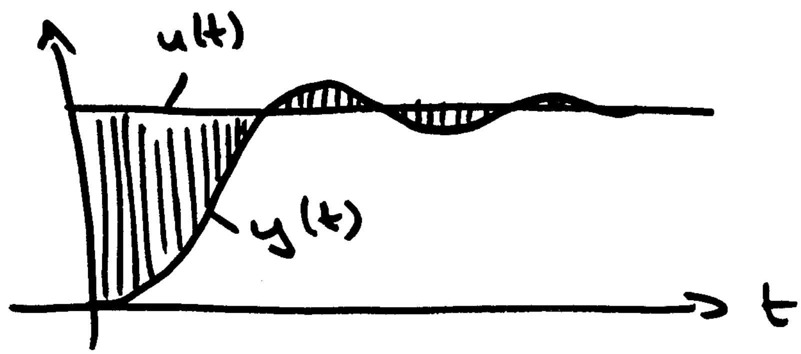
State Feedback through Optimal Control:

Problem: Sometimes, the pole locations may be inconvenient to specify, because all our assumptions base on the analysis of second order systems. Can these assumptions be generally adapted to higher order systems. Studies indicate that this is only true with good accuracy if the dominant poles are five times closer to the imaginar axis than any other poles:

This, we can hardly afford as the feedback gains will become too large. Otherwise, we may be in for some suprises when simulating the step response (e.g. overshoot >> 5%, etc.).

Question: Can we come up with a design that does not base on second-order system analysis?

Answer: Formulate as an optimization problem:



$e = u - y$

⇒ e.g. $$PI = \int_0^{t_f} e^2(\tau)\,d\tau) \overset{!}{=} \min_u$$

(ISE criterion),

or: $$PI = \int_0^{t_f} |e(\tau)|\,d\tau \overset{!}{=} \min_u$$

(IAE criterion),

or: $$PI = \int_0^{t_f} \tau \cdot |e(\tau)|\,d\tau \overset{!}{=} \min_u$$

(ITAE criterion).

We shall see that this optimization problem leads to a state feedback solution if:

- → system is linear
- → performance index is quadrati[c]
- → $t_f \to \infty$

Let us start by looking at the more general problem:

$$\left|\begin{array}{l} \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t) \\ PI = \int_0^{t_f} \varphi(\underline{x}, \underline{u}, t)\, dt \stackrel{!}{=} \underset{\underline{u}(t)}{Min} \end{array}\right.$$

Plug in the system equation into the performance equation:

$$PI = \int_0^{t_f} \varphi(\underline{x}, \underline{u}, t)\, dt \equiv \int_0^{t_f} \tilde{\varphi}(\underline{x}, \dot{\underline{x}}, t)\, dt = PI(\underline{x}, \dot{\underline{x}},$$

$$\Rightarrow PI(\underline{x}, \dot{\underline{x}}, t_f) \doteq \underset{\dot{\underline{x}}(t)}{Min}$$

We apply a <u>variation</u> around the optimum:

$$\Delta PI \geq \phi$$

<u>and</u>: $\Delta PI \equiv \phi$ at optimum.

$$\Delta PI = PI(\underline{x} + \delta\underline{x}, \dot{\underline{x}} + \delta\dot{\underline{x}}, t_f + \delta t_f) - PI(\underline{x}, \dot{\underline{x}}, t_f$$

$$\equiv \int_0^{t_f+\delta t_f} \tilde{\varphi}\,(\underline{x}+\delta\underline{x},\, \underline{\dot{x}}+\delta\underline{\dot{x}}, t)\, dt - \int_0^{t_f} \tilde{\varphi}\,(\underline{x}, \underline{\dot{x}}, t)\, dt$$

$$\equiv \int_0^{t_f} \left\{ \tilde{\varphi}\,(\underline{x}+\delta\underline{x},\, \underline{\dot{x}}+\delta\underline{\dot{x}}, t) - \tilde{\varphi}\,(\underline{x}, \underline{\dot{x}}, t) \right\} dt \; +$$

$$+ \int_{t_f}^{t_f+\delta t_f} \tilde{\varphi}\,(\underline{x}+\delta\underline{x},\, \underline{\dot{x}}+\delta\underline{\dot{x}}, t)\, dt$$

