- Other applications of these two canonical forms will be shown later.

Relation between Time-domain representation, Frequency-domain representation, and the solution space.

Example:

$$J\dot{\omega} + H_r \omega = T(t) \quad ; \quad \omega(t=\emptyset) = \omega_0$$

$$T(t) = \begin{cases} \sin(2t) \; ; \; t \geq \\ \emptyset \quad ; \; t < \end{cases}$$

$$\Rightarrow \omega(t) = \omega_p(t) + C_0 \cdot \omega_h(t)$$

$\omega_p(t)$:. a particular solution of the inhomogeneous problem with arbitrary initial condition (I.C.)

$\omega_h(t)$:. general solution of the homogeneous problem

C_0 :. constant to satisfy I.C.

$$\Rightarrow \omega_h(t) = C_0 e^{\lambda t}$$

where: λ is the solution of the characteristic equation:

$$J\lambda + H_r = 0 \Rightarrow \lambda = -\frac{H_r}{J}$$

$$\Rightarrow \omega_h(t) = C_0 e^{(-H_r/J)\cdot t}$$

satisfies the homogeneous equation:

$$J\dot{\omega}_h + H_r \omega_h = 0$$

for any value of C_0.

$\omega_p(t)$: We try:

$$\omega_p(t) = C_1 \sin(2t) + C_2 \cdot \cos(2t)$$

$$\Rightarrow \dot{\omega}_p(t) = 2C_1 \cdot \cos(2t) - 2C_2 \cdot \sin(2t)$$

plug into: $J\dot{\omega}_p(t) + H_r \omega_p(t) = T(t)$

$$\Rightarrow J\left(-2C_2 \sin(2t) + 2C_1 \cos(2t)\right) + H_r\left(C_1 \sin(2t)\right.$$
$$\left. C_2 \cos(2t)\right) = \sin(2t) \quad ; \forall t \geq 0$$

$\Rightarrow$

$$\left|\begin{array}{l} -2J C_2 + H_r C_1 = 1 \\ 2J C_1 + H_r C_2 = \emptyset \end{array}\right|$$

$$\Rightarrow \left|\begin{array}{l} C_1 = -\dfrac{\lambda}{J(\lambda^2+4)} \\ C_2 = -\dfrac{2}{J(\lambda^2+4)} \end{array}\right|$$

I.C.:

$$w(t) = C_0 e^{\lambda t} + C_1 \sin(2t) + C_2 \cos(2t)$$

$$w(t=\emptyset) = w_0 = C_0 + C_2$$

$$\Rightarrow C_0 = w_0 - C_2$$

$$\Rightarrow w(t) = \left[w_0 + \frac{2}{J(\lambda^2+4)}\right] e^{-\frac{H_r}{J} t} - \frac{\lambda}{J(\lambda^2+4)} \sin(2t$$

$$- \frac{2}{J(\lambda^2+4)} \cos(2t)$$

Generalization:

$T(t)$ is unknown.

$$\Rightarrow \omega_h(t) \neq f(T(t)) = C_0 e^{\lambda t}$$

as before

$\omega_p(t)$: We try:

$$\omega_p(t) = \gamma(t)\, e^{\lambda t}$$

$\gamma(t)$ unknown, to be found.

$$\Rightarrow \dot{\omega}_p(t) = \dot{\gamma}(t)\, e^{\lambda t} + \gamma(t)\lambda e^{\lambda t}$$

$$\Rightarrow J\left(\dot{\gamma}(t) e^{\lambda t} + \gamma(t)\lambda e^{\lambda t}\right) + H_r\left(\gamma(t) e^{\lambda t}\right) =$$

$$\Rightarrow J\dot{\gamma}(t) e^{\lambda t} + [\underbrace{J\lambda + H_r}_{\equiv \phi}]\gamma(t) e^{\lambda t} = T(t)$$

as $\lambda = -\frac{H_r}{J}$

$$\Rightarrow \dot{\gamma}(t) = \frac{1}{J} e^{-\lambda t} \cdot T(t)$$

$$\Rightarrow \gamma(t) = \int_{\phi}^{t} e^{-\lambda \tau} \cdot \left(\frac{1}{J}\right) \cdot T(\tau)\, d\tau$$

