not observable.

⟹ If a multiple eigenvalue leads to a multiple eigenvector, that mode belongs to sub=system S_4 (neither controllable nor observab

<u>Analysis in the frequency domain</u>

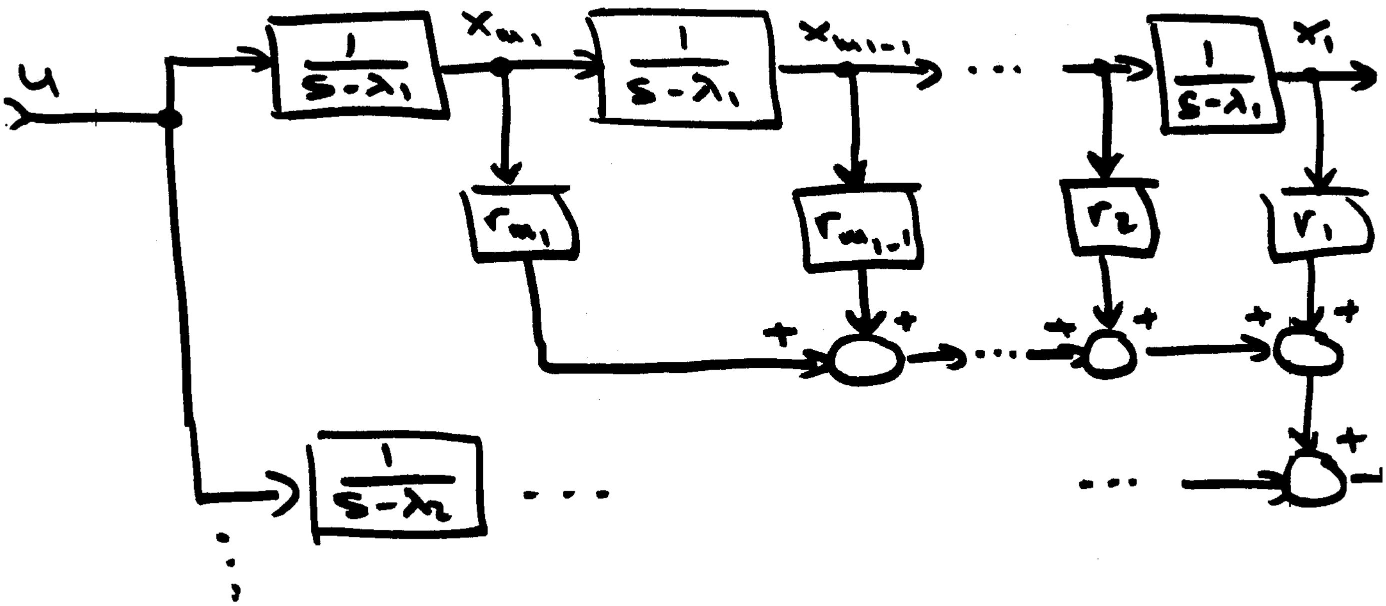
A multiple pole that is controllable and observable may not cancel with a zero:

$$G(s) = \frac{P(s)}{(s-\lambda_1)^{m_1}(s-\lambda_2)\ldots}$$

$P(s)$ has <u>no</u> root at λ_1 ⟺ the system has a pole at λ_1 with $m_1 = k$ (multiplici

We apply Partial Fraction Expansion:

$$G(s) = \frac{r_{m_1}}{s-\lambda_1} + \frac{r_{m_1-1}}{(s-\lambda_1)^2} + \cdots + \frac{r_1}{(s-\lambda_1)^{m_1}}$$



$$\Rightarrow \left| \begin{aligned} \dot{x}_1 &= \lambda_1 x_1 + x_2 \\ \dot{x}_2 &= \lambda_1 x_2 + x_3 \\ &\vdots \\ \dot{x}_{m_1-1} &= \lambda_1 x_{m_1-1} + x_{m_1} \\ \dot{x}_{m_1} &= \lambda_1 x_{m_1} + u \\ &\vdots \end{aligned} \right|$$

$$A = \left[\begin{array}{cccc|ccc} \lambda_1 & 1 & & & & & \\ & \lambda_1 & 1 & & & \emptyset & \\ & & \ddots & 1 & & & \\ & & & \lambda_1 & & & \\ \hline & & & & \lambda_2 & & \\ & \emptyset & & & & \ddots & \\ & & & & & & \lambda_n \end{array}\right] ; \quad \underline{b} = \left[\begin{array}{c} \emptyset \\ \vdots \\ \emptyset \\ 1 \\ \hline 1 \\ \vdots \\ 1 \end{array}\right]$$