$$\int_{t_f}^{t_f+\delta t_f} \tilde{\varphi}\,(\underline{x}+\delta\underline{x},\, \underline{\dot{x}}+\delta\underline{\dot{x}}, t)\, dt \approx \tilde{\varphi}\,(\underline{x}, \underline{\dot{x}}, t_f)\cdot \delta t_f$$

$$\tilde{\varphi}\,(\underline{x}+\delta\underline{x},\, \underline{\dot{x}}+\delta\underline{\dot{x}}, t) \approx \tilde{\varphi}\,(\underline{x}, \underline{\dot{x}}, t) \; +$$

$$+ \frac{\partial \hat{\varphi}\,(\underline{x}, \underline{\dot{x}}, t)}{\partial \underline{x}} \cdot \delta\underline{x} + \frac{\partial \hat{\varphi}\,(\underline{x}, \underline{\dot{x}}, t)}{\partial \underline{\dot{x}}} \cdot \delta\underline{\dot{x}}$$

$$\Rightarrow \Delta PI \approx \tilde{\varphi}\,(\underline{x}, \underline{\dot{x}}, t_f)\cdot \delta t_f \; +$$

$$+ \int_0^{t_f} \left\{ \frac{\partial \tilde{\varphi}\,(\underline{x}, \underline{\dot{x}}, t)}{\partial \underline{x}} \cdot \delta\underline{x} + \frac{\partial \hat{\varphi}\,(\underline{x}, \underline{\dot{x}}, t)}{\partial \underline{\dot{x}}} \cdot \delta\underline{\dot{x}} \right\} dt$$

$$\int_0^{t_f} \frac{\partial \hat{\varphi}\,(\underline{x}, \underline{\dot{x}}, t)}{\partial \underline{\dot{x}}} \cdot \delta\underline{\dot{x}} \cdot dt = \left[\frac{\partial \hat{\varphi}\,(\underline{x}, \underline{\dot{x}}, t)}{\partial \underline{\dot{x}}} \cdot \partial\underline{x} \right]_0^{t_f}$$

$$- \int_0^{t_f} \left[\frac{\partial \hat{\varphi}\,(\underline{x}, \underline{\dot{x}}, t)}{\partial \underline{\dot{x}}} \right]^{\bullet} \cdot \delta\underline{x} \cdot dt$$

$$\left[\frac{\partial\tilde{\varphi}\;(\underline{x},\dot{\underline{x}},t)}{\partial\dot{\underline{x}}}\cdot\delta\underline{x}\right]_0^{t_f} = \frac{\partial\tilde{\varphi}\;(\underline{x},\dot{\underline{x}},t_f)}{\partial\dot{\underline{x}}}\cdot\delta\underline{x}(t_f)$$

$$- \frac{\partial\tilde{\varphi}\;(\underline{x},\dot{\underline{x}},0)}{\partial\dot{\underline{x}}}\cdot\underbrace{\delta\underline{x}(0)}_{\equiv 0}$$

as the initial condition is **fixed**.

$$\Rightarrow \Delta PI \approx \hat{\varphi}\;(\underline{x},\dot{\underline{x}},t_f)\cdot\delta t_f + \frac{\partial\tilde{\varphi}\;(\underline{x},\dot{\underline{x}},t_f)}{\partial\dot{\underline{x}}}\cdot\delta\underline{x}$$

$$+ \delta\underline{x}\cdot\left\{\int_0^{t_f}\left\{\frac{\partial\tilde{\varphi}\;(\underline{x},\dot{\underline{x}},t)}{\partial\underline{x}} - \frac{d}{dt}\left[\frac{\partial\tilde{\varphi}\;(\underline{x},\dot{\underline{x}},t)}{\partial\dot{\underline{x}}}\right]\right\}d\right.$$

