \rceil$ $i = k+1,\dots,d$ are all distinct, that proves the result. Suppose that $W^i = W^j$ for some i, j with $k+1 \le i < j \le d$. Noting that $e_j \in \underline{X}^j \setminus \underline{X}^i$ and $e_i, e_j \notin \underline{W}^i$, $W^i \prec W^i \circ X^i \prec X$ contradicting that W^i is a facet of X. This completes the proof. □

Immediate corollaries are:

(10.20) <u>Corollary</u> Every d-dimensional face (or cell) of C has at least (d+1) facets, for $d \geq 0$.

(10.21) <u>Corollary</u> For every d-dimensional face X (or cell) of C and its vertex (i.e. 0-dimensional face) V, there are at least d facets of X which V is a face of.

(10.22) <u>Corollary</u> Every d-dimensional face X of C is the comoposition $X^1 \circ X^2$ of any two distinct facets X^1 and X^2 of X, for $d \geq 1$.

C. Edges

It has been remarked that the vertices $\mathcal{V} = \mathcal{V}(C)$ of C are precisely the 0-dimensional faces of C .

We define the <u>edges</u> of C as the 1-dimensional faces of C .

(10.23) <u>Proposition</u> Every edge of C has exactly two vertices as its faces.

<u>Proof</u> Let W be an edge of C . By (10.20) W has at least two vertices. Suppose that W has three vertices, V^1, V^2, V^3. By (10.22), $V^1 \circ V^2 = V^2 \circ V^3 = V^3 \circ V^1 = W$. Let $T^i = \underline{W} \setminus \underline{V}^i$ for $i = 1,2,3$. Clearly $T^i \cap T^j = \phi$ if $i \neq j$. Setting $X^1 = V^1$, $X^2 = -V^2$ and $I = T^3$, it follows from (1.8) that there exists $X \in C$ and $j \in T^3$ s.t.

$$X_j = 0$$

$$X_{T^3} \preceq W_{T^3}$$

$$X_{T^1} = -W_{T^1}$$

$$X_{T^2} = W_{T^2} .$$

Remark that $X_{T^3} = \underset{\sim}{0}$, since otherwise $V^3 \prec V^3 \circ X \prec W$, contradicting W being an edge. Thus $\underline{X} \subseteq \underline{V}^3$ and hence $X = V^3$ or $-V^3$. This cannot happen because $X_e = -V^3_e \neq 0$ for $e \in T^1$ and $X_f = V^3_f \neq 0$ for $f \in T^2$. This completes the proof. □

(10.24) <u>Corollary</u> If $X, X' \in C$ and $X' \prec X$ with $d(X) = d(X')$ then there are exactly two faces between X' and X.

<u>Proof</u> Let $R = \underline{X}'$. Consider the minor $C' = C \setminus R$ of C. Clearly there is one to one correspondance between the faces of C between X' and X and the vertices of C' below the edge $X_{E \setminus R}$ of C'. Hence the result follows from (10.23). □

(10.25) One important consequence of (10.23) is that there is a graph structure underlying oriented matroids. Let t be a subset of C with the property that

$$\text{(a)} \quad \langle X' \in C \text{ and } X' \preceq X \in t \rangle \Rightarrow \langle X' \in t \rangle \quad .$$

Let $K^i(t)$ be the set of all i-dimensional faces of C in t, for $-1 \leq i \leq r(C)-1$. The <u>1-skeleton</u> $S^1(t)$ is the pair $(K^o(t), K^1(t))$ Proposition (10.23) implies that

(10.26) The 1-skeleton $S^1(t)$ is a graph in which the incidence relation is induced by the conformal relation $\preceq$ in C .

In particular,

(10.27) if t is a 0-dimensional cell then $S^1(t)$ is a graph consisting of a single vertex ; and

(10.28) if t is a 1-dimensional cell then $S^1(t)$ is a graph consisting of a single edge and its two vertices.

D. Supercells

We shall introduce a new notion generalizing flats and cells.

A supercell in $\mathcal{C}$ is a subset t of $\mathcal{C}$ which is of the form $t = \mathcal{C}[W]$ for an sd vector W on E.

Following properties are obvious.

(10.29) Every flat and every cell in $\mathcal{C}$ are supercells;

(10.30) The intersection of supercells is a supercell;

(10.31) Supercells are closed under composition;

(10.32) Supercells satisfy the elimination property.

It is also clear that if t is a supercell,

(10.33) the poset L[t] is graded by the dimension function d (in (10.7)).

Since the dimension function

$$d(X) = r(\mathcal{C}) - r(E \setminus \underline{X}) - 1 \qquad X \in \mathcal{C}$$

depends only on the support $\underline{X}$ of X, and by (10.31), we know that for a supercell t

(10.34) every maximal faces of L[t] have the same dimension.

The dimension d(t) of a supercell t is defined as the dimension of any maximal face of t.

(10.35) Proposition Let t be a d-dimensional supercell, and let X be a (d-1)-dimensional face of t. Then there exists at most two faces of t covering X in L[t] .

Furthermore, if t is a flat then there exists exactly two faces of t covering X.

The last statement easily follows from the fact that flats are closed under the composition and the negation. The first statement is implied by the following.

(10.36) <u>Lemma</u> Let $X \in \mathcal{C}$ be a common face of three distinct faces $X^1, X^2, X^3 \in \mathcal{C}$ having the same support. Then

$$d(X^i) \geq d(X) + 2 \quad \text{for} \quad i = 1,2,3 .$$

<u>Proof</u> Since X^1, X^2, X^3 are distinct, we may assume $X^1 \neq \pm X^2$. Using (1. 8) with setting I be the set of elements separating X^1 and X^2, we obtain $X^4 \in \mathcal{C}$ s.t. $X \prec X^4 \prec X^1$. This together with (10.33) implies $d(X^1) \geq d(X) + 2$. Since $d(X^1) = d(X^2) = d(X^3)$, the proof is complete. □

11. SHELLING

Bruggesser and Mani [SD] showed that the facets of any d-dimensional convex polytope P can be arranged in a sequence $F_1, F_2, \ldots, F_r$ (r = the number of facets of P) such that for each i with $2 \le i < r$, $F_i \cap (\bigcup_{j=1}^{i-1} F_j)$ is a topological (d-2)-ball. This property, called the shellability of polytopes was necessary to complete Schläfli's computation of the Euler characteristic for convex polytopes. Since then, the shellability, naturally considered as a property of geometrical cell complexes, has been studied by many authors. One of the most significant result using shellability is the proof of the upper bound conjecture (see McMullen and Shepherd [UB]).

In this chapter we introduce the notion of shellability for posets which abstracts the same notion for cell complexes. It will be shown that the Euler relation is still valid for shellable posets. The main theorem is the shellability of OM cells (ordered by the conformal relation).

It can be also shown as a consequence of shellability of OM cells that the l-skelton of an d-dimensional OM cell is d-connected. (This result was first proved by Mandel [TO] by a different argument.)

It is worth remarking that Lawrence (unpublished) recently found an OM cell which is not isomorphic to any convex polytope. This implies that the above results really generalize the corresponding results on convex polytopes. (See, Balinski [GC] for the proof of d-connectivity of a d-dimensional convex polytope).

A. PM Posets

First of all we note that every poset L studied in this chapter has the least element $0 = 0_L$, and assume that the dimension function $d = d_L$ of L is the height function $h = h_L$ of L minus one.

(11.1) A poset L (ordered by a partial order $\leq$) is called a PM poset of dimension d if the conditions (PM-1) ~(PM-5) are satisfied

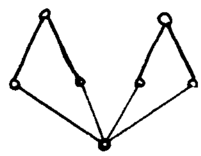
- (PM-1) L has the least element $0 = 0_L$;
- (PM-2) L satisfies the J-D property, or equi-alently L is graded by its dimension function ;
- (PM-3) Every maximal element of L has dimension d;
- (PM-4) Each element of dimension d-1 is covered by at most two (at least one from (PM-3)) maximal elements ;
- (PM-5) For any two ordered elements $X' < X$ with $d(X) = d(X')+2$, there are exactly two elements between X' and X.

(11.2) The boundary elements D_L of a PM poset L of dimension d are the (d-1)-dimensional elements (i.e. coatoms) of L which are covered by only one maximal element.

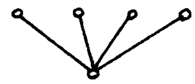
(11.3) The <u>boundary</u> ∂L of a PM poset L is the lower ideal of D_L i.e.,

$$\partial L = \{X \in L : X \leq Z \text{ for some } Z \in D_L\} .$$

It is easy to see that the boundary of a PM poset is not in general PM. For example :

L =  is PM ,

but

∂L =  is not PM

because this violates (PM-4).

However the following property holds:

(11.4) If L is a PM poset with the greatest element 1_L then the boundary ∂L of L is a PM poset of dimension one less than L, and furthermore $\partial\partial L = \phi$.

PM posets of small dimensions are easy to characterize:

(11.5) Every PM poset of dimension (-1) is a singleton $\{0 \equiv 1\}$;

(11.6) Every PM poset of dimension 0 is either a poset of two ordered elements $\{0, 1\}$ or a poset of three elements $\{0, M_1, M_2\}$ where M_1 and M_2 are the maximal elements covering 0.

Therefore the PM posets L of dimension 0 is isomorphic to either the face lattice of a 0-dimensional convex polytope or the boundary poset of a 1-dimensional convex polytope.

It can be shown that the face lattice of any d-dimensional convex polytope is a PM poset of dimension d and the boundary ∂L is a PM poset of dimension d-1. However a more general statement is true.

By an <u>OM cell of dimension d</u> we mean a d-dimensional cell in some OM. By (10.9) and (10.24) the following proposition holds:

(11.7) <u>Proposition</u> The lattice L[t] of an OM cell t of dimension d ($d \geq -1$) is a PM poset of dimension d and the boundary $\partial L[t]$ is a PM posets of dimension (d-1).

B. Shellability of PM Posets

For a poset L with a least element 0_L, and for a k-dimensional element $X \in L$, the interval $[0_L, X] = \{X' \in L : X' \leq X\}$ is called a k-dimensional cell (or k-cell) of L.

(11.8) The shellability of PM posets is defined inductively, using two different types of shellable posets namely shellable d-balls and shellable d-spheres:

(B) A poset L is defined to be a shellable d-ball ($d \geq 0$) if it is a PM poset of dimension d and either

(B-1) L has the greatest element 1_L and the boundary ∂L is a shellable (d-1)-sphere ; or

(B-2) every d-cell of L is a shellable d-ball and the d-cells of L can be arranged in a sequence:

$$t^1, t^2, \ldots, t^r$$

in such a way that

(B-2a) $\left(\bigcup_{i=1}^{j-1} t^i \right) \cap t^j$ is a shellable (d-1)-ball for all $1 \leq j \leq r$.

(S) A poset L is defined to be a <u>shellable d-sphere</u> ($d \geq -1$) if it is a PM poset of dimension d and either

(S-1) $d = -1$ (i.e. $L = \{O_L\}$) ; or

(S-2) $\partial L = \phi$ and every d-cell of L is a shellable d-cells of L can be arranged in a sequence:

$$t^1, t^2, \ldots, t^r$$

in such a way that

(S-2a) $\left(\bigcup_{i=1}^{j-1} t^i\right) \cap t^j$ is a shellable (d-1)-ball for all $1 \leq j \leq r$

and

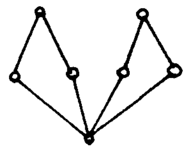
(S-2b) $\left(\bigcup_{i=1}^{r-1} t^i\right) \cap t^r$ is a shellable (d-1)-sphere.

(11.9) If L is a PM poset of dimension d a sequence $t^1, t^2, \ldots, t^r$ of the d-cells of L is said to be a <u>shelling sequence of L</u> if it satisfies either (B-2a), or (S-2a) and (S-2b). A PM poset of dimension d is said to be <u>shellable</u> if it is either a shellable d-ball or a shellable d-sphere.

The following is straightforward from the definitions , (11.5) and (11.6).

(11.10) The PM posets of dimension (-1) are the shellable (-1)-spheres. The PM-posets of dimension 0 are shellable and those of from {0, 1} are the shellable 0-balls and those of form $\{0, M_1, M_2\}$ are the shellable 0-spheres.

It should be clear that not all PM posets are shellable. For example,

L =  is not shellable.