$\Rightarrow$ The Jordan-form of a system with multiple poles takes the form:

$$A = \begin{bmatrix} \Lambda_1 & & & \emptyset \\ & \Lambda_2 & & \\ & & \ddots & \\ \emptyset & & & \Lambda_k \end{bmatrix} ; \quad \underline{b} = \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \\ \underline{b}_k \end{bmatrix}$$

where: $\Lambda_i = \begin{bmatrix} \lambda_i & 1 & & \emptyset \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ \emptyset & & & \lambda_i \end{bmatrix} ; \quad \underline{b}_i = \begin{bmatrix} \emptyset \\ \vdots \\ \emptyset \\ 1 \end{bmatrix}$

- The Jordan-form is block-diagonal, and the Λ_i are called the Jordan blocks.

- A system that is completely controllable and observable (no pole/zero cancellation) has exactly one Jordan block associated with each eigenvalue.

- For a system to be totally controllable and observable, each eigenvalue can have and must have exactly one eigenvector associated with it.

- If the nullity of the matrix $(\lambda_i I - A)$ is > 1 for <u>any</u> $\lambda_i \Rightarrow$ there is at least one uncontrollable and unobservable mode in the system.

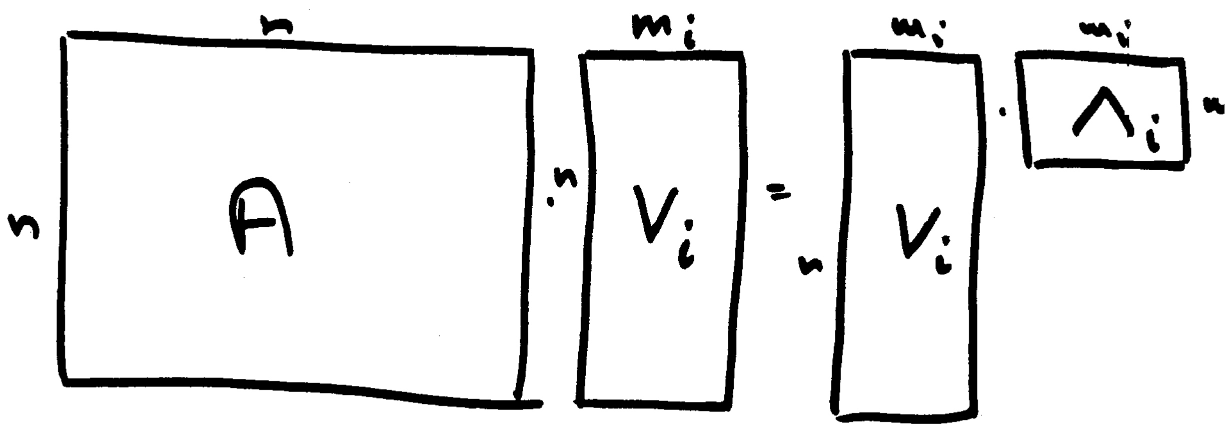
Question: Which similarity transformation will get us into Jordan-canonical form?

- From: $AV = V\Lambda$

  $\Longleftrightarrow \quad A\underline{v}_i = \underline{v}_i \lambda_i$,

  we realize that, due to the complete input/output decoupli[ng] we need to look at one Jordan block at a time only:

$$\underline{\underline{A V_i = V_i \cdot \Lambda_i}}$$



will get us into the desired form. However, we don't know yet what V_i is.