$$\Rightarrow \omega_p(t) = \int_0^t e^{\lambda(t-\tau)} \cdot \left(\frac{1}{J}\right) T(\tau)\, d\tau$$

$$\Rightarrow \omega(t) = C_0 e^{\lambda t} + \int_0^t e^{\lambda(t-\tau)} \left(\frac{1}{J}\right) T(\tau)\, d$$

$$\omega(t=0) = \omega_0 = C_0$$

$$\Rightarrow \omega(t) = \underbrace{\omega_0 e^{\lambda t}}_{\substack{= f(\text{I.C.}) \\ (\text{eigensolution})}} + \underbrace{\int_0^t e^{\lambda(t-\tau)} \cdot \left(\frac{1}{J}\right) T(\tau)\, d\tau}_{\substack{f(\text{input}) \\ (\text{forced solution})}}$$

Generalization:

$$\dot{\underline{x}}(t) = A \cdot \underline{x}(t) + \underline{b}\, u(t) \;;\; \underline{x}(0) =$$

$$\Rightarrow \underline{x}(t) = \underline{x}_R(t) + \underline{x}_p(t)$$

Homogeneous Problem:

$$\underline{\dot{x}} = A\underline{x}$$

scalar case: $\dot{x} = ax$

$$\Rightarrow x(t) = C_0 e^{at}$$

$$= C_0 \left[1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \cdots \right.$$

matrix case:

We try:

$$\underline{x}(t) = \left[I^{(n)} + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \right] \cdot$$

$$\Rightarrow \underline{\dot{x}}(t) = \left[\phi^{(n)} + A + A^2 \cdot \frac{2t}{2!} + A^3 \cdot \frac{3t^2}{3!} + \cdots \right.$$

$$= \left[A + A^2t + A^3\frac{t^2}{2!} + A^4\frac{t^3}{3!} + \cdots \right]$$

$$\equiv A \left[I^{(n)} + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \right] \cdot \underline{c}$$

$$= A\underline{x}(t) \quad \checkmark$$

We define:

$$e^{At} := I^{(n)} + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$\Rightarrow \quad \underline{x}_h(t) = e^{At} \cdot \underline{c}_0$$

<u>Warning</u>: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$e^{A} \neq \begin{bmatrix} e^1 & e^2 \\ e^3 & e^4 \end{bmatrix} \quad !$$

We have not yet discussed ho to practically compute the exponential of a matrix.

<u>particular solution</u>:

We try: $\underline{x}_p(t) = e^{At} \cdot \underline{\gamma}(t)$

$$\Rightarrow \dot{\underline{x}}_p(t) = Ae^{At}\underline{\gamma}(t) + e^{At}\dot{\underline{\gamma}}(t)$$

$$\Rightarrow \cancel{Ae^{At}\underline{\gamma}(t)} + e^{At}\dot{\underline{\gamma}}(t) = \cancel{A\left(e^{At}\underline{\gamma}(t)\right)} + \underline{b}u(t)$$

$$\Rightarrow \dot{\underline{\gamma}}(t) = e^{-At}\,\underline{b}\,u(t)$$

$$\Rightarrow \underline{\gamma}(t) = \int_0^t e^{-A\tau}\,\underline{b}\,u(\tau)\,d\tau$$

$$\Rightarrow \underline{x}_p(t) = \int_0^t e^{A(t-\tau)} \underline{b}\, u(\tau)\, d\tau$$

$$\Rightarrow \underline{x}(t) = e^{At}\underline{c} + \int_0^t e^{A(t-\tau)} \underline{b}\, u(\tau)\, d\tau$$

$$\underline{x}(t=0) = \underline{x}_0 = \underline{c}_0$$

$$\Rightarrow \underline{x}(t) = \underbrace{e^{At}\underline{x}_0}_{= f(\text{I.C.})} + \underbrace{\int_{0^-}^t e^{A(t-\tau)} \underline{b}\, u(\tau)\, d}_{= \text{function (inpu}}$$

convolution integral

$$\boxed{y = \underline{c}'\underline{x} + du}$$

$$\Rightarrow y(t) = \underline{c}' e^{At}\underline{x}_0 + \underline{c}'\int_0^t e^{A(t-\tau)} \underline{b}\, u(\tau)\, d$$
$$+ d\, u(t)$$