C. Euler's Relation

Let L be a PM poset of dimension d ($d \geq -1$). We denote by $f_i(L)$ the number of i-deminsional elements of L for $-1 \leq i \leq d$. The Euler characteristic $\chi(L)$ of L is defined by

$$(11.11) \qquad \chi(L) = \sum_{i=0}^{d} (-1)^i f_i(L) .$$

It is well-known as Euler's relation that if L is the face lattice of a d-dimensional convex polytope then

$$(11.12) \qquad \chi(L) = 1$$

and if L is the boundary of the face lattice of a (d+1)-dimensional polytope then

$$(11.13) \qquad \chi(L) = 1 + (-1)^d .$$

In this section we shall prove a generalization of this result:

(11.14) **Theorem** The shellable PM posets satisfy the Euler's relation, that is,

(a) if L is a shellable d-ball then $\chi(L) = 1$; and

(b) if L is a shellable d-sphere then $\chi(L) = 1 + (-1)^d$.

Proof Let L be a shellable PM posets. If $d(L) = -1$, by (11.10), $L \equiv \{O_L\}$ is a shellable sphere and $\chi(L) = 0 = 1 + (-1)^{-1}$ follows. We use induction on d(L). Assume that the Euler-Poincare relation holds for all shellable PM posets of dimension $\le d-1$ $(d \ge 0)$. Consider the case $d(L) = d$. There are two cases (a) L is a shellable d-ball and (b) L is a shellable d-sphere.

Case (a): Since L is a shellable d-ball, (B-1) or (B-2) holds. If (B-1) holds, then clearly $L = \partial L + 1_L$ and

$$\chi(L) = \chi(\partial L) + (-1)^d$$
$$= 1 + (-1)^{d-1} + (-1)^d = 1$$

where the second equality holds because ∂L is a shellable (d-1)-sphere and by the inductive hypothesis. Suppose (B-2) holds. Then there exists a shelling sequence

$$t^1, t^2, \ldots, t^r$$

of the d-cells of L satisfying (B-2a). Let $U^{j-1} = \bigcup_{i=1}^{j-1} t^i$ for $1 \le j \le r + 1$. We shall prove that

(*) $\chi(U^{j-1}) = 1$ for $1 \le j \le r + 1$

which proves $\chi(L) = 1$. We use induction on j to prove (*). If $j = 1$, the result (*) follows from the fact that t^1 is a shellable d-ball and from the first observation. Thus, we assume that $\chi(U^{j-1}) = 1$ for $1 \le j \le k$, and calculate $\chi(U^k)$ as follows :

$$\chi(U^k) = \chi(U^{k-1}) + \chi(t^k) - \chi(U^{k-1} \cap t^k) .$$

Since t^k is a shellable d-ball with a greatest element, $\chi(t^k) = 0$.

By the (second) induction $\chi(U^{k-1}) = 1$. By (B-2a), $U^{k-1} \cap t^k$ is a shellable (d-1)-ball and by the first induction $\chi(U^{k-1} \cap t^k) = 0$. Therefore $\chi(U^k) = 0$ and (*) follows. The completes the proof of (a) part.

Case (b): Since L a shellable d-sphere with $d \geq 0$, the condition (S-2) holds. Let $t^1, t^2, \ldots, t^r$ be a shelling sequence satisfying (S-2a) and (S-2b). Let $U^{r-1} = \bigcup_{i=1}^{r-1} t^i$. Since U^{r-1} and t^r are shellable d-balls and by the Case (a), we have

(**) $\chi(U^{r-1}) = 1$ and $\chi(t^r) = 1$.

Since $U^{r-1} \cap t^r$ is a shellable (d-1)-sphere, the inductive hypothesis implies

(***) $\chi(U^{r-1} \cap t^r) = 1 + (-1)^{d-1}$.

Using (**) and (***), we obtain

$$\begin{aligned} \chi(L) &= \chi(U^{r-1}) + \chi(t^r) - \chi(U^{r-1} \cap t^r) \\ &= 1 + 1 - 1 - (-1)^{d-1} \\ &= 1 + (-1)^d \ . \end{aligned}$$

This completes the proof. □

D. The Shellability of OM Cells

In this section, we assume that sets of signed vectors (on a finite set)are ordered by the conformal relation $\preceq$. So we may simply write t for the poset L[t] of an OM cell t .

We have already remarked in (11.7) that every OM cell t of dimension d and its boundary ∂t are both PM posets of dimension d and (d-1) respectively. In this section we shall prove the main result of the present chapter :

(11.15) Theorem (Shellability of OM Cells)
Every OM cell of dimension d is a shellable d-ball and its boundary is a shellable (d-1)-sphere.

An important corollary of this theorem and Theorem (11.14) is a generalization of Euler's relation for convex polytopes:

(11.16) Corollary (Euler's Relation for OM Cells)
For every OM cell t of dimension d,

$$\chi(t) = 1 \quad \text{and}$$
$$\chi(\partial t) = 1 + (-1)^d \quad .$$

First of all, we shall reduce the theorem (11.15) to an equivalent statement which is easier to handle.

Let $\mathcal{C}$ be an OM on a finite set E. The maximal vectors (or faces) of $\mathcal{C}$ are said to be the <u>topes</u> of $\mathcal{C}$. Hence, if $\mathcal{C}$ has no loops, a vector $X \in \mathcal{C}$ is a tope iff $\underline{X} = E$. For a tope T of $\mathcal{C}$, the cell $\mathcal{C}[T] = \{ X \in \mathcal{C}: X \preceq T \}$ is said to be a <u>tope cell</u> of $\mathcal{C}$. Clearly the tope cells of $\mathcal{C}$ are the maximal cells of $\mathcal{C}$, and topes and tope cells have dimension $r(\mathcal{C}) - 1$.

(11.17) <u>Proposition</u> Let t be an OM cell of dimension d. Then there exists an OM $\mathcal{C}$ with a tope $\underset{\sim}{+} \in \mathcal{C}$ such that the tope cell $\mathcal{C}[\underset{\sim}{+}]$ is isomorphic to t .

<u>Proof</u> Let t be a d-dimensional cell of an OM $\mathcal{C}'$ on E', and let $W \in \mathcal{C}'$ be the maximal face of t. Thus $t = \mathcal{C}'[W]$. Let

$$A = \{ e \in E' : W_e = 0 \}$$

$$B = \{ e \in E' : W_e = - \} ,$$

and let $\mathcal{C} = {}_B(\mathcal{C}' / A)$, $E = E' \setminus A$. It is easy to see that $\mathcal{C}$ has the desired properties. □

By the proposition above, Theorem (11.15) is equivalent to the following :

(11.18) <u>Theorem</u> Let $\mathcal{C}$ be an OM on E with $\underset{\sim}{+} \in \mathcal{C}$. Then, the tope cell $\mathcal{C}[\underset{\sim}{+}]$ is a shellable d-ball and its boundary $\partial\mathcal{C}[\underset{\sim}{+}]$ is a shellable (d-1)-sphere, where $d = r(\mathcal{C}) - 1$.

We shall prove a more general form of Theorem (11.18) in the rest of the section, and hence obtain the main theorem (11.15) as a consequence. The proof technique we use here is a combinatorial analog of the "line shelling" of convex polytopes given by Bruggesser and Mani [SD]. The idea of line shelling can be explained as follows. Suppose P is a d-dimensional convex polytope in $\mathbb{R}^d$ with facets $F_1, F_2, \ldots, F_r$. Let H_i be the hyperplane in $\mathbb{R}^d$ spanned by F_i for each i. Take any line ℓ in the space passing through an interior point of P, which intersects with the hyperplanes at distinct points. Now we trace the line in one direction starting from a interior point of P, and pass the first hyperplane, say H_{i_1}, and then the second H_{i_2}, the third H_{i_3} and so on till infinity, and comes back from the other end of infinity (i.e., trace the line ℓ projectively), and again pass the remaining hyperplanes $H_{i_k}, H_{i_{k+1}}, \ldots, H_{i_r}$ to return to the initial point. (See Fig 11.1 in the next page.) Then, the sequence $F_{i_1}, F_{i_2}, \ldots, F_{i_r}$ of facets of P is known to be a shelling of P, i.e., for each $2 \le j < r$, $F_{i_j} \cap (\bigcup_{k=1}^{j-1} F_{i_k})$ is a topological (d-2)-dimensional ball.

It turns out that for the purpose of obtaining a shelling of a convex polytope, we need not take a "straight" line as described above, but we may as well take a topological line segment provided that (a) it contains an interior point

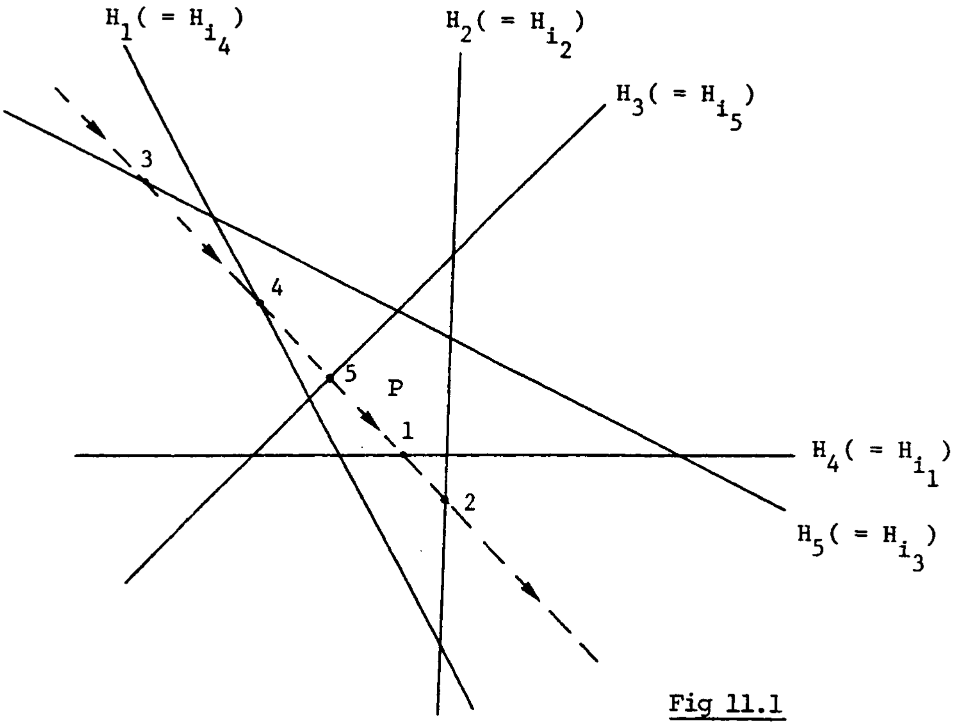


Fig 11.1

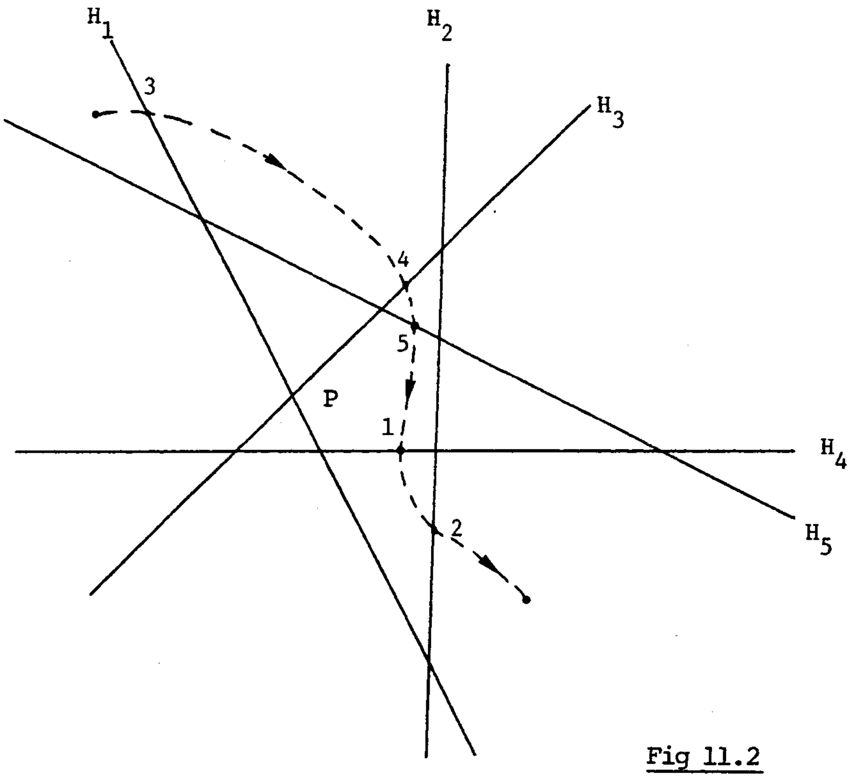


Fig 11.2

of P, (b) it passes through each hyperplane exactry once, (c) the intersections of the segment and hyperplanes are all distinct, and (d) it connects two d-dimensional unbounded polyhedra that are opposite, i.e., they are determined by intersecting opposite sides (halfspaces) of each hyperplanes. (See Fig 11.2.) For shelling of an OM cell, we shall use a combinatorial analog of such a segment, which is a directed path in a certain graph that is to be described now.