$$A \begin{bmatrix} \vdots & & \vdots \\ \underline{v}_{i_1} & \cdots & \underline{v}_{i_{m_i}} \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & & \vdots \\ \underline{v}_{i_1} & \cdots & \underline{v}_{i_{m_i}} \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \lambda_i & 1 \\ \emptyset & & & \lambda_i \end{bmatrix}$$

$$\Rightarrow \left| \begin{array}{l} A\underline{v}_{i_1} = \lambda_i \underline{v}_{i_1} \\ A\underline{v}_{i_2} = \lambda_i \underline{v}_{i_2} + \underline{v}_{i_1} \\ \quad \vdots \\ A \cdot \underline{v}_{i_{m_i}} = \lambda_i \underline{v}_{i_{m_i}} + \underline{v}_{i_{m_i-1}} \end{array} \right|$$

$$\Rightarrow \left| \begin{array}{l} (A - \lambda_i I)\underline{v}_{i_1} = \emptyset \\ (A - \lambda_i I)\underline{v}_{i_2} = \underline{v}_{i_1} \\ \quad \vdots \\ (A - \lambda_i I)\underline{v}_{i_{m_i}} = \underline{v}_{i_{m_i-1}} \end{array} \right|$$

$\Rightarrow$ $\underline{v}_{i_1}$ is an eigenvector associated with λ_i.

The other vectors are so-called <u>generalized eigenvectors</u>.

- To simplify, we can multiply the second equation with $(A-\lambda_i I)$:

$$(A-\lambda_i I)^2 \underline{v}_{i_2} = (A-\lambda_i I)\underline{v}_{i_1} = \phi$$

- The next equation is multiplied with $(A-\lambda_i I)^2$:

$$(A-\lambda_i I)^3 \underline{v}_{i_3} = (A-\lambda_i I)^2 \underline{v}_{i_2} = \phi$$

<u>etc</u>.

- Thus, the set of equations can also be written as:

$$\left|\begin{array}{c} (A-\lambda_i I)\underline{v}_{i_1} = \phi \\ (A-\lambda_i I)^2 \underline{v}_{i_2} = \phi \\ \vdots \\ (A-\lambda_i I)^{m_i} \underline{v}_{i_{m_i}} = \phi \end{array}\right|$$

Of course:

$$\left| \begin{array}{l} (A-\lambda_i I)^{m_i} \, \underline{v}_{i_1} = \emptyset \\ (A-\lambda_i I)^{m_i} \, \underline{v}_{i_2} = \emptyset \\ \quad \vdots \\ (A-\lambda_i I)^{m_i} \, \underline{v}_{i_{m_i}} = \emptyset \end{array} \right|$$

is also correct.

- As a transformation into Jordan form must exist

$$\Longrightarrow \boxed{\text{Nullity}\left\{ (A-\lambda_i I)^{m_i} \right\} \equiv m_i}$$

- Among all these, we are interested to find the one generalized eigenvector of grade m_i, which satisfies the conditions:

$$\left| \begin{array}{l} (A-\lambda_i I)^{m_i} \, \underline{v}_{i_{m_i}} = \emptyset \\ (A-\lambda_i I)^{m_i-1} \cdot \underline{v}_{i_{m_i}} \neq \emptyset \end{array} \right|$$

Once, this generalized eigenvector is found, the <u>chain</u> of related eigenvectors of lower grade can be computed immediately:

$$
\left|\begin{aligned}
\underline{v}_{i_{m_i-1}} &= (A-\lambda_i I)\cdot \underline{v}_{i_{m_i}} \\
\underline{v}_{i_{m_i-2}} &= (A-\lambda_i I)\cdot \underline{v}_{i_{m_i-1}} \\
&\ \vdots \\
\underline{v}_{i_1} &= (A-\lambda_i I)\cdot \underline{v}_{i_2}
\end{aligned}\right|
$$

The general algorithm will be demonstrated at hand of an example:

$$
\Lambda = \left[\begin{array}{cccc|cc|cc|c|c}
\lambda_1 & 1 & \phi & \phi & & & & & & \\
\phi & \lambda_1 & 1 & \phi & & & & & & \\
\phi & \phi & \lambda_1 & 1 & & & & \Phi & & \\
\phi & \phi & \phi & \lambda_1 & \phi & & & & & \\
\hline
 & & & & \lambda_1 & 1 & & & & \\
 & & & & \phi & \lambda_1 & \phi & & & \\
\hline
 & & & & & & \lambda_1 & 1 & & \\
 & \Phi & & & & & \phi & \lambda_1 & \phi & \\
\hline
 & & & & & & & & \lambda_2 & \phi \\
\hline
 & & & & & & & & & \lambda_3
\end{array}\right]
$$

$\Rightarrow$ There are three Jordan blocks associated with λ_1:

$$\Rightarrow \Lambda = \begin{bmatrix} \Lambda_1^{(4)} & & & & \\ & \Lambda_1^{(2)} & & \emptyset & \\ \emptyset & & \Lambda_1^{(2)} & & \\ & & & \lambda_2 & \\ & & & & \lambda_3 \end{bmatrix}$$

$\Rightarrow$ There are uncontrollable / unobservable modes.

$$\Rightarrow \qquad \gamma_1 = \text{Nullity}\{(A-\lambda_1 I)\} = 3$$

$\Leftrightarrow$ There exist three eigenvectors for λ_1 $\Leftrightarrow$ there exist three Jordan blocks for λ_1.