Let us compare this solution with the one obtained through Laplace Transform:

$$\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + \underline{b}u \\ y = \underline{c}'\underline{x} + du \end{array} \right| \qquad \underline{x}(t=\emptyset) = \underline{x}_0$$

$$\left| \begin{array}{l} s\underline{X}(s) - \underline{x}_0 = A\underline{X}(s) + \underline{b}U(s) \\ Y(s) = \underline{c}'\underline{X}(s) + dU(s) \end{array} \right|$$

$$\Rightarrow [sI - A]\underline{X}(s) = \underline{x}_0 + \underline{b}U(s)$$

$$\Rightarrow \underline{X}(s) = [sI-A]^{-1}\underline{x}_0 + [sI-A]^{-1}\underline{b}U(s)$$

$$\Rightarrow Y(s) = \underline{c}'[sI-A]^{-1}\underline{x}_0 + \underline{c}'[sI-A]^{-1}\underline{b}U(s) + dU(s)$$

$$\Rightarrow e^{At} = \mathcal{L}^{-1}\{[sI-A]^{-1}\}$$

is one technique to compute e^{At}.

Example:

$$A = \begin{bmatrix} -1 & \emptyset \\ +2 & -3 \end{bmatrix}$$

$$\Rightarrow [sI - A] = \begin{bmatrix} (s+1) & \emptyset \\ -2 & (s+3) \end{bmatrix}$$

$$\Rightarrow [sI-A]^{-1} = \frac{1}{(s+1)(s+3)} \begin{bmatrix} (s+3) & \emptyset \\ 2 & (s+1) \end{bmatrix}$$

$$\equiv \begin{bmatrix} \frac{1}{s+1} & \emptyset \\ \frac{2}{(s+1)(s+3)} & \frac{1}{s+3} \end{bmatrix} \equiv \begin{bmatrix} \frac{1}{s+1} & \emptyset \\ \left(\frac{1}{s+1} - \frac{1}{s+3}\right) & \frac{1}{s+3} \end{bmatrix}$$

partial fraction expansion

$$\Rightarrow e^{At} = \mathcal{L}^{-1}\left\{[sI-A]^{-1}\right\} = \begin{bmatrix} e^{-t} & \emptyset \\ \left(e^{-t} - e^{-3t}\right) & e \end{bmatrix}$$

The Markov Parameters:

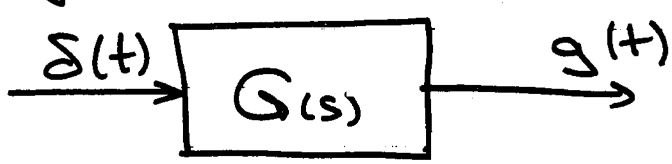
Definition: Develop the Transfer Function into a MacLaurin Series

$$\boxed{G(s) = \sum_{i=1}^{\infty} \beta_i s^{-i} + d}$$

The resulting coefficients β_i are called the Markov Parameters of the system.

Relation to the Impulse Response:

$$g(t) = \mathcal{L}^{-1}\{G(s)\}$$



$\delta(t)$: Dirac Impulse

$$\Rightarrow g(t) = \mathcal{L}^{-1}\{\underline{c}'(sI-A)^{-1}\underline{b} + d\}$$

$$= \underline{c}'\mathcal{L}^{-1}\{(sI-A)^{-1}\}\underline{b} + d\cdot\delta(t)$$

$$\Rightarrow \underline{\underline{g(t) = \underline{c}' e^{At} \underline{b} + d \cdot \delta(t)}}$$

$$\Rightarrow g(t) = \underline{c}' \left[I + At + A^2 \frac{t^2}{2!} + \cdots \right] \underline{b} + d \cdot \delta(t$$