For the simplicity of our discussion, in what follows, we assume that the assumption of Theorem (11.18) is satisfied:

(11.19) Assumption $\mathcal{C}$ is an OM on a finite set E having the tope $\underset{\sim}{+}$ and $r(\mathcal{C}) = d + 1$.

Thus we have

(11.20) $\underset{\sim}{-} \in \mathcal{C}$

(11.21) $X \in \mathcal{C}$ is a tope iff $\underline{X} = E$ iff $d(X) = d$.

(11.22) Proposition Let W be a (d-1)-dimensional vector of $\mathcal{C}$. Then there exist exactly two topes of $\mathcal{C}$ having W as their facets.

Proof Since there are at least two topes $W \circ \underset{\sim}{+}$ and $W \circ \underset{\sim}{-}$ having W as their facets, the result immediatly follows from Lemma (10.36). □

Using Proposition(11.22), we can construct a directed graph as follows :

(11.23) The <u>tope graph</u> TG = TG($\mathcal{C}$) of $\mathcal{C}$ is the directed graph in which the <u>nodes</u> are the topes, the <u>edges</u> are the (d-1)-dimensional vectors of $\mathcal{C}$, and each edge W is directed from the tail t(W) to the head h(W) defined by

$$t(W) = W \circ \underset{\sim}{-}$$

$$h(W) = W \circ \underset{\sim}{+} \quad .$$

One important property of the tope graph that immediatly follows from the definition is :

(11.24) TG is acyclic.

For singed vectors X and X' on E, let D(X,X') denote the set of elements in E separating X and X'. Two elements e and e' are said to be <u>equivalent</u> in $\mathcal{C}$ if $X_e = X'_e$ for all X in $\mathcal{C}$. Clearly, the set E can be partitioned into the equivalent classes of elements by this equivalence ralation. The following properties are easily verified :

(11.25) Topes T and T' are adjacent in TG iff the set D(T,T') is an equivalent class of element and $T + D(T,T')^{o}$ is a common facet of T and T' ;

(11.26) A tope T' is dircted from a tope T in TG iff they are adjacent in TG and $T' = T + D(T, T')^{+}$;

(11.27) If e is a facet element of T then there exists a uniqe tope T' adjacent to T such that $T'_e = -T_e$.

The following proposion is very useful.

(11.28) <u>Proposition</u> For distinct topes T and T', there exists a facet element of T separating T and T'.

<u>Proof</u> Suppose that there exists no facet element of T separating T and T'. Then, by (10.17), we obtain

$$C[T] = C[T + D(T,T')^{\bullet}] .$$

This is a contradiction, because T' is in RHS but not in LHS. Therefore the result holds. □

Two immediate corollaries of (11.28) are :

(11.29) <u>Corollary</u> The tope $\underset{\sim}{+}$ ($\underset{\sim}{-}$, respectively) is the only sink (source) of the tope graph TG.

(11.30) <u>Corollary</u> Let T and T' be distinct topes of such that $T'_e = +$ for all $e \in E$ with $T_e = +$. Then there exists a directed path from T to T' in TG.

It is clear from (11.30) that there exists a directed path from the source $\underset{\sim}{-}$ to the sink $\underset{\sim}{+}$ in TG. This path, which can be considered as a combinatorial analog of a line described previously for the line shelling of convex polytopes, will play a significant role in determining a shelling sequence of the tope cell $\mathcal{C}[\underset{\sim}{+}]$ (and its boundary $\partial\mathcal{C}[\underset{\sim}{+}]$), and hence in proving the main result (11.18).

One more term, which is merely a combinatorial analog of "invisible points" given by Bruggesser and Mani [SD], is necessary here. For a tope T of $\mathcal{C}$, the <u>umbrella</u> U(T) of T is the subset of $\mathcal{C}[\underset{\sim}{+}]$ defined by

(11.31) $$U(T) = \{ X \in \mathcal{C}[\underset{\sim}{+}] : X_e = 0 \text{ for some } e \in E \text{ with } T_e = + \} ,$$

or equivalently,

(11.32) $$U(T) = \cup \{ \mathcal{C}[\underset{\sim}{+} + e^o] : T_e = + \text{ and } e \in E \} .$$

From the definition, we have

(11.33a) $U(\underset{\sim}{-}) = \emptyset$;

(11.33b) $U(\underset{\sim}{+}) = \partial\mathcal{C}[\underset{\sim}{+}]$;

(11.33c) $U(T) \subseteq \partial\mathcal{C}[\underset{\sim}{+}]$ for every tope T .

A more interesting property of an umbrella is as follows :

(11.34) <u>Lemma</u> Let T be a tope of $\mathcal{C}$ different from $\underset{\sim}{-}$. Then the umbrella U(T) of T is a nonempty union of facets of $\mathcal{C}[\underset{\sim}{+}]$, and it is a PM poset of dimension (d-1).

<u>Proof</u> Let f be an element of E such that $T_f = +$. To prove the first statement, it is enough to show that there exists a facet element g of $\underset{\sim}{+}$ such that $T_g = +$ and $\mathcal{C}[\underset{\sim}{+} + f^o] \subseteq \mathcal{C}[\underset{\sim}{+} + g^o]$. Let

$$T^o = \lceil \underset{\sim}{+} + f^o \rceil \circ (-T) ,$$

where $\lceil X \rceil$ denotes the maximal face of $\mathcal{C}$ conforming to a vector X. Clearly T^o is a tope of $\mathcal{C}$. By (11.28) (with setting $T = \underset{\sim}{+}$ and $T' = T^o$), we obtain a facet element g of $\underset{\sim}{+}$ such that $T'_g = -$. It follows that $\lceil \underset{\sim}{+} + f^o \rceil_g = 0$, $T_g = +$, and $\mathcal{C}[\underset{\sim}{+} + f^o] \subseteq \mathcal{C}[\underset{\sim}{+} + g^o]$. This element g is what we needed. The second result easily follows from the first one. □

Let T be a tope of $\mathcal{C}$ different from $\underset{\sim}{-}$. By (11.30), there exists a directed path P from $\underset{\sim}{-}$ to T in TG :

(11.35) $\underset{\sim}{-} = T^o \to T^1 \to T^2 \to \quad \to T^{\ell-1} \to T^{\ell} = T$.

For $i = 1,2,\dots,\ell$, we denote

(11.36) $S_i = D(T^{i-1},T^i)$,

(11.37) $t^i = \mathcal{C}[\underset{\sim}{+} + S_i^o]$.

It follows from (11.25) that

(11.38) $\{S_1, S_2, \ldots, S_\ell\}$ is the partition of $D(T,\underset{\sim}{-})$ into equivalent classes of elements in C .

Therefore,

$$U(T) = \bigcup_{i=1}^{\ell} t^i \quad . \tag{11.39}$$

Note that each t^i is a face of $C[\underset{\sim}{+}]$ but not necessarily a facet The <u>facet sequence of $C[\underset{\sim}{+}]$ induced by a directed path (11.35)</u> is the subsequence of $t^1, t^2, \ldots, t^\ell$:

$$t^{i_1}, t^{i_2}, \ldots, t^{i_r} \qquad (r \leq \ell) \tag{11.40}$$

consisting of all facets in the original sequence.

From (11.34) and (11.39), we obtain

$$U(T) = \bigcup_{j=1}^{r} t^{i_j} \quad , \tag{11.41}$$

and furthermore,

$$U(T^{i_k}) = \bigcup_{j=1}^{k} t^{i_j} \qquad \text{for} \quad k = 1,2,\ldots,r \ . \tag{11.42}$$

(11.43) <u>Theorem</u> Let T be a tope of C different from $\underset{\sim}{-}$, an let P be a directed path from $\underset{\sim}{-}$ to T in TG. Then the following statements hold :

(a) If $T \neq \underset{\sim}{+}$ then the umbrella U(T) of T is a shellable (d-1)-ball ;

(b) If $T = \underset{\sim}{\pm}$ then the umbrella $U(T) \equiv \partial C[\underset{\sim}{\pm}]$ of T is a shellable (d-1)-sphere (thus, $C[\underset{\sim}{\pm}]$ is a shellable d-ball);

(c) In both cases (a) and (b) above, if $d \geq 1$, the facet sequence of $C[\underset{\sim}{\pm}]$ induced by the directed path P is a shelling sequence of $U(T)$.

It is clear that the statement (b) of the above theorem implies the main results, Theorem (11.18) and (11.15). We shall use somewhat complicated induction to prove (11.43). The following lemma will serve as the core of the inductive proof.

(11.44) Lemma Let T and T' be topes of C with $T \to T'$ in TG, and let $S = D(T,T')$. Then the following statements hold :

(a) $T_{E\setminus S}$ ($\equiv T'_{E\setminus S}$) is a tope of C / S ;

(b) If $C[\underset{\sim}{\pm} + S^o]$ is a facet of $C[\underset{\sim}{\pm}]$ then $\underset{\sim}{\pm}$ is a tope of C / S and the poset $U(T) \cap C[\underset{\sim}{\pm} + S^o]$ is isomorphic to the umbrella $U_{C/S}(T_{E\setminus S})$ of the tope $T_{E\setminus S}$ in C / S .

Proof The statement (a) is clear from (11.25). We shall prove (b). Let $F = \underset{\sim}{\pm} + S^o$, and suppose $C[F]$ is a facet of

$C[+]$. Since $T + S^o \in C$, and since $\lceil F \rceil$ has the same dimension as $T + S^o$, and by the fact that $\underline{\lceil F \rceil \subseteq T + S^o}$, we have $\lceil F \rceil = F \in C$. Thus, $F_{E \setminus S} = \underset{\sim}{+} \in C / S$ is a tope of C / S . For the proof of the last statement, take $X \in U(T) \cap C[F]$. Noting that $X_e = 0$, $T_e = +$ for some $e \in E \setminus S$ and $X_S = \underset{\sim}{0}$, we have $X_{E \setminus S} \in U_{C/S}(T_{E \setminus S})$. Conversely, if $X' \in U_{C/S}(T_{E \setminus S})$ then $X' + S^o \in U(T) \cap C[\underset{\sim}{+} + S^o]$. Thus there exists a bijection between the two posets, and this obviously preserves the conformal relation. This completes the proof. □

Now we shall prove Theorem (11.43), and thus we complete the proof of the main theorem.

Proof of (11.43) We use induction on $r(C)$ $(= d - 1)$. If $r(C) = 1$, $C = \{ \underset{\sim}{+}, \underset{\sim}{-}, \underset{\sim}{0} \}$ and the result (a) and (b) are obvious. Assume that (a) and (b) are true when $1 \le r(C) \le m-1$.

Let $r(C) = m$. Let $T \neq \underset{\sim}{-}$ be a tope of C , and let P be any directed path : $\underset{\sim}{-} = T^o \to T^1 \to \cdots \to T^{\ell} = T$ in the tope graph TG. We describe the facet sequence induced by the path P as in (11.35) ~ (11.40). Then we have

$$U(T^{i_k}) = \bigcup_{j=1}^{k} t^{i_j} \qquad \text{for } 1 \le k \le r .$$

Remarking that each t^{i_j} $(1 \le j \le r)$ is a cell of dimension (m-2) and clearly a shellable (m-2)-ball by the inductive hypothesis. Hence, $U(T^{i_1}) = t^{i_1}$ is a shellable (m-2)-ball.

We shall prove that $U(T^{i_k})$ is a shellable (m-2)-ball for $1 \le k < r$ using induction on k , and finally show that $U(T^{i_r})$ is a shellable (m-2)-ball if $T \neq \underset{\sim}{+}$, and a shellable (m-2)-sphere if $T = \underset{\sim}{+}$.