Let:

$$\Delta PI_f \triangleq \hat{\varphi}\;(\underline{x},\dot{\underline{x}},t_f)\cdot\delta t_f + \frac{\partial\tilde{\varphi}\;(\underline{x},\dot{\underline{x}},t_f)}{\partial\dot{\underline{x}}}\cdot\delta\underline{x}($$

$$\Rightarrow \boxed{\Delta PI \approx \Delta PI_f + \delta\underline{x}\cdot\int_0^{t_f}\left\{\frac{\partial\tilde{\varphi}}{\partial\underline{x}} - \frac{d}{dt}\left(\frac{\partial\tilde{\varphi}}{\partial\dot{\underline{x}}}\right)\right\}dt}$$

Let us now look at some special cases of this general formulae:

(1) t_f = <u>fixed</u> ; $\underline{x}(t_f)$ = <u>fixed</u>

(meet your spouse at 5 pm in Park Mall $\Longleftrightarrow$ both the final time and the final position are frozen)

$$\Longrightarrow \quad \delta t_f \equiv \phi \quad ; \quad \delta \underline{x}(t_f) \equiv \phi$$

$$\Longrightarrow \quad \Delta PI_f \equiv \phi$$

$$\Rightarrow \Delta PI = \delta \underline{x} \cdot \int_0^{t_f} \left\{ \frac{\partial \tilde{\varphi}}{\partial \underline{x}} - \frac{d}{dt}\left(\frac{\partial \tilde{\varphi}}{\partial \dot{\underline{x}}} \right) \right\} dt \overset{!}{=} \phi$$

over <u>all</u> variations $\delta \underline{x}$

$$\Rightarrow \quad \boxed{\frac{\partial \tilde{\varphi}}{\partial \underline{x}} - \frac{d}{dt}\left(\frac{\partial \tilde{\varphi}}{\partial \dot{\underline{x}}} \right) \equiv \phi}$$

<u>Euler Equation</u>

Example :

$$\left| \begin{array}{c} \dot{x} = u \\ PI = \int_0^{t_f} (x^2 + \lambda u^2)\, dt \stackrel{!}{=} \underset{u}{\mathrm{Min}} \end{array} \right|$$

given :

$$\left| \begin{array}{l} \underline{x}(t=\emptyset) = \underline{x}_0 \\ \underline{x}(t=t_f) = \underline{x}_f \\ \quad t_f \end{array} \right|$$

$$\Rightarrow \varphi(x,u,t) = x^2 + \lambda u^2$$

$$\Rightarrow \tilde{\varphi}(x,\dot{x},t) = x^2 + \lambda \dot{x}^2$$

$$\Rightarrow \frac{\partial \tilde{\varphi}}{\partial x} = 2x \quad ; \quad \frac{\partial \tilde{\varphi}}{\partial \dot{x}} = 2\lambda \dot{x}$$

$$\Rightarrow 2x - \frac{d}{dt}(2\lambda \dot{x}) = \emptyset$$

$$\Rightarrow x - \frac{d}{dt}(\lambda u) = \emptyset$$

$$\Rightarrow \boxed{\dot{u} = \frac{1}{\lambda} x}$$

$$\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} \emptyset & 1 \\ 1/\lambda & \emptyset \end{bmatrix} \cdot \begin{bmatrix} x \\ u \end{bmatrix}$$

$x(t=\emptyset) = x_0$; $x(t=t_f) = x_f$

are given ; $u(t=\emptyset)$ is unknown.

⟹ We have transformed our optimization problem into a <u>boundary value problem</u>

We can solve either of the two. The answers must be the same.

- This simple problem still has an analytic solution:

$$\dot{u} = \frac{1}{\lambda} x \equiv \ddot{x}$$

$$\Rightarrow \boxed{\ddot{x} - \frac{1}{\lambda} x = \emptyset}$$

$$\Rightarrow x(t) = a \cdot \sinh\left(\sqrt{1/\lambda} \cdot t\right) + b \cdot \cosh\left(\sqrt{1/\lambda} \cdot t\right)$$

Find $\{a, b\}$ that satisfy the boundary conditions.