$$= \underline{c}' \mathcal{L}^{-1} \left\{ I\left(\tfrac{1}{s}\right) + A\left(\tfrac{1}{s^2}\right) + A^2\left(\tfrac{1}{s^3}\right) + \cdots \right\} \underline{b} + d \cdot \delta$$

$$\Rightarrow g(t) = \mathcal{L}^{-1} \left\{ \underline{c}' \left[I\left(\tfrac{1}{s}\right) + A\left(\tfrac{1}{s^2}\right) + A^2\left(\tfrac{1}{s^3}\right) + \cdots \right] \underline{b} + d \right.$$

$$= \mathcal{L}^{-1} \{ G(s) \}$$

$$\Rightarrow G(s) = \underline{c}' \left[I \cdot s^{-1} + A s^{-2} + A^2 s^{-3} + \cdots \right] \underline{b} + d$$

$$= \sum_{i=1}^{\infty} (\underline{c}' A^{i-1} \underline{b}) s^{-i} + d$$

$$= \sum_{i=1}^{\infty} \beta_i s^{-i} + d$$

$$\Longrightarrow \boxed{\beta_i = \underline{c}' A^{i-1} \underline{b}}$$

for $\underline{\underline{\text{any}}}$ state-space representation.

$$g(t) = \underline{c}' e^{At} \underline{b} + d \cdot \delta(t)$$

$$\Rightarrow \underline{\underline{g(t=0^{+})}} = \underline{c}' \underline{b} = \underline{\underline{\beta_1}}$$

$$\dot{g}(t) = \underline{c}' A e^{At} \underline{b} + d\,\dot{\delta}(t)$$

$$\Rightarrow \underline{\underline{\dot{g}(t=0^{+})}} = \underline{c}' A \underline{b} = \underline{\underline{\beta_2}}$$

$$\ddot{g}(t) = \underline{c}' A^2 e^{At} \underline{b} + d \cdot \ddot{\delta}(t)$$

$$\Rightarrow \underline{\underline{\ddot{g}(t=0^{+})}} = \underline{c}' A^2 \underline{b} = \underline{\underline{\beta_3}}$$

<u>etc.</u>

The Markov Parameters of a system are directly related to the initial condition of the impulse response and its derivative (the Nordsieck vector):

$$\underline{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} = \mathcal{N}\left\{ g(t=0^{+}) \right\}$$

Relation to Transfer Function:

Question: What is the relation between the Markov Parameters and the coefficients of the transfer function?

$$G(s) = \frac{P(s)}{Q(s)} = \frac{b_0 + b_1 s + \cdots + b_{n-1} s^{n-1}}{a_0 + a_1 s + \cdots + a_{n-1} s^{n-1} +}$$

Definition: $\underline{p}_s = [\, b_0; b_1; \cdots; b_{n-1} \,]$

$\underline{q}_s = [\, a_0; a_1; \cdots; a_{n-1}; 1$

We go through the controller-canoni form:

$$A_{CCF} = \mathcal{C}_{Lo}\{\underline{q}_s\} = \begin{bmatrix} & 1 & & \emptyset \\ & & \ddots & \\ \emptyset & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}$$

$\mathcal{C}_{Lo}$: lower Companion Matrix

$$\underline{b}_{CCF} = \underline{e}_n = \begin{bmatrix} \emptyset \\ \vdots \\ \emptyset \\ 1 \end{bmatrix}$$

$\underline{e}_n$: unity vector in direction n

$$\underline{c}'_{CCF} = \underline{\beta}_s' = [\,b_0, b_1, b_2, \dots, b_{n-1}\,]$$
$$d_{CCF} = \phi$$

$$\Rightarrow \quad \beta_1 = \underline{c}'\underline{b} = b_{n-1}$$
$$\beta_2 = \underline{c}'A\underline{b} = [\,b_0, b_1, b_2, \dots, b_{n-1}\,]\begin{bmatrix}\phi\\ \phi\\ \vdots\\ \phi\\ 1\\ -a_{n-1}\end{bmatrix}$$
$$= b_{n-2} - b_{n-1}a_{n-1}$$
$$= b_{n-2} - a_{n-1}\beta_1$$