- There must exist one generalized eigenvector of grade 4 leading to a chain: $4 \to 3 \to 2 \to 1$. There exist two more generalized eigenvectors of grade 2, leading each to a chain $2 \to 1$.

$\Rightarrow \quad n = 1\emptyset \quad ; \quad m_1 = 8$

Algorithm:

- We start by computing the nullities of $(A - \lambda_i I)^k$ for increasing k until $\underline{\nu\{(A-\lambda_i I)^k\}}$

Example:

$$\rho\{(A-\lambda_1 I)^{\emptyset}\} = 1\emptyset \quad ; \quad \nu\{(A-\lambda_1 I)^{0}\} = $$

$$\rho\{(A-\lambda_1 I)^{1}\} = 7 \quad ; \quad \nu\{(A-\lambda_1 I)^{1}\} = 3$$

$$\rho\{(A-\lambda_1 I)^{2}\} = 4 \quad ; \quad \nu\{(A-\lambda_1 I)^{2}\} = 6$$

$$\rho\{(A-\lambda_1 I)^{3}\} = 3 \quad ; \quad \nu\{(A-\lambda_1 I)^{3}\} = 7$$

$$\rho\{(A-\lambda_1 I)^{4}\} = 2 \quad ; \quad \underline{\underline{\nu\{(A-\lambda_1 I)^{4}\} = 8}}$$

$$\Rightarrow \underline{\underline{k = 4}}$$

Abbreviations: $\rho\{(A-\lambda_i I)^k\} \equiv \rho_i^{(k)}$

$\nu\{(A-\lambda_i I)^k\} \equiv \nu_i^{(k)}$

Obviously, the following rules apply always:

(1) $\varsigma_i^{(\phi)} \equiv n \; ; \; \nu_i^{(\phi)} \equiv \phi \; ; \; \forall i$

(2) $\varsigma_i^{(j)} + \nu_i^{(j)} \equiv n \; ; \; \forall i, j$

(3) $\nu_i^{(k)} \equiv m_i$

- After we have determined k, we look for:

$$\left| \begin{array}{l} (A - \lambda_1 I)^4 \underline{v}_4 = \phi \\ (A - \lambda_1 I)^3 \underline{v}_4 \neq \phi \end{array} \right|$$

$$\Longrightarrow \underline{\underline{v_4}}$$

Then:
$$\left| \begin{array}{l} \underline{v}_3 = (A - \lambda_1 I) \underline{v}_4 \\ \underline{v}_2 = (A - \lambda_1 I) \underline{v}_3 \\ \underline{v}_1 = (A - \lambda_1 I) \underline{v}_2 \end{array} \right|$$

As $\nu_1^{(4)} - \nu_1^{(3)} = 1 \Rightarrow$ there exists exactly <u>one</u> generalized eigenvector of grade 4 ($\underline{v}_4$)

As $\nu_1^{(3)} - \nu_1^{(2)} = 1 \Rightarrow$ there exists exactly one generalized eigenvector of grade 3 ($\underline{v}_3$) which has already been found.

As $\nu_1^{(2)} - \nu_1^{(1)} = 3 \Rightarrow$ there exist two more generalized eigenvectors of grade 2 beside from $\underline{v}_2$:

$$\left| \begin{array}{l} (A-\lambda_1 I)^2 \underline{v}_6 = \phi \\ (A-\lambda_1 I)^1 \underline{v}_6 \neq \phi \end{array} \right|$$

and: $\underline{v}_6$ lin. indep. from $\underline{v}_2$

$$\Rightarrow \underline{\underline{v_6}}$$

$$\Rightarrow \left| \underline{v}_5 = (A-\lambda_1 I)\underline{v}_6 \right|$$

Then: $\left| \begin{array}{l} (A-\lambda_1 I)^2 \underline{v}_8 = \phi \\ (A-\lambda_1 I)^1 \underline{v}_8 \neq \phi \end{array} \right|$

and: $\underline{v}_8$ lin. indep. from $\underline{v}_2, \underline{v}_6$

$\Rightarrow$ $\underline{\underline{v_8}}$ ✓

$$\Rightarrow \quad \boxed{\underline{v}_7 = (A - \lambda_1 I)\,\underline{v}_8}$$