Assume that $U(T^{i_k})$ is a shellable (m-2)-ball if $1 \le k \le q-1$ $(q \le r)$. In order to prove that $U(T^{i_q})$ is a shellable (m-2)-ball (-sphere, respectively), it is sufficient to show that $U(T^{i_{q-1}}) \cap t^{i_q}$ is a shellable (m-3)-ball (-sphere). First of all, we observe that

(i) $$U(T^{i_{q-1}}) = U(T^{i_q - 1}) .$$

Applying Lemma (11.44) with setting $T = T^{i_q - 1}$ and $T' = T^{i_q}$, we obtain

(ii) $$U(T^{i_q - 1}) \cap t^{i_q} \text{ is isomorphic to } \overline{U}(\overline{T}^{i_q}) ,$$

where we set $S = S_{i_q}$, $\overline{X}$ denotes the subvector $X_{E \setminus S}$ of a signed vector X, and $\overline{U}(\cdot)$ denotes the umbrella $U_{C/S}(\cdot)$ in C / S .
It is easy to see that $r(C / S) = r(C) - 1$,and by the first inductive hypothesis we know that $\overline{U}(\overline{T}^{i_q})$ is a shellable (m-3)-ball if $\overline{T}^{i_q} \neq \underset{\sim}{+}$, and a shellable (m-3)-sphere if $\overline{T}^{i_q} = \underset{\sim}{+}$. Since $\overline{T}^{i_q} = \underset{\sim}{+}$ iff $T^{i_q} = \underset{\sim}{+}$ iff $q = r$ and $T^{i_r} = T = \underset{\sim}{+}$, and by (i) and (ii), one obtains

(iii) $U(T^{i_{q-1}}) \cap t^{i_q}$ is a shellable (m-3)-ball if $T \neq \underset{\sim}{+}$ and $q \neq r$; and

(iv) $U(T^{i_{q-1}}) \cap t^{i_q}$ is a shellable (m-3)-sphere if $T = \underset{\sim}{+}$ and $q = r$.

Therefore, by the second induction, U(T) is a shellable (m-2)-ball if $T \neq \underset{\sim}{+}$, and a shellable (m-2)-sphere if $T = \underset{\sim}{+}$. The result (c) is clear from the proof. □

Before ending this section, we shall show that there are several different ways of shelling OM cells. We know from (11.43) that any directed path from $\underset{\sim}{-}$ to $\underset{\sim}{+}$ in the tope graph TG induces a shelling sequence of $C[\underset{\sim}{+}]$. Hence, by showing the existence of a directed path with certain properties, one can obtain a shelling sequence with special requirements.

(11.45) <u>Proposition</u> Let $V \in C$ be any vertex of $C[\underset{\sim}{+}]$, and let $F, F' \in C$ be any two distinct facets of $C[\underset{\sim}{+}]$. Then, there exists a directed path :

$$\underset{\sim}{-} = T^o \to T^1 \to T^2 \to \cdot \cdot \cdot \to T^\ell = \underset{\sim}{+}$$

from - to + in the tope graph TG with a prescribed property, either Property 1 or Property 2 :

<u>Property 1</u> : $T^1 = -(F \circ \underset{\sim}{-})$ and $T^{-1} = F' \circ \underset{\sim}{-}$;

<u>Property 2</u> : $V \not\preceq T^i$ for $i = 1,2,...,k$ and
$V \preceq T^i$ for $i = k+1,...,\ell$
for some $1 \le k < \ell$.

Proof Since F and F' are facets of $C[\underset{\sim}{+}]$, $F \circ \underset{\sim}{-}$ and $F' \circ \underset{\sim}{-}$ are topes of C adjacent to $\underset{\sim}{+}$. Thus, $-(F' \circ \underset{\sim}{-})$ is a tope and there exists a directed path from $-(F' \circ \underset{\sim}{-})$ to $F \circ \underset{\sim}{-}$ by (11.30) . Since $\underset{\sim}{-} \to -(F' \circ \underset{\sim}{-})$ and $F \circ \underset{\sim}{-} \to \underset{\sim}{+}$, the result with Property 1 follows.

Let $T = V \circ \underset{\sim}{-}$. Since T is a tope of C , there exists a directed path from $\underset{\sim}{-}$ to $\underset{\sim}{+}$ through T by (11.30). For such a path, Property 2 is clearly satisfied. This completes the proof. □

(11.46) Proposision Let $\underset{\sim}{-} = T^o \to T^1 \to \cdot \cdot \cdot \to T^\ell = \underset{\sim}{+}$ be a directed path from $\underset{\sim}{-}$ to $\underset{\sim}{+}$ in TG. Then, the reflection $\underset{\sim}{-} = -T^\ell \to \cdot \cdot \cdot \cdot \to -T^1 \to -T^o = \underset{\sim}{+}$ is also a directed path from $\underset{\sim}{-}$ to $\underset{\sim}{+}$ in TG.

Proof Trivial. □

The following proposision is implied by (11.45) and (11.46).

(11.47) Proposition Let $V \in C$ be any vertex of $C[\underset{\sim}{+}]$, and let $t, t' \in C$ be any two distinct facets of $C[\underset{\sim}{+}]$. Then, the statements (a), (b) and (c) hold :

(a) There exists a shelling sequence of $C[\underset{\sim}{+}]$, in which t is the first and t' is the last ;

(b) There exists a shelling sequence of $C[\underset{\sim}{\pm}]$, in which no facet of $C[\underset{\sim}{\pm}]$ containing V appears befor any facet of $C[\underset{\sim}{\pm}]$ not containing V ;

(c) There exists a shelling sequence of $C[\underset{\sim}{\pm}]$, in which every facet of $C[\underset{\sim}{\pm}]$ containing V appears before any facet of $C[\underset{\sim}{\pm}]$ not containing V.

12. PERTURBATIONS

Let E be a finite set and let A be an $m \times E$ real matrix having A^e as the e-column of A for each $e \in E$. We denote by $\mathcal{C} = \mathcal{C}_A$ the oriented matroid (OM) obtained from the row space of A, i.e.,

$$(12.1) \qquad \mathcal{C}_A = \{ S(\lambda A) : \lambda \in \mathbb{R}^m \} ,$$

where S(x) is the signed vector of $x \in \mathbb{R}^E$. For two fixed elements f and g of E, and for a real number ε, define the $m \times E$ matrix $A(\varepsilon)$ by

$$(12.2) \qquad A(\varepsilon)^e = \begin{cases} A^e & \text{if } e \in E \setminus f \\ A^e + \varepsilon A^g & \text{if } e = f. \end{cases}$$

Then, one can verify that there exists a positive number $\bar{\varepsilon}$ such that the OM $\mathcal{C}_{A(\varepsilon)}$ is invariant over all $0 < \varepsilon < \bar{\varepsilon}$. We shall denote this OM by $\mathcal{C}'$. A more interesting observation is that the OM $\mathcal{C}'$ is obtainable from $\mathcal{C}$ itself without knowing its representaion matrix A, once two elements f and g are fixed More explicitly, $\mathcal{C}'$ is uniquely determined by

(12.3a) $X' \in \mathcal{C}'$ and $X'_g = 0$ iff $X' \in \mathcal{C}$ and $X'_g = 0$;

(12.3b) $X' \in C'$ and $X'_g = +$ iff either (1),(2) or (3) holds :

(1) $X' \in C$, $X'_g = +$ and $X'_f \neq 0$,

(2) $X' = X + f^+$ for some $X \in F^+$,

(3) $X' = X + f^+$ or $X' = X + f^o$ for some $X \in C$ such that $X_f = -$ and there exists $Z \in F^+$ with $X \succ Z$,

where $F^+ = \{ Z \in C : Z_g = + \text{ and } Z_f = 0 \}$; and

(12.3c) $C' = -C'$.

Hence, for any linear OM C and for any two elements f and g, the operation for obtaining C' by using the relations (12.3a) ~ (12.3c) preserves the OM-ness (and the linearity). This observation leads us to the question as to whether the same operation preserves the OM-ness for any non-linear OM C . We shall answer to this question positively in this chapter. Namely, we show that considerably more general operations can preserve the OM-ness. Such operations, which will be called perturbations, are specified by choosing two elements f and g together with a certain flat F of C lying on f, and modify the f-components of vectors in the "neighborhood" of the flat F. Two important specializations of perturbations are the sliding operation (described above for linear OM's) in which F is chosen to be $C(f) = \{ X \in C : X_f = 0 \}$, and the point perturbation in which F is chosen to be a point $\{ \pm V \}$ for some vertex V of C.

There are interesting applications of point perturbations such as constructions of well-known nonlinear OM's (e.g. the non-Desargues OM, the Vamos OM) from certain linear OM's, and constructions of non-BOM's (see, Chapter 7). Sliding operations can be used for an OM programming problem to disolve degeneracy so that every optimal (respectively, unbounded) basis of a resulting new problem is optimal (unbounded) for the original problem, just like the perturbation of a linear programming problem. Unfortunately, no interesting applications of perturbations excluding slidings and point perturbations have been found.

A. GENERAL PERTURBATION THEOREM

Let $\mathcal{C}$ be an OM on a finite set E. For a subset $\mathcal{D}$ of $\mathcal{C}$, we define the star $st(\mathcal{D})$ of $\mathcal{D}$ (in $\mathcal{C}$) is the set of vectors in $\mathcal{C} \setminus \mathcal{D}$ having some vector in $\mathcal{D}$ as its face, i.e.,

(12.4) $st(\mathcal{D}) = \{ X \in \mathcal{C} \setminus \mathcal{D} : X \succcurlyeq X' \text{ for some } X' \in \mathcal{D} \}.$

The following is the main theorem of this chapter.

(12.5) Theorem (General Perturbation Theorem)

Let $\mathcal{C}$ be an OM on E, let f and g be two elements of E and let $\mathcal{F}$ be a flat of $\mathcal{C}$ satisfying the following conditions (a) and (b):

(a) $\mathcal{F}$ lies on f (i.e., $X_f = 0$ for all $X \in \mathcal{F}$);

(b) $< X \in st(\mathcal{F}^+)$ and $X_f = 0 > \Rightarrow < X + f^+ \in \mathcal{C} >$,

where $\mathcal{F}^+ = \{ Z \in \mathcal{F} : Z_g = + \}$. Then the set $\mathcal{C}'$ is an OM on E, where

$\mathcal{C}' = \mathcal{C} \setminus \{\pm\mathcal{F}^+\} \cup \{\pm\hat{\mathcal{F}}^+\} \cup \{\pm\mathcal{N}\}$

$\hat{\mathcal{F}}^+ = \{ X + f^+ : X \in \mathcal{F}^+\}$

$\mathcal{N} = {}^{o}\mathcal{N} \cup {}^{+}\mathcal{N}$

${}^{+}\mathcal{N} = \{ X + f^+ : X \in st(\mathcal{F}^+) \text{ and } X_f = - \}$

${}^{o}\mathcal{N} = \{ X + f^o : X \in st(\mathcal{F}^+) \text{ and } X_f = - \}.$

(12.6) Remark: It is easy to see that the condition (b) of the theorem (under the other conditions) implies

$$\text{(b')} \quad < X \in st(F^+) \text{ and } X_f = 0 > \; \Rightarrow \; < X + f^- \in \mathcal{C} >.$$

Moreover, this condition is independent of the orientation of $\mathcal{C}$ but it is in fact equivalent to the following condition on the underlying matroid $M = \{ E \setminus \underline{X} : X \in \mathcal{C} \}$ of $\mathcal{C}$:

$$\text{(b")} \quad < F' \in M \text{ and } f \in F' \subset F > \; \Rightarrow \; < F' \setminus \{f\} \in M >,$$

where $F = \cap \{ E \setminus \underline{X} : X \in \mathcal{F} \}$.

(12.7) Remark: In Theorem (12.5), the new OM $\mathcal{C}'$ is different from the original OM $\mathcal{C}$ iff

$$\exists \quad X \in \mathcal{F}^+ \text{ such that } X + f^+ \notin \mathcal{C} .$$

As we remarked before, the following two corollaries of the main theorem can be obtained immediately from (12.5) by taking $\mathcal{F}$ to be (i) a point (0-dimensional flat) of $\mathcal{C}$ or (ii) the minimal flat $\{ X \in \mathcal{C}: X_f = 0 \}$ of $\mathcal{C}$ lying on f (that is a hyperplane of $\mathcal{C}$ unless f is a loop).

(12.8) Corollary (Point Perturbation Theorem)

Let $\mathcal{C}$ be an OM on E, let $f \in E$ and let V be a vertex of $\mathcal{C}$ satisfying the following conditions (a) and (b):

(a) $V_f = 0$;

(b) $\langle X \in C \, , \, X_f = 0$ and $X \succ V \rangle \Rightarrow \langle X + f^+ \in C \rangle$.

Then $C' = C \setminus \{V,-V\} \cup \{\hat{V},-\hat{V}\} \cup \{\pm N\}$ is an OM on E, where

$V = V + f^+$

$N = {}^{0}N \cup {}^{+}N$

${}^{\alpha}N = \{ X + f^{\alpha} : V \prec X \in C$ and $X_f = - \}$ $(\alpha \in \{0,+\})$.

(12.9) <u>Corollary</u> <u>(Sliding Theorem)</u>

Let C be an OM on E and let f and g be two distinct elements of E. Then the set C' defined as in (12.5) is an OM on E for $F = \{ X \in C : X_f = 0 \}$.