We continue in the same manner and find:

$$\beta_3 = b_{n-3} - a_{n-1}\beta_2 - a_{n-2}\beta_1$$
$$\beta_4 = b_{n-4} - a_{n-1}\beta_3 - a_{n-2}\beta_2 - a_{n-3}\beta_1$$
etc.

$$\Leftrightarrow \quad b_{n-1} = \beta_1$$
$$b_{n-2} = \beta_2 + a_{n-1}\beta_1$$
$$b_{n-3} = \beta_3 + a_{n-1}\beta_2 + a_{n-2}\beta_1 \qquad \text{etc.}$$

$$\Rightarrow \begin{bmatrix} b_0\\ b_1\\ b_2\\ \vdots\\ b_{n-2}\\ b_{n-1}\end{bmatrix} = \begin{bmatrix} a_1 & \cdots & \cdots & a_{n-2} & a_{n-1} & 1\\ \vdots & & & & 1 & \\ \vdots & & a_{n-2} & 1 & & \\ a_{n-2} & a_{n-1} & 1 & & & \\ a_{n-1} & 1 & & & \phi & \\ 1 & & & & & \end{bmatrix} \cdot \begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \vdots\\ \beta_{n-1}\\ \beta_n\end{bmatrix}$$

<u>Definition</u>: A matrix that is constant along its antidiagonals is called <u>Hankel-Matrix</u>.

$$\Rightarrow \quad \underline{p_s = \mathcal{H}_{up}\{\underline{q}_s\} \cdot \underline{\beta}}$$

$$\Longleftrightarrow \quad \underline{\underline{\beta} = \mathcal{H}_{up}^{-1}\{\underline{q}_s\} \cdot \underline{p}_s}$$

We can also reverse the $\underline{p}_s$-vector

$$\underline{p}_s^r = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix} = \mathcal{R}\{\underline{p}_s\}$$

$$\Rightarrow \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 & & & & \phi \\ a_{n-1} & 1 & & & \\ a_{n-2} & a_{n-1} & 1 & & \\ & a_{n-2} & \ddots & 1 & \\ \ddots & \ddots & \ddots & \ddots & 1 \\ a_1 & \cdots & a_{n-2} & a_{n-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \\ \beta_{n-1} \\ \beta_n \end{bmatrix}$$

<u>Definition</u>: A matrix that is constant along its diagonal is called <u>Toeplitz-Matrix</u>.

$$\Rightarrow \quad \underline{\underline{\beta_s^r = T_{l_0}\{\underline{q}_s\} \cdot \underline{\beta}}}$$

$$\Longleftrightarrow \quad \underline{\underline{\underline{\beta} = T_{l_0}^{-1}\{\underline{q}_s\} \cdot \underline{\beta}_s^r}}$$

Relation to Controllability and Observability Matrices:

$$\beta_i = \underline{c}' A^{i-1} \underline{b}$$

$$\Rightarrow [\beta_1, \beta_2, \ldots, \beta_{n-1}, \beta_n] = \underline{c}' [\underline{b}, A\underline{b}, \ldots, A^{n-1}\underline{b}]$$

$$\Rightarrow \quad \underline{\beta}' = \underline{c}' \cdot Q_c$$

$$\Longleftrightarrow \quad \underline{\underline{\underline{\beta} = Q_c' \cdot \underline{c}}}$$

$\underline{\beta}$:- Markov-Vect

or: $[\beta_1; \beta_2; \ldots; \beta_{n-1}; \beta_n] = [\underline{c}'; \underline{c}'A; \ldots; c'A^{n-1}] \cdot \underline{b}$

$$\Rightarrow \quad \underline{\underline{\underline{\beta} = Q_o \cdot \underline{b}}}$$

In <u>any</u> state-space representation

$$\Longrightarrow \quad \underline{\underline{Q_o \cdot \underline{b} \equiv Q_c' \cdot \underline{c}}}$$