Of course:

$$\left| \begin{array}{l} (A - \lambda_2 I)\,\underline{v}_9 = \phi \\ (A - \lambda_3 I)\,\underline{v}_{10} = \phi \end{array} \right|$$

as usual.

$$\Rightarrow V = [\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_{10}]$$

is the generalized right modal matrix, and

$$T = V^{-1}$$

will get us into Jordan-canonical form.

Warning:

[V, Lambda] = eig(A)

will not give you a

generalized modal matrix in matlab !!!

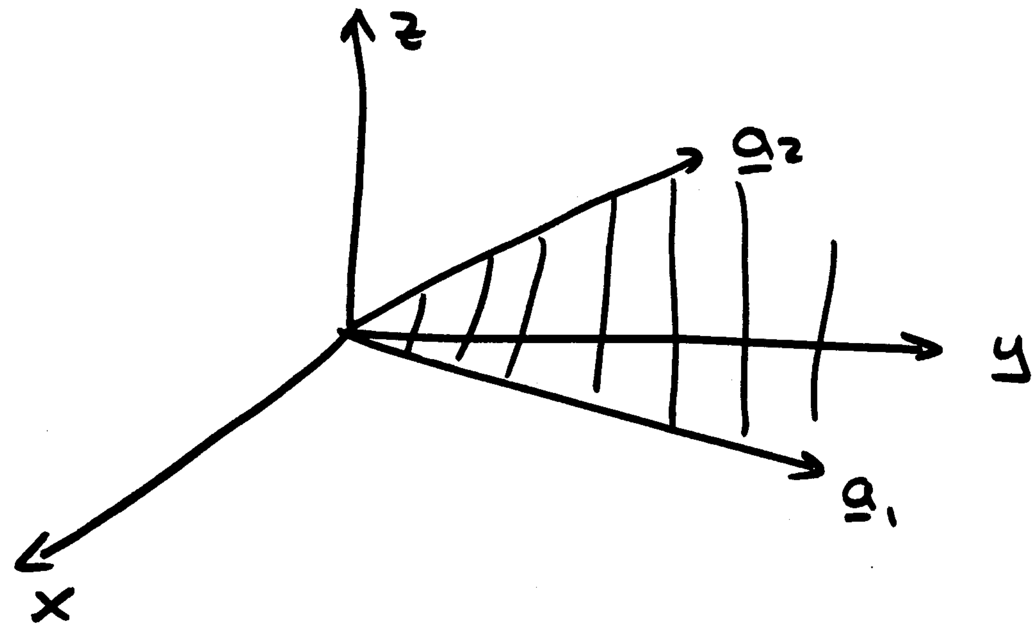
- We have not yet discussed efficient ways to find eigenvectors and generalized eigenvectors.

Projections (Images & Nullspaces)

Example:

$$A = \begin{bmatrix} 1 & \phi & 2 \\ 2 & 7 & 11 \\ -5 & 8 & -2 \end{bmatrix} = [\underline{a}_1, \underline{a}_2, \underline{a}_3]$$

$\det(A) = \phi \implies \text{Rank}(A) < 3$
but $\underline{a}_1$ is linearly independent of $\underline{a}_2 \implies \underline{\text{Rank}(A) = 2}$.



$\Rightarrow$ $\underline{a}_3$ must lie in the plane that is spanned by $\underline{a}_1$ and $\underline{a}_2$.

$\Leftrightarrow$ There exist values α_1 and α_2 such that:

$$\underline{a}_3 = \alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2$$

In our example:

$$\alpha_1 = 2 \;;\; \alpha_2 = 1$$

as:

$$\begin{bmatrix} 2 \\ 11 \\ -2 \end{bmatrix} \equiv 2 \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} + 1 \begin{bmatrix} \phi \\ 7 \\ 8 \end{bmatrix}$$

$\Longleftrightarrow$ $A = [\underline{a}_1, \underline{a}_2, \underline{a}_3]$ does not qualify for a base spanning the threedimensional space, that is: there are points in the threedimensional space that cannot be reached through a linear combination of $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$.