It should be remarked that the above two corollaries were first proved by Mandel. In fact, Mandel [TO] gave a very interesting generalization of the Point Perturbation Theorem, and a topological proof using a topological representation theorem of oriented matroids.

We pointed out earier that a sliding operation preserves the linearity of OM's. However, a general perturbation does not necessarily preserve this property. Some examples of nonlinear OM's are constructed from a linear OM by means of a sequence of point perturbations in Chapter 7.

B. Proof of the General Perturbation Theorem

In this section we shall give a proof of the main theorem (12.5). Obviously the direct proof of the theorem is to verify that the OM axioms (1.2 OM-0) ~ (OM-3) are satisfied by the new C' in (12.5) under the conditions of the theorem. While the axioms (OM-0) and (OM-1) are clearly satisfied by C' because of the construction, the verification of the closedness under composition (OM-2) and the elimination property (OM-3) is not straightforward.

Here we introduce a new property, called the P-connectivity property, which is equivalent to the property (OM-2) together with (OM-3). We shall verify that C' has this property instead of showing the last two axioms are satisfied. Although the proof is not too simple, it is much more tedious to follow the direct proof.

Let E be a finite set. We say that two signed vectors X and X' on E are adjacent if either X is a face of X' or X' is a face of X (i.e., $X \succeq X'$ or $X' \succeq X$). For a set C of vectors on E, and two signed vectors X and X' in C, a path from X to X' (in C) is a sequence of vectors in C :

(12.10) $\quad X = X^0, X^1, X^2, \ldots, X^k = X'$

such that every two consecutive vectors in the sequence are adjacent.

Such a path is called a <u>proper</u> (or <u>P-</u>) path if

(12.11) $X^i_e = (X \circ X')_e$ for every $e \in E$ not separating X and X', for each $i = 1,2,\ldots,k-1$.

(12.12) A set $\mathcal{C}$ of vectors on E is said to be <u>properly connected</u> (or <u>P-connected</u>) if

(OM-4) there exists a proper path between any two vectors of $\mathcal{C}$.

(12.13) <u>Proposition</u> For a set $\mathcal{C}$ of vectors on E,

$\langle$ (OM-4) $\rangle$ $\Longleftrightarrow$ $\langle$(OM-2) and (OM-3)$\rangle$.

<u>Proof</u> (=>) Suppose (OM-4) is satisfied. It is easy to see that (OM-3) holds. We shall prove that (OM-2) holds. Let $X, X' \in \mathcal{C}$, and prove $X \circ X' \in \mathcal{C}$. We may assume $X \neq X \circ X'$. Take any element h s.t. $X_h = 0$. By (OM-4), there exists in $\mathcal{C}$ a proper path from X to X' : $X = X^0, X^1, \ldots, X^k = X'$. By the properness,

$$(*) \qquad X^1_h = (X \circ X')_h = X'_h$$

From the assumption $X \neq X \circ X'$, there exists $i \in E$ s.t. $X_i = 0$ and $X'_i \neq 0$. By (*), we have $X^1_i \neq 0$. Since X and X^1 are adjacent and $X_i \prec X^1_i$, we know $X \prec X^1$ and this shows $X^1 = X \circ X' \in \mathcal{C}$. Hence (OM-2) follows.

(<=) Suppose C satisfies (OM-2) and (OM-3). Let X^1 and X^2 be in C. By using the comformal elimination property (1.8 OM-3') ,which is equivalent to (OM-3), repeatedly, we can obtain a sequence : $X=X^0, X^1, \ldots, X^k=X'$ of vectors in C such that no element separates each X^{i-1} and X^i, and

$$X^i_e = (X^1 \circ X^2)_e \qquad \text{for all } e \in E \text{ not separating } X^1 \text{ and } X^2.$$

By (OM-2), $W^i \equiv X^{i-1} \circ X^i \in C$ and clearly

$$X^0,\ W^1,\ X^1,\ W^2,\ X^2,\ \ldots\ ,\ X^{k-1},\ W^k,\ X^k$$

is a proper path from X^1 to X^2. This completes the proof. □

It follows immediately from (12.13) that

(12.14) a set C of vectors on E is an OM iff it satisfies (OM-0), (OM-1) and the P-connectivity (OM-4).

For the rest of the section we assume that

(12.15a) C is an OM on a finite set E;

(12.15b) f and g are given distinct elements of E;

(12.15c) F is a flat of C lying on f.

Using (12.14), in order to prove Theorem (12.5) it is enough to show

(12.16) C' is P-connected under the condition (12.5b), where C' is given in(12.5).

The rest of the section will be devoted to achieving (12.16). We shall use all the sets defined in (12.5), such as $\mathcal{C}'$, $\hat{\mathcal{F}}^+$, $\mathcal{N}$, ${}^o\mathcal{N}$, ${}^+\mathcal{N}$, and also we set

$$\mathcal{F}^\alpha = \{ X \in \mathcal{F} : \quad X_g = \alpha \} \quad \text{for } \alpha \in \{+,0,-\} .$$

Let

(12.17) $\mathcal{T} = \text{st}(\mathcal{F}^+) \cup \mathcal{F}^+$.

(12.18) Proposition $\mathcal{T}$ is P-connected.

For the proof of (12.18) we need the following lemma :

(12.19) Lemma Let X, X', Z and Z' be vectors of $\mathcal{C}$ with $X \succeq Z$ and $X' \succeq Z'$. Then there exists a P-path from X to X' :

$$X = X^0,\ X^1,\ X^2,\ .\ .\ .\ ,\ X^k = X'$$

in $\mathcal{C}$ and a path from Z to Z' :

$$Z = Z^0,\ Z^1,\ Z^2,\ .\ .\ .\ ,\ Z^k = Z'$$

in $\mathcal{C}$ satisfying (a) and (b) for each i = 1,2,...,k-1 :

(a) $Z^i_e = (Z \circ Z')_e$ for all $e \in E$ with $Z_e = Z'_e$;

(b) $Z^i \preceq X^i$.

Proof Since $\mathcal{C}$ is an OM, $\mathcal{C}$ is P-connected. Let $Z = Z^0, Z^1, . . . , Z^r = Z'$ be any shortest P-path from Z to Z'. We use induction on r to prove the result.

Suppose r = 0, i.e., Z = Z'. Let $X = X^0, X^1, . . . , X^k = X'$ be a P-path from X to X'. Then, setting $\bar{X}^i = Z \circ X^i$ for i = 0,1,2,...,k,

$$(*)\quad \begin{array}{l} X = \bar{X}^0, \bar{X}^1, \bar{X}^2, \ldots, \bar{X}^k = X' \\ Z = Z, Z, Z, \ldots, Z = Z \end{array}$$

are paths we needed.

Consider tha case $r = 1$. Without loss of generality we can assume $Z \prec Z'$. Then $Z \prec X'$. Using the first case, we have two paths as in (*). Adding X' and Z' to the end of each paths in (*) we obtain paths we needed.

Now we assume by induction that the result is true if $0 \leq r < p$ $(p \geq 2)$, and consider the case $r = p$. If $Z^{p-1} \preceq X'$, then the result immediately follows. Suppose that $Z^{p-1} \npreceq X'$. Let $X^* = Z^{p-1} \circ X' \circ X$ and $Z^* = Z^{p-1}$. Note that $X^*_e = (X \circ X')_e$ for all $e \in E$ not separating X and X', and $X^* \geq Z^*$. By the inductive hypothesis there exists a P-path from X to X' and a path from Z to Z' :

$$\begin{array}{l} X = X^0, X^1, \ldots, X^s (= X^*), X^{s+1}, \ldots, X^k = X' \\ Z = Z^0, Z^1, \ldots, Z^s (= Z^*), Z^{s+1}, \ldots, Z^k = Z' \end{array}$$

satisfying for $i = 1, 2, \ldots, k-1$,

$Z^i_e = (Z \circ Z')_e$ for all $e \in E$ with $Z_e = Z'_e$, and

$X^i \succeq Z^i$.

This completes the proof. □

Now we are ready to prove (12.18).

<u>Proof of (12.18)</u>: Let X , $X' \in \mathcal{T}$. Then we can always take $Z, Z' \in F^+$ such that $X \succsim Z$ and $X' \succsim Z'$. It follows from (12.19) that there exists a P-path in $\mathcal{T}$, and therefore $\mathcal{T}$ is P-connected. This completes the proof. □

Define the set $\mathcal{T}'$ by

(12.20) $$\mathcal{T}' = st(F^+) \cup \hat{F}^+ \cup N$$
$$(= \mathcal{T} \setminus F^+ \cup \hat{F}^+ \cup N) .$$

We shall see later that the set $\mathcal{T}'$ is P-connected as well as 1
under the condition (12.5b). For the proof of this property, it will be convenient to observe the following :

(12.21) <u>Lemma</u> Suppose that the condition (12.5b) is satisfied. Then the following properties hold :

(a) $< Z \in \mathcal{T} > \;\Rightarrow\; < Z + f^+ \in \mathcal{T}' >$;

(b) $< X \in \mathcal{T}'$ and $X_f = + >$

$\Rightarrow\; < X = Z + f^+$ for some $Z \in \mathcal{T} >$;

(c) $< X \in \mathcal{T}'$ and $X_f = 0 >$
$\Rightarrow\; < X + f^+ \in \mathcal{T}'$ and $X + f^- \in \mathcal{T}' >$;

(d) $< X \in \mathcal{T}'$ and $X_f = - >$
$\Rightarrow\; < X + f^0 \in \mathcal{T}'$ and $X + f^+ \in \mathcal{T}' >$.

Proof (a) Let $Z \in T$. If $Z_f = +$ then $Z + f^+ = Z \in T \setminus F^+ \subseteq T'$. If $Z_f = -$ then $Z + f^+ \in {}^+N \subseteq T'$. Finally, suppose $Z_f = 0$. If $Z \in F^+$ then $Z + f^+ \in \hat{F}^+ \subseteq T'$. If $Z \notin F^+$ (i.e., $Z \in \text{st}(F^+)$) then $Z + f^+ \in \text{st}(F^+) \subseteq T'$ by the assumption (12.5b).

(b) This is straightforward from the construction.

(c) Let $X \in T'$ and $X_f = 0$. Then either $X \in {}^0N$ or $X \in \text{st}(F^+)$. If $X \in {}^0N$, $X + f^- \in \text{st}(F^+) \subseteq T'$ and also $X + f^+ \in {}^+N \subseteq T'$. If $X \in \text{st}(F^+)$ then by the assumption (12.5b) we have $X + f^\alpha \in \text{st}(F^+) \subseteq T'$ for $\alpha = +$ and $-$ (see Remark (12.6)).

(d) Let $X \in T'$ and $X_f = -$. Then $X \in \text{st}(F^+)$ and $X + f^\alpha \in {}^\alpha N \subseteq T'$ for $\alpha = +$ and 0. This completes the proof. □

(12.22) Proposition T' is P-connected.

Proof Let X and X' be in T'. We shall show that there exists a P-path from X to X' in T'. There are five cases to consider : (1) $X_f = X'_f = +$, (2) $X_f = X'_f = -$, (3) $X_f \neq 0$ and $X'_f = 0$, (4) $X_f = -X'_f \neq 0$, (5) $X_f = X'_f = 0$.

Case 1 : $X_f = X'_f = +$

By (12.21b) there exist vectors Z and $Z' \in T$ such that $X = Z + f^+$ and $X' = Z' + f^+$. Then there exists a P-path $Z = Z^0, Z^1, \ldots, Z^k = Z'$ in T by (2.18). It follows from (12.21a) that $X^i \equiv Z^i + f^+ \in T'$, and hence $X = X^0, X^1, \ldots, X^k = X'$ is a P-path from X to X' in T'.

<u>Case 2</u> : $X_f = X'_f = -$

In this case, X and X' are in $\mathcal{T} \setminus F^+$ and any P-path from X to X' in $\mathcal{T}$ is a P-path in $\mathcal{T}'$, since every vector in the path has negative f-component.

<u>Case 3</u> : $X_f \neq 0$ and $X'_f = 0$

It follows from (12.21c) that $X' + f^+$ and $X' + f^-$ are in $\mathcal{T}'$. If $X_f = +$, then we know from Case 1 that there exists a P-path in $\mathcal{T}'$ from X to $X' + f^+$ and this path together with X' is a P-path from X to X'. Similarly the same thing follows from Case 2 when $X_f = -$.

<u>Case 4</u> : $X_f = -X'_f \neq 0$ (w.l.g., $X_f = +$ and $X'_f = -$)

We have $X^* \equiv X' + f^0 \in \mathcal{T}'$ by (12.21d). Using Case 3 a P-path from X to X* exists in $\mathcal{T}'$, and this path together with X' is a P-path we are looking for.