In general, we need to solve the boundary value problem <u>numerically</u> $\Rightarrow$ quite difficult

$\boxed{\text{cf. ECE 4726}}$

(2) t_f = <u>variable</u> and/or $\underline{x}(t_f)$ = <u>variable</u>

$$\Rightarrow \Delta PI_f = \tilde{\varphi}(\underline{x},\dot{\underline{x}},t_f)\cdot\delta t_f + \frac{\partial\tilde{\varphi}(\underline{x},\dot{\underline{x}},t_f)}{\partial\dot{\underline{x}}}\cdot\delta\underline{x}(t_f) \stackrel{!}{=} \phi$$

for <u>all</u> possible variations.

$$\delta\underline{x}_f \approx \delta\underline{x}(t_f) + \dot{\underline{x}}(t_f)\cdot\delta t_f$$

$$\Leftrightarrow \quad \delta\underline{x}(t_f) \cong \delta\underline{x}_f - \dot{\underline{x}}(t_f)\cdot\delta t_f$$

$$\Rightarrow \Delta PI_f = \left\{\tilde{\varphi}(\underline{x},\dot{\underline{x}},t_f) - \dot{\underline{x}}(t_f)\cdot\frac{\partial\tilde{\varphi}(\underline{x},\dot{\underline{x}},t_f)}{\partial\dot{\underline{x}}}\right\}\delta t_f + \frac{\partial\tilde{\varphi}(\underline{x},\dot{\underline{x}},t_f)}{\partial\dot{\underline{x}}}\cdot\delta\underline{x}_f \stackrel{!}{\equiv} \phi$$

(2a) t_f = fixed ; $\underline{x}_f$ = Variable

(Hit the foreign missil at 10 am wherever you can get hold of it)

$\Rightarrow \quad \delta t_f \equiv 0$

$$\Rightarrow \quad \boxed{\frac{\partial \tilde{\varphi}(\underline{x}, \dot{\underline{x}}, t_f)}{\partial \dot{\underline{x}}} \equiv 0}$$

(2b) t_f = variable ; $\underline{x}_f$ = fixed

(Hit the foreign missil site whenever you are ready)

$\Rightarrow \quad \delta \underline{x}_f \equiv 0$

$$\Rightarrow \quad \boxed{\tilde{\varphi}(\underline{x}, \dot{\underline{x}}, t_f) - \dot{\underline{x}}(t_f) \cdot \frac{\partial \tilde{\varphi}(\underline{x}, \dot{\underline{x}}, t_f)}{\partial \dot{\underline{x}}} \equiv 0}$$

Lagrange Equation

Example (continued):

(2a) $\quad \tilde{g} = x^2 + \lambda \dot{x}^2$

$$\Rightarrow \frac{\partial \tilde{g}}{\partial \dot{x}} = 2\lambda \dot{x} \equiv \phi \quad @\ t = t_f$$

$$\Rightarrow \dot{x} = \phi \quad \Rightarrow \quad u = \phi$$

$$\Rightarrow \boxed{u(t_f) \equiv \phi}$$

is now an additional equation for the additional unknown: $x(t_f)$:

given: $\left| \begin{array}{c} x(t=\phi) = x_0 \\ u(t=t_f) = \phi \\ t_f \end{array} \right|$

$\Rightarrow$ still a boundary value problem.

(2b) $$(x^2+\lambda\dot{x}^2) - \dot{x}\cdot(2\lambda\dot{x}) \equiv \emptyset$$

$$\Rightarrow x^2 - \lambda\dot{x}^2 = \emptyset$$

$$\Rightarrow \boxed{x^2(t_f) - \lambda u^2(t_f) = \emptyset}$$

given:

$$x(t=\emptyset) = x_0$$
$$x(t=t_f) = x_f$$
$$u(t=t_f) = \sqrt{1/\lambda}\cdot x_f$$

but t_f is unknown

$\Rightarrow$ Another type of boundary value problem.

(3) t_f = variable ; $\underline{x}_f$ = variable

but: $\underline{x}_f(t) = \underline{r}(t)$

(tracking problem)

[I want to fly to the moon. I don't care when I arrive, I don't care where the moon is, but when I pass the lunar trajectory, the moon better be there.

$$\Rightarrow \quad \boxed{\partial \underline{x}_f \approx \underline{\dot{r}}(t_f) \cdot \delta t_f}$$

$$\Rightarrow \boxed{\tilde{g}(\underline{x}, \underline{\dot{x}}, t_f) + \left(\underline{\dot{r}}(t_f) - \underline{\dot{x}}(t_f)\right) \cdot \frac{\partial \tilde{g}(\underline{x}, \underline{\dot{x}}, t_f)}{\partial \underline{\dot{x}}} =}$$

$\Rightarrow$ Transversality condition

$\Rightarrow$ Yet another boundary value problem.