Def: A Nullspace of a matrix is a base of unity vectors that are all perpendicular to each other and that are perpendicular to the subspace spanned by all vectors in A. The number of such vectors is the Nullity of the matrix A.

In our example:

$$\varrho(A) = 2 \Rightarrow \underline{\underline{\nu(A) = 1}}$$

$\Rightarrow$ There exists one such vector.

$$\Rightarrow \left| \begin{array}{l} \underline{n}_1' \cdot \underline{a}_1 = \emptyset \\ \underline{n}_1' \cdot \underline{a}_2 = \emptyset \\ |\underline{n}_1| = 1 \end{array} \right|$$

$$\left| \begin{array}{c} n_{11} + 2n_{12} - 5n_{13} = \emptyset \\ 7n_{12} + 8n_{13} = \emptyset \end{array} \right|$$

We choose: $n_{12} = 8$; $n_{13} = -7$

$$\Rightarrow n_{11} = -51$$

Normalization:

$$\sqrt{n_{11}^2 + n_{12}^2 + n_{13}^2} = \sqrt{2714}$$
$$= 52.\emptyset961$$

$$\Rightarrow n_{11} = \frac{-51}{52.0961} = -0.979$$

$$n_{12} = 0.1536$$

$$n_{13} = -0.1344$$

$$\Rightarrow \quad \underline{n}_1 = N(A) = \begin{bmatrix} -0.979 \\ +0.1536 \\ -0.1344 \end{bmatrix}$$

Def: The Image of A is a set of unity vectors that are perpendicular to each other and span the same subspace as the vectors in A.

$\Longrightarrow$ The Image of A is the Nullspace of the Nullspace of A:

$$\boxed{I(A) = N(N(A))}$$

Example:

e.g. $\left| \begin{array}{l} \underline{i}_1 = \alpha \cdot \underline{a}_1 \\ |\underline{i}_1| = 1 \end{array} \right|$

$$\Rightarrow \sqrt{a_{11}^2 + a_{12}^2 + a_{13}^2} = \sqrt{30} = 5.4772$$

$$\Rightarrow i_{11} = \frac{a_{11}}{5.4772} = 0.1826$$

$$i_{12} = 0.3651$$

$$i_{13} = -0.9129$$

$$\Rightarrow \underline{i}_1 = \begin{bmatrix} 0.1826 \\ 0.3651 \\ -0.9129 \end{bmatrix}$$

$$\left| \begin{array}{l} \underline{i}_2' \cdot \underline{i}_1 = 0 \\ \underline{i}_2' \cdot \underline{n}_1 = 0 \\ |\underline{i}_2| = 1 \end{array} \right|$$

$$\cdots\!> \quad \underline{i}_2 = \begin{bmatrix} 0.0911 \\ 0.9182 \\ 0.3855 \end{bmatrix}$$

$$\Rightarrow I(A) = \begin{bmatrix} 0.1826 & 0.0911 \\ 0.3651 & 0.9182 \\ -0.9129 & 0.3855 \end{bmatrix}$$

is the Image of A.

- Notice the following properties of Images and Nullspaces:

(1) $\rho(I(A)) \equiv \rho(A)$

(2) $\rho(N(A)) \equiv \nu(A)$

(3) $A' \cdot N(A) = N(A)' \cdot A = 0$

(4) $I(A)' \cdot N(A) = N(A)' \cdot I(A) = 0$

(5) $\| I(A) \|_2 = 1$

(6) $\| N(A) \|_2 = 1$

- Of course, $I(A)$ and $N(A)$ are not unique.

- The matrix:

$$Q = \left[I(A) , N(A) \right]$$

is a square matrix of the same dimension as A where each column vector is of length 1, thus:

$$\|Q\|_2 = 1$$

and where each vector is perpendicular to each other vector. Such a matrix is called. <u>orthonormal</u> or <u>unitary</u>. Unitary matrices play an important role in the numerical linear algebra due to their benign behavior of error propagation.

Def: A <u>unitary transformation</u> is a similarity transformation with T being a unitary matrix.