<u>Case 5</u> : $X_f = X'_f = 0$

By (12.21c), $W \equiv X + f^-$ and $W' \equiv X' + f^-$ are in $\mathcal{T}'$. It follows from Case 2 that there exists a P-path $W = W^0, W^1, \ldots, W^k = W'$ in $\mathcal{T}'$. Since $W^i_f = -$, and by (12.21d), $X^0, X^1, \ldots, X^k$ is a P-path from X to X' in $\mathcal{T}'$, where $X^i = W^i + f^0$.

This completes the proof. □

For a subset $\mathcal{D}$ of $\mathcal{C}$, the neighborhood nb($\mathcal{D}$) of $\mathcal{D}$ in $\mathcal{C}$ is defined to be the set of vectors in $\mathcal{C} \setminus \mathcal{D}$ adjacent to (some vector in) $\mathcal{D}$, i.e.,

(12.23) $$\mathrm{nb}(\mathcal{D}) = \{ X' \in \mathcal{C} \setminus \mathcal{D} : \exists\ X \in \mathcal{D} \text{ such that } X \preceq X' \text{ or } X \succeq X' \} .$$

It is easy to see that

(12.24) $$\mathrm{nb}(F^+) = \mathrm{st}(F^+) \cup F^o .$$

Let us define the following set $\mathcal{B}$:

(12.25) $$\mathcal{B} = \mathrm{nb}(F^+) \setminus F^+ \cup \hat{F}^+ \cup N$$
$$(\equiv T' \cup F^o) .$$

(12.26) Proposition

Under the condition (12.5b), the set $\mathcal{B}$ is P-connected.

Proof Since F^o is a flat of $\mathcal{C}$, it is P-connected as well as T'. Let $X \in F^o$ and $X' \in T'$. It is enough to show (*) the existence of a P-path between X and X'. We claim that $X^* \equiv X \circ X' \in T'$. This together with the P-connectivity of T' implies (*). We consider three cases : (1) $X' \in \mathrm{st}(F^+)$, (2) $X' \in F^+$, and (3) $X' \in N$.

If (1) is the case, $X^* \in \mathrm{st}(F^+) \subset T'$. Suppose we have the case (2), and let $X' = Z + f^+$ for some $Z \in F^+$. Since $X \circ Z \in F^+$, $X^* = X \circ Z + f^+ \in \hat{F}^+ \subset T'$. Finally suppose $X' \in N$ and let $X' = Z + f^\alpha$ for some $Z \in \mathrm{st}(F^+)$ with $Z_f = -$

and $\alpha = 0$ or $+$. Noting that $X \circ Z \in st(F^+)$ and $(X \circ Z)_f = -$, we have $X^* = X \circ Z + f^\alpha \in N \subset T'$. This proves the claim and completes the proof. $\square$

Now we are ready to prove the maim theorem.

Proof of (12.5) : As we have already remarked, it is sufficient to show the P-connectivity of C' under the condition (12.5b). Let us define the subset $\mathcal{D}$ of C' by

$$\mathcal{D} = C \setminus \{ \pm F^+ \} .$$

Then the set C' has a partition :

$$C' = \mathcal{D} \cup \mathcal{B} \cup -\mathcal{B} .$$

It will be useful to keep the following relations in mind :

$$\begin{aligned} \mathcal{D} \cap \mathcal{B} &\subseteq nb(F^+) ; \\ \mathcal{D} \cap -\mathcal{B} &\subseteq -nb(F^+) = nb(F^-) ; \\ \mathcal{B} \cap -\mathcal{B} &= F^o ; \\ \mathcal{B} \setminus \mathcal{D} &\subseteq \hat{F}^+ \cup N . \end{aligned}$$

Take any two vectors X and X' from C'. We shall show that there exists a P-path from X to X' in C' for each of the four cases :

(1) $X, X' \in \mathcal{D}$, (2) $X, X' \in \mathcal{B}$ or $X, X' \in -\mathcal{B}$,

(3) $X \in \mathcal{D} \setminus \{\pm\mathcal{B}\}$ and $X' \in \{\pm\mathcal{B}\} \setminus \mathcal{D}$; and

(4) $X \in \mathcal{B} \setminus \mathcal{D}$ and $X' \in -\mathcal{B} \setminus \mathcal{D}$.

<u>Case 1</u> : $X, X' \in \mathcal{D}$

Since $\mathcal{C}$ is P-connected, there exists a P-path $X = X^0, X^1, X^2, \ldots, X^k = X'$ in $\mathcal{C}$. If $X^i \in \mathcal{D}$ for all i, this path is in $\mathcal{C}'$. Suppose $X^s \in \mathcal{F}^+$ (or $\mathcal{F}^-$) for some s. Then there exist vectors X^r and $X^t \in nb(\mathcal{F}^+)$ (or $nb(\mathcal{F}^-)$) with $0 < r < s < t < k$, since the path must meet the neighborhood of $\mathcal{F}^+$ (or $\mathcal{F}^-$) before and after it meets $\mathcal{F}^+$ (or $\mathcal{F}^-$). Since $\mathcal{B}$ (and $-\mathcal{B}$) is P-connected and contains $nb(\mathcal{F}^+)$ (and $nb(\mathcal{F}^-)$, respectively), there exists a P-path from X^r to X^t in $\mathcal{B}$ (or $-\mathcal{B}$). Replacing $X^r, X^{r+1}, \ldots, X^t$ with this path, and repeating a similar replacement for every other intersection of a new path and $\mathcal{F}^+ \cup \mathcal{F}^-$, one can obtain a P-path from X to X' in $\mathcal{C}'$.

<u>Case 2</u> : $X, X' \in \mathcal{B}$ or $X, X' \in -\mathcal{B}$

Straightforward from Proposition (12.16).

<u>Case 3</u> : $X \in \mathcal{D} \setminus \{\pm\mathcal{B}\}$ and $X' \in \{\pm\mathcal{B}\} \setminus \mathcal{D}$

Since $\mathcal{C}'$ is symmetric, we may assume $X' \in \mathcal{B} \setminus \mathcal{D}$, and thus we have $X' \in \hat{\mathcal{F}}^+ \cup \mathcal{N}$.

Let X" be any vector in $\mathcal{F}^+$ with $X'' \precsim X'$ if $X' \in \mathcal{N}$, and let X" be the vector in $\mathcal{F}^+$ with $X' = X'' + f^+$ if $X' \in \hat{\mathcal{F}}^+$. Let $X^* = X'' \circ X \circ X'$. We claim that $X^* \in \mathcal{C}$. Once this is shown, it is easy to see that $X^* \in st(\mathcal{F}^+) \subseteq \mathcal{D} \cap \mathcal{B}$, and it follows from the fact : $X^*_e = (X \circ X')_e$ for all $e \in E$ not separating

X and X' , from Case 1 and Case 2 that there exists a P-path from X to X' through X* in C'. So we shall prove the claim.

First we suppose $X_f = 0$. Since $X' = Z + f^{\alpha}$ for some $\alpha \in \{0,+\}$, and since C is closed under composition, $X^* = X'' \circ X \circ (Z + f^{\alpha}) = X'' \circ X \circ Z \in C$.

Suppose $X_f \neq 0$. There are two cases to consider: (1) $X' \in \hat{F}^+$ and (2) $X' \in N$. Let (1) be the case. Thus $X' = X'' + f^+$, and $X^* = X'' \circ X + f^+$. Since $X'' \circ X \succeq X'' \in F^+$,and since $X \nsubseteq F$, we know that $X'' \circ X \in st(F^+)$ and hence $X^* \in C$ by the condtion (12.5b). Suppose (2) is the case. Then there exists a vector $Z \in st(F^+)$ such that $Z_f = -$ and $X' = Z + f^{\alpha}$ for $\alpha = 0$ or + according as $X' \in {}^{0}N$ or ${}^{+}N$. Noting that $X'' \circ X \in st(F^+)$ and $(X'' \circ X)_f = 0$, the condition (12.5b) implies $X^1 \equiv X'' \circ X + f^+ \in C$. Let $X^2 = X'' \circ X \circ Z$. Using the elimination property (1.2 OM-3) for X^1 and X^2 , together with the closedness under composition (1.2 OM-2), we obtain $X^2 + f^{\alpha}$ ($= X^*$) $\in C$ for $\alpha = 0$ and + .

<u>Case 4</u> : $X \in B \setminus D$ and $X' \in -B \setminus D$

Note $X, -X' \in \hat{F}^+ \cup N \subseteq T'$. Therefore $X_f \neq -$ and $X'_f \neq +$. If $X_f = 0$ and $X'_f = -$, then by (12.21c) we have $X + f^- \in D$ and a P-path from X to X' through $X = f^-$ exists in C' from Case 3. By the symmetry of C' , a desired path exists when $X_f = +$ and $X'_f = 0$. Suppose $X_f = -X'_f = +$. If either $X \in N$ or $X' \in -N$, we have either $X + f^-$ or $X' + f^+ \in D$, and a required path through one of these vectors exists by Case If $X \in \hat{F}^+$ and $-X' \in \hat{F}^+$, then $X + f^0 \in F$ and $X' + f^0 \in F$, and using a P-path from $X + f^0$ to $X' + f^0$ in F, one can easily construct a P-path from X to X' in $F^o \cup \{\pm\hat{F}^+\} \subseteq C'$. The case $X_f = X'_f = 0$ (i.e., $X \in {}^oN$ and $-X' \in {}^oN$) is left.

Suppose $X \in {}^oN$ and $-X \in {}^oN$. Then there are vecto: $Z, -Z' \in st(F^+)$ such that $Z_f = -Z'_f = -$, and $X = Z + f^0$ and $X' = Z' + f^0$. Let $\hat{Z}, -\hat{Z}'$ be vectors in F^+ such that $Z \succeq \hat{Z}$ and $Z' \succeq \hat{Z}'$. Let $Z^* = \hat{Z} \circ Z' \circ Z \in C$. Note that $Z^* \in st(F^+)$, and that $Z^*_e = (Z \circ Z')_e = (X \circ X')_e$ for all $e \in E$ not separating X and X'. Since $Z^*, Z \in T$, and by (12.18), there exists a P-path from Z to Z^* in T. Noting that $Z_f = -Z^*_f = -$, there exists a vector Z^0 in the path such that $Z^0_f = 0$. Remark that $Z^0_e = (X \circ X')_e$ for all $e \in E$ not separating X and X'. If $Z^0 \notin F$ then $Z^0 \in T \setminus F = st(F^+) \subseteq B \cap D$, and hence by Case 3 there exists a P-path from X to X' through Z^0 in C'. Finally suppose $Z^0 \in F$. By the P-connectivity of F, there exists a P-path from Z^0 to $\hat{Z}'$ in F. Since $Z^0_g = -Z'_g = +$, there exists a vector $Z^1 \in F$ satisfying $Z^1_g = 0$ in the path. Observing that $Z^1 \in F^o = B \cap -B$, and $Z^1_e = (X \circ X')_e$ for all $e \in E$ not separating X and X', a required path from X to X' through Z' exists in C' by Case 2.

This completes the proof of (12.5). □

13. BOP'S AND BOM'S

In Chapter 7 we constructed some examples of nonlinear oriented matroid programming problems (OP), for which the simplex method produces a cycling of pivots, not all of which are non-degenerate. These examples were used to show that finding a finite pivot method for OP's is not a simple matter of adapting an existing finite simplex method for linear programming and that we need a new approach to the subject.

Independently, Mandel [TO] used a slight modification of our examples to show that an oriented matroid generalization of the separation theorem (Hahn-Banach theorem) proposed by Las Vergnas [CV] was false.

Thus it is suggested that the oriented matroid generalization of vector subspaces of $\mathbb{R}^n$ and that of linear programming be too general in some aspects. This motivates us to define a subclass of OP's (and a subclass of OM's) to which more properties of linear programming (and vector subspaces of $\mathbb{R}^n$) can generalize.

In this chapter we shall introduce a subclass of OP's called the BOP's, and a subclass of OM's called the BOM's. The class of BOP's (BOM's, respectively) includes the class of linear OP's (OM's) and excludes the class of OP's (OM's) for th simplex method can produce nondegenerate cycling (for some selection of the infinity and the objective element).

A. Definitions

Let C be an OM on a finite set E.

For a fixed element $g \in E$, consider an affine OM $(C; g)$ with the affine space $A = \{X \in C : X_g > 0\}$ and the infinite space $A^\infty = \{X \in C: X_g = 0\}$. Let $\mathcal{D}(X,X')$ be the set of directions from X to X' in $(C; g)$ (see (2.21) for the definition). First we observe:

(13.1) Proposition If $X^1, X^2 \in A$ and $X^1 \neq X^2$ then there is at most one line ℓ of C containing both X^1 and X^2.

Proof Suppose that $X^1, X^2 \in A$ and $X^1 \neq X^2$ and that there exist two lines ℓ^1 and ℓ^2 of C containing both X^1 and X^2. Since $\ell^1 \cap \ell^2 \neq \{\underset{\sim}{0}\}$ and $\ell^1 \cap \ell^2 \subset \ell^i$ $(i = 1,2)$. the flat $\ell^1 \cap \ell^2$ must be a point of C. This is not possible because $\{X^1, X^2\} \subseteq \ell^1 \cap \ell^2$ and $X^1 \neq \pm X^2$. □

(13.2) Proposition If X^1 and X^2 are distinct vectors of with $X^1 \in A$ and $X^2 \in A$. and if there exists a line of C containing both X^1 and X^2 then $D(X^1,X^2) = \{Z\}$ for some vertex Z of C.

Proof Suppose that ℓ is a line containing X^1 and X^2. Since $A \neq \phi$, A^∞ is a hyperplane. Clearly $\ell \nsubseteq A^\infty$ and hence $\ell \cap A^\infty$ is a point, say $\{Z, -Z, \underset{\sim}{0}\}$ for some vertex Z of C. Since both $\mathcal{D}(X^1, X^2)$ and $\mathcal{D}(X^2, X^1)$ are contained in any flat containing both X^1 and X^2, by (2.22.a) and (2.22.b), we have either $D(X^1,X^2) = \{Z\}$ or $\mathcal{D}(X^1,X^2) = \{-Z\}$. This completes the proof. □

(13.3) For two elements g and f of E, an OP $(\mathcal{C};\ g,f)$ is said to be a <u>non-BOP</u> if there exists a finite sequence of vertices of $\mathcal{C}$:

$$V^o,\ V^1,\ V^2,\dots,V^k$$

such that

(a) $V^o = V^k$;

(b) $V^i \in A \equiv \{X \in \mathcal{C} :\ X_g > 0\}$
for all $i = 0,1,\dots,k$;

(c) $V^i_{E\setminus f}$ is a vertex of $\mathcal{C}\setminus\{f\}$
for all $i = 0,1,\dots,k$;

(d) V^{i-1} and V^i are on a line for all $i = 1,\dots,k$;

(e) $Z^i_f \geq 0$ for all $i = 1,\dots,k$ and
$Z^j_f > 0$ for at least one $1 \leq j \leq k$
where Z^i is the unique direction in $(\mathcal{C};\ g)$ from V^{i-1} to V^i for $i = 1,\dots,k$.

(13.4) An OP is said to be a <u>BOP</u> if it is not a non-BOP.

(13.5) An OM $\mathcal{C}$ on E is said to be a <u>BOM</u> if the OP $(\mathcal{C};\ g,\ f)$ is a BOP for every choice of g and f, and a <u>non-BOM</u> otherwise - i.e. if there exist two elements g and f of E such that the OP $(\mathcal{C};\ g,\ f)$ is a non-BOP.

B. Characterizations

One elementary property on non-BOP's is the following:

(13.6) Suppose that an OP $(C;\ g,f)$ is a non-BOP and let $V^o, V^1, \ldots, V^k$ be a sequence of vertices of satisfying the conditions (13.3a) ~ (13.3e). Then

$$V_f^o = V_f^1 = \cdots = V_f^k \neq 0 \quad .$$

Proof Suppose the result is false. Since $V^o = V^k$, at least one of the following conditions must hold:

(1) $V_f^{i-1} = +$ and $V_f^i = 0$ or $-$ for some $1 \leq i \leq k$;

(2) $V_f^{i-1} = 0$ and $V_f^i = -$ for some $1 \leq i \leq k$.

It follows that each of cases (1) and (2), we must have $Z_f^i < 0$ for the direction Z^i from V^{i-1} to V^i. This violates the condition (13.3e). Thus the result follows. □

Let $\mathcal{B}$ be the set of bases of an OM C and let $\mathcal{B}_1$ be the set of bases of an OP $(C;g,f)$. Thus,

$$\mathcal{B}_1 = \{B \subseteq E_1 : B \cup \{g\} \in \mathcal{B}\}$$

where $E_1 = E \setminus \{g,f\}$. We shall give a characterization of non-BOP's using a certain sequence of bases of $(C;g,f)$. For this purpose we need several observations. Let $\mathcal{V}$ be the set of vertices of C and let $\mathcal{V}_1$ be

$$\mathcal{V}_1 = \{V \in \mathcal{V} : V \in \mathcal{A} \text{ and } V_{E\setminus\{f\}} \text{ is a vertex of } \mathcal{C}\setminus\{f\}\},$$

where $\mathcal{A}$ is the affine space of $(\mathcal{C};g)$.

(13.7) Proposition For each vertex $V \in \mathcal{V}_1$, there exists a basis $B \in \mathcal{B}_1$ such that $X(B) = V$. (Where $X(B)$ is the basic solution of $(\mathcal{C};g,f)$ with respect to B, see (5.4))

Proof Let $V \in \mathcal{V}_1$. Since V is a vertex of $\mathcal{C}$ and $V_g > 0$, the set $T \equiv E \setminus \underline{V} \cup \{g\}$ contains a basis of $\mathcal{C}$. It is enough to show that there exists a basis B of $\mathcal{C}$ s.t. $g \in B \subseteq T\setminus \{f\}$. Let B^o be any basis of $\mathcal{C}$ contained in T. Since $\underset{\sim}{0} \neq V \in \mathcal{C}(T\setminus\{g\})$, $g \in B^o$. If $f \notin B^o$, nothing to prove, and hence we can assume $f \in B^o$. Let X be the fundamental vertex $X(B^o;f)$ of f in B^o. If $X_e \neq 0$ for some $e \in T\setminus \{f\}$ then by (4.12) $B \equiv B^o\setminus \{f\} \cup \{e\}$ is a basis of $\mathcal{C}$ we are looking for. Otherwise $X_{T\setminus\{f\}} = 0$ and it implies that $\phi \neq \underline{X} \setminus\{f\} \subset \underline{V}$ and hence $V_{E\setminus\{f\}}$ is not a vertex of $\mathcal{C}\setminus\{f\}$. Since $V \in \mathcal{V}_1$, this case never occurs. This completes the proof. □

(13.8) Proposition If V and V' are distinct vertices in, $\mathcal{V}_1$ and if there exists a line ℓ containing both V and V' then there exists bases B and $B' \in \mathcal{B}_1$ such that

$$V = X(B) \quad , \quad V' = X(B') \quad \text{and}$$
$$|B \setminus B'| = 1 .$$

Proof Let $T = E \setminus (\underline{V} \cup \underline{V}')$ and let S be a basis of T in $\mathcal{C}$. Clearly $\mathcal{C}(T)$ is the smallest flat of $\mathcal{C}$ containing V and V' and hence by (13.1), $\ell = \mathcal{C}(T)$. It follows from (9.36) that $|S| = r(\mathcal{C})-2$. Since $V \neq \pm V'$, there exists an element $e' \in \underline{V} \setminus V'$ and $e \in \underline{V}' \setminus \underline{V}$. Let $B = S \cup \{e\}$ and $B' = S \cup \{e'\}$. Then we have that B and B' are independent in $\mathcal{C}$, since $e \notin cl(S) = T$ and $e' \notin cl(S) = T$. It is easy to verify that $B, B' \in \mathcal{B}_1$ and they satisfy the conditions of the result. □

(13.9) Proposition If $V, V' \in \mathcal{V}_1$ are distinct and $B, B' \in \mathcal{B}_1$ satisfying $V = X(B)$, and $|B \setminus B'| = 1$, then there exists a unique line containing both V and V' and

(a) $V_{e'} \neq 0$ for $\{e'\} = B' \setminus B$;

(b) $Z^e(B)_{e'} \neq 0$ and the unique direction, say Z, from V to V' is determined by:

$$Z = \begin{cases} Z^e(B) & \text{if} \quad V_{e'} = -Z^e(B)_{e'} \\ -Z^e(B) & \text{if} \quad V_{e'} = Z^e(B)_{e'}, \end{cases}$$

where $\{e\} = B \setminus B'$ and $Z^j(B)$ is the basic feasible direction for $j \in B$.

Proof Suppose the assumptions are satisfied. Since $|B \cap B'| = r(\mathcal{C})-2$, the flat $\mathcal{C}(B \cap B')$ is the line containing V and V'. The part (a) follows from the fact that $X(B) \neq X(B')$. The first statement of (b), $Z^e(B)_{e'} \neq 0$, is satisfied because $V \neq V'$. Let $\ell = \mathcal{C}(B \cap B')$. Since $\ell \cap A^\infty = \{Z, -Z, \underset{\sim}{0}\}$ and

$Z^e(B) \in \mathcal{L}$. $Z = \pm Z^e(B)$. Remarking that $V' \circ V = V' \circ (-Z)$ by the definition of directions, and that $V'_{e'} = 0$, the second statement follows. □

If $B \in \mathcal{B}_1$, $j \in B$ and $i \in E_1 \setminus B$, and if $Z^j(B)_1 \neq 0$, a pivot at (i,j) in B is said to be <u>nondecreasing</u> if either

(13.10.a) $X(B)_i = 0$;

(13.10.b) $Z^j(B)_f = 0$; or

(13.10.c) $X(B)_i \neq 0$, $Z^j(B)_f \neq 0$ and exactly one or three of $\{X(B)_i, Z^j(B)_f, Z^j(B)_i\}$ are negative,

and <u>increasing</u> if (13.10.c) holds.

(13.11) <u>Proposition</u> Let $B \in \mathcal{B}_1$, $j \in B$ and $i \in E_1 \setminus B$, and let $Z^j(B)_i \neq 0$. Then the following properties hold)

(a) $B' \equiv B \setminus \{j\} \cup \{i\} \in \mathcal{B}_1$;

(b) $Z^i(B')_j \neq 0$;

(c) A pivot at (i,j) in B is nondecreasing iff a pivot at (j,i) in B' is nonincreasing.

<u>Proof</u> The parts (a) and (b) are trivial.

We shall prove (c). Remark that

(1) $Z^j(B)_i = + \Rightarrow X(B)_i = -X(B')_j$ and $Z^j(B) = Z^i(B')$;

(2) $Z^j(B)_i = - \Rightarrow X(B)_i = X(B')_j$ and $Z^j(B) = -Z^i(B')$.

This implies that

(3) $(13.10.a) \iff X(B')_i = 0$;

(4) $(13.10.b) \iff z^i(B')_f = 0$;

(5) $(13.10.c) \iff X(B')_j \neq 0$, $z^i(B')_f \neq 0$ and exactly two of $\{X(B')_i, z^i(B')_f, z^i(B')_j\}$ are negative.

This implies the statement (c). □

The following theorem gives a characterization of non-BOP's using bases.

(13.13) Theorem An OP $(C;g,f)$ is a non-BOP iff there exists a sequence of bases of $(C;g,f)$:

$$B^o, B^1, B^2, \ldots, B^\ell$$

such that

(a) B^j is obtained from B^{j-1} by a nondecreasing pivot for $j = 1,2,\ldots,\ell$; and

(b) for at least one $1 \leq s \leq \ell$. the pivot replacing B^{s-1} by B^s is increasing.

Proof (Necessity): Suppose that there exists a sequence $B^o, B^1, \ldots, B^\ell$ of bases satisfying (a) and (b). Let $V^j = X(B^j)$ for $j = 0,1,\ldots,\ell$. For each j, the pivot replacing B^{j-1} by B^j is nondecreasing, one of (13.10.a), (13.10.b) and (13.10.c) holds. If (13.10.a) is the case. the pivot is degenerate and $V^j = V^{j-1}$. Suppose that (13.10.a) is not the case. Then the pivot is nondegenerate and $V^j \neq V^{j-1}$ (by (6.)). By (13.9), there exists a unique direction, say z^j, from V^{j-1} to V^j, which is determined by

$$(*) \qquad z^j = \begin{cases} z^e(B^{j-1}) & \text{if} \quad V^j_{e'} = -z^e(B^{j-1})_{e'} \\ -z^e(B^{j-1}) & \text{if} \quad V^j_{e'} = z^e(B^{j-1})_{e'} \end{cases}$$

where $\{e\} = B^{j-1} \setminus B^j$ and $\{e'\} = B^j \setminus B^{j-1}$.

We claim that $z^j_f \geq 0$ for each $j = 1,\ldots,\ell$ for which $V^j \neq V^{j-1}$. If (13.10.b) with $j = e$ is the case, then clearly $z^j_f = 0$. Otherwise exactly one or three of $\{V^j_{e'}\ ,\ z^e(B^{j-1})_{e'}\ ,\ z^e(B^{j-1})_f\}$ are negative, and by (*), $z^j_f \geq 0$ follows. By the condition (b), we know that $z^s_f > 0$ for some $1 \leq s \leq \ell$. Therefore, the subsequence $V^{j_o}, V^{j_1},\ldots,V^{j_k}$ of $V^o, V^1,\ldots,V^\ell$, obtained by replacing two or more consequtive identical vertices by the single vertex satisfies (13.3.a) ~ (13.3.e). This completes the proof. □

(Sufficiency): Let $V^o, V^1,\ldots,V^k$ be a sequence of vertices of C satisfying (13.3.a) ~ (13.3.e). It follows from (13.3.b) ~ (13.3.e) and from (13.7) ~ (13.9) that for any $1 \leq j \leq k$, there exist bases $B^{j-1}(j)$ and $B_j(j) \in \mathcal{B}_1$ such that $V^{j-1} = X(B^{j-1}(j))$, $V^j = X(B^j(j))$ and $B^j(j)$ is obtained from B^{j-1} by a nondecreasing pivot, and such that for some $1 \leq s \leq k$ the pivot replacing $B^{s-1}(s)$ by $B^s(s)$ is increasing. It is easy to show that for each $1 \leq j \leq k$ if $B^{j-1}(j-1) \neq B^{j-1}(j)$ then $B^{j-1}(j)$ can be obtained from $B^{j-1}(j-1)$ by a sequence of nondecreasing pivots of type (13.10.a). This implies that there exists a sequence $B^o, B^1, \ldots, B^\ell$ of bases satisfying (a) and (b). This completes the proof. □

C. Duality

The dual of an OP $(C;g,f)$ was defined in Chapter 8 as an OP $(C^*; f,g)$ where C^* is the dual of C . The following theorem states that the duality holds for the class of BOP's.

(13.14) Theorem An OP $(C; g,f)$ is a BOP iff the dual is a BOP.

As an immediate corollary, we get:

(13.15) Corollary An OM C is a BOM iff the dual C^* is a BOM.

For the proof of (13.14), we need some observations. Suppose that an OP $(C; g,f)$ is given, and let $\mathcal{B}_1$ and $\mathcal{D}_1$ be respectively the set of bases of $(C; g,f)$ and the set of bases of the dual OP. We know from (8.) that

$$\mathcal{D}_1 = \{E_1 \setminus B \ : \ B \in \mathcal{B}_1\}$$

where $E_1 = E \setminus \{g,f\}$. For a basis $B \in \mathcal{B}_1$ of the primal and for each $j \in B$, $X(B)$ indicates the basic solution and $Z^j(B)$ the basic feasible direction in $(C; g,f)$. And for each basis $D \in \mathcal{D}_1$ of the dual OP and for each $i \in D$, $Y(D)$ indicates the basic solution and $W^i(D)$ the basic feasible direction in the dual OP. We have

(13.16.a) $X(B) = X(B \cup \{g\} ; g)$

(13.16.b) $Z^j(B) = X(B \cup \{g\} ; j)$

(13.16.c) $Y(D) = Y(D \cup \{f\} ; f)$

(13.16.d) $W^i(D) = Y(D \cup \{f\} ; i)$

for $B \in \mathcal{B}_1$, $D \in \mathcal{D}_1$, $j \in B$, $i \in D$, where $X(B';\ell)$ is the fundamental vertex of $\ell(\in B')$ in a basis B' of C and $Y(D';k)$ is the fundamental vertex of k $(\in D')$ in a basis D' of C^*.
By (8.12),

(13.17) $Y(D \cup \{f\} ; s)_r = - X(B \cup \{g\} ; r)_s$
for each $B \in \mathcal{B}_1$ and $D = E_1 \setminus B$ and for each
$r \in B \cup \{g\}$ and $s \in D \cup \{f\}$.

This together with (13.16.b) and (13.16.d) implies that

(13.18) $W^i(D)_j = -Z^j(B)_i$ for all $B \in \mathcal{B}_1$, $D = E_1 \setminus B$
and for $j \in B$, $i \in D$.

Hence a pivot at (i,j) in a basis $B \in \mathcal{B}_1$ for a primal OP exists iff a pivot at (j,i) in basis $D \equiv E \setminus B \in \mathcal{D}_1$ for the dual OP.

Also it follows from (13.14) and (13.15) that for any $B \in \mathcal{B}_1$, $D = E_1 \setminus B$, $i \in D$, $j \in B$, with $Z^j(B)_j \neq 0$,

(13.19.a) The statement (13.10.a) holds
$\iff W^i(D)_g = 0$

(13.19.b) The statement (13.10.b) holds
$\iff Y(D)_j = 0$

(13.19.c) The statement (13.10.c) holds
$\iff$ exactly two of $\{W^i(D)_g, Y(D)_j, W^i(D)_j\}$
are negative.

Hence we obtain:

(13.20) <u>Lemma</u> Let $B \in \mathcal{B}_1$, $D = E_1 \setminus B$, $i \in D$, $j \in B$ and let $z^j(B)_i \neq 0$. Then a pivot at (i,j) in B for the primal OP is nondecreasing iff a pivot at (j,i) in D for the dual OP is nonincreasing.

<u>Proof of (13, 14)</u> Suppose that $P = (C; g,f)$ is a non-BOP. Then, by (13.13) there exists a sequence of bases $B^o, B^1, \ldots, B^\ell$ of P satisfying (13.13) (a) and (b). Letting $D^{\ell-i} = E_1 \setminus B^i$ for $i = 0,1,\ldots,\ell$. It follows from (13.20) that a pivot replacing D^{j-1} is nondecreasing for each $1 \le j \le \ell$, and for at least one $1 \le s \le \ell$ a pivot replacing D^{s-1} by D^s is increasing. Therefore, again by (13.13), the OP $P^* = (C^*; f,g)$ is a non-BOP. Since $P^{**} = P^*$, the theorem follows. □

D. Cycling of Non-degenerate Simplex Pivots and non-BOP's.

By vertue of the characterization (13.13) of non-BOP's, it is easy to find a closed relationship between the class of BOP's and the class of OP's for which the simplex method can produce a cycle of pivot, at least one of which is non-degenerate.

For this section we assume that $P = (C; g,f)$ is an OP, $\mathcal{B}_1$ is the set of bases of P.

We defined in Chapter 5 that a basis $B \in \mathcal{B}_1$ is feasible if the basic solution $X(B)$ is feasible for P, i.e.

(13.21.a) $X(B)_{E_1} \geq 0$

where $E_1 = E \setminus \{g,f\}$. Recall that for any feasible basis $B \in \mathcal{B}_1$ and for $j \in B$ and $i \in E_1 \setminus B$ with $Z^j(B)_i \neq 0$, a pivot at (i,j) in B is a simplex pivot if

(13.21.b) $Z^j(B)_f > 0$;

(13.21.c) $Z^j(B)_i < 0$; and

(13.21.d) $B' \equiv B \setminus \{j\} \cup \{i\}$ is feasible. And if in addition

(13.21.e) $X(B)_i > 0$

a simplex pivot at (i,j) in B is nondegenerate, otherwise (i.e., $X(B)_i = 0$) degenerate. Comparing (13.10.a) ~ (13.10.c) with (13.21.a) ~ (13.21.e), we can easily see that

(13.22) Every simplex pivot is a nondecreasing pivot and every nondegenerate simplex pivot is a increasing pivot.

This together with (13.13) implies:

(13.23) Every OP for which the simplex method can produce a cycle of pivots, at least one of which is non-degenerate is a non-BOP.

By this property we know that the OP we constructed in Chapter 7 is a non-BOP. And by using the same idea of the construction but using different linear OM's to start with we can easily obtain an infinite class of non-BOP's. Thus,

(13.24) The class of non-BOP's and the class of non-BOM's are infinite.

References

Balinski, M.L.

[CA] Notes on a constructive approach to linear programming, Mathematics of the Decision Sciences, eds. G.B. Dantzig and A.F. Veinott, Jr., Amer. Math. Soc.,

[GC] On the graph structure of convec polyhedra in n-space, Pacific J. Math., 11 (1961), 431-434.

Beale, E.M.L.

[CD] Cycling in the Dual Simplex Aglorithm, Naval Res. Logist. Quart., Vol. 2, No. 4, December, 1955, pp. 269-276.

Birkhoff, G.

[LT] "Lattice Theory", A.M.S. Collog. Publ. (3 rd ed.), Providence, 1967.

Bland, R.G.

[OT] Complementary Orthogonal Subspaces of $\mathbb{R}^n$ and Orientability of Matroids, Ph.D. Dissertation, Cornell University, Ithaca, 1974.

[AL] A Combinatorial abstraction of linear programming J. Combinatorial Theory, Ser. B 23 (1977), 33-57.

[FS] New finite pivot rules for the simplex method, Math, Operations Res.

[LM] Linear Programming duality and Minty's lemma, Technical Report No. 449, School of Operations Res. and Industrial Engl., Cornell Univ. (1980).

__________ and M. Las Vergnas

[OR] Orientability of Matroids, J. Combinatorial Theory, Ser. B 24 (1978) 94-123.

Bruggesser, H. and P. Mani

[SD] Shellable decomposition of cells and spheres, Math. Scand. 29 (1972) 197-205.

Camion, P

[MU] Modules Unimodularies, J. Combinatorial Theory 4 (1968), 301-362.

Cordovil, R.

[QP] "Quelques Properties Algebriques des Matroides", Ph.D. dissertation, L'Univ. Pierre et Marie Curie, Paris, 1981.

____________, M. Las Vergnas and A. Mandel

[ER] Euler's relation, Möbis functions and Matroid identities, Geom. Dedicata, 9 (1980), 243-246.

Crapo, H.H. and G.-C. Rota

[CG] "Combinatorial Geometries:, M.I.T. Press, Cambridge, Mass., 1970.

Dantzig, G.B.

[LP] "Linear Programming and Extensions", Princeton Univ. Press, Princeton, N.J. 1963.

Folkman, J. and J. Lawrence

[OM] Oriented Matroids, J. Combinatorial Theory, Ser. B 25 (1978), 199-236.

Fulkerson, D.R.

[NT] Networks, frames, blocking systems, in Mathematics of the Decision Sciences, eds. G.B. Dantzig and A.F. Veinott, Jr., Amer. Math. Soc. (1968), 303-334.

Grunbaum, B.

[CP] "Convex Polytopies", John Wiley and Sons, London-New York-Sydney, 1967.

[AS] "Arrangements and Spreads", Amer. Math. Soc., Providence, R.I., 1972.

Las Vergnas, M.

[MO] Matroides orientables, C.R. Acad. Sc. Paris, t.280 (1975), Serie A, 61-64.

[CV] Convexity in oriented matroids, J. Combinatoria Theory, Ser. B 29 (1980) 231-243.

Lawrence, J.

[OM] "Oriented Matroids", Ph.D. Dissertation, University of Washington, Seattle, 1975.

Lindstrom, B.

[RC] On the realization of Convex Polytopes, Euler's Formula, Möbius Functions, Aeg. Math. 6 (1971) 235-240.

Mandel, A.

[TO] "The Topology of Oriented Matroids", Ph.D Dissertation, University of Waterloo, Waterloo,

McMullen, P. and G.C. Shephard

[UB] "Convex Polytopies and the Upper Bound Conjecture", Cambridge Univ. Press, London, 197

Minty, G.J.

[AF] On the axiomatic foundations of the theories of directed linear graphs, electrical networks, and network programming, J.Math.Mech. 15 (1966), 485-520.

Munson, B.S.

[FL] "Face Lattices of Oriented Matroids", Ph.D. Dissertation, Cornell University, Ithaca, 1981.

Rockafellar, R.T.

[EV] The elementary vectors of a subspace of $\mathbb{R}^n$, in "Combinatorial Mathematics and Its Applications", Proc. of the Chapel Hill Conf. (R.C. Bose and T.A. Dowling, Eds.), pp.104-127, University of North Carolina Press, Chapel Hill, 1969.

Tucker, A.W.

[CL] Combinatorial theory underlying linear programms, Recent Advances in Mathematical Programming, (R.E. Graves and P. Wolfe, edcs.), McGraw-Hill, New York, (1963) 1-16.

Welsh, D.J.A.

[MT] "Matroid Theory", Academic Press, London, 1976.

Whitney, H.

[AP] On the abstract properties of linear dependence, Amer. J. Math. 57 (1935), 507-553.

Zaslavsky, T.

[FA] "Facing up to Arrangements: Face Count Formulas for Partitioning of Space by Hyperplanes", Memoirs Amer. Math. Soc. 154, 1975.