There is another way to look at the same problem.

$$\left|\begin{array}{c} \underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t) \\ PI = \int_0^{t_f} g(\underline{x}, \underline{u}, t) \doteq \underset{\underline{u}(t)}{Min} \end{array}\right|$$

can be viewed as an optimization problem with equality constraint.

Lagrange transformed this into another optimization problem without constraint:

$$\tilde{PI} = \int_0^{t_f} \phi(\underline{x}, \dot{\underline{x}}, \underline{u}, \underline{\psi}, t) \overset{!}{=} \underset{\substack{\underline{u}(t) \\ \underline{\psi}(t) \\ \underline{x}(t)}}{\mathrm{Min}}$$

where:

$$\phi(\underline{x}, \dot{\underline{x}}, \underline{u}, \underline{\psi}, t) = \underline{\varphi}(\underline{x}, \underline{u}, t) + \underline{\psi}'(t) \cdot [\underline{f} - \dot{\underline{x}}]$$

ϕ :. Lagrange Function

$\underline{\psi}$:. Lagrange Multipliers

Proof: Apply the Euler equation to this new problem:

(1) $$\underbrace{\frac{\partial \Phi}{\partial \underline{x}}}_{\frac{\partial \varphi}{\partial \underline{x}}} - \frac{d}{dt}\Big(\underbrace{\frac{\partial \Phi}{\partial \underline{\dot{x}}}}_{-\underline{\psi}(t)}\Big) = 0$$

$$\Rightarrow \boxed{\underline{\dot{\psi}}(t) = -\frac{\partial \varphi}{\partial \underline{x}}}$$

$\Rightarrow$ condition for $\underline{\psi}(t)$

(2) $$\underbrace{\frac{\partial \Phi}{\partial \underline{\psi}}}_{\underline{f}-\underline{\dot{x}}} - \frac{d}{dt}\Big(\underbrace{\frac{\partial \Phi}{\partial \underline{\dot{\psi}}}}_{0}\Big) = 0$$

$$\Rightarrow \boxed{\underline{\dot{x}}(t) = \underline{f}(\underline{x}, \underline{u}, t)}$$ state equat

$\Rightarrow$ System equations are satisfied. $\Longleftrightarrow \Phi_{opt} \equiv \varphi_{opt}$

$\Longleftrightarrow \widetilde{PI} = PI$

(3) $$\underbrace{\frac{\partial \Phi}{\partial \underline{u}}}_{\frac{\partial \varphi}{\partial \underline{u}}} - \frac{d}{dt}\Big(\underbrace{\frac{\partial \Phi}{\partial \underline{\dot{u}}}}_{0}\Big) = 0$$

$$\Rightarrow \boxed{\frac{\partial \varphi}{\partial \underline{u}} = 0}$$ optim condi

$$\boxed{\underline{\dot{\psi}}(t) = -\frac{\partial \varphi}{\partial \underline{x}}}$$

is often called the adjugate system.

Modification by Hamilton:

$$\Phi(\underline{x}, \underline{\dot{x}}, \underline{u}, \underline{\psi}, t) = \varphi(\underline{x}, \underline{u}, t) + \underline{\psi}'(t) \cdot \underline{f}(\underline{x}, \underline{u}, t) - \underline{\psi}'(t) \cdot \underline{\dot{x}}(t)$$

$$\equiv H(\underline{x}, \underline{u}, \underline{\psi}, t) - \underline{\psi}'(t) \cdot \underline{\dot{x}}(t)$$

$$H(\underline{x}, \underline{u}, \underline{\psi}, t) = \varphi(\underline{x}, \underline{u}, t) + \underline{\psi}'(t) \cdot \underline{f}(\underline{x}, \underline{u}, t)$$

H : Hamilton function

$$\Rightarrow \frac{\partial \Phi}{\partial \underline{u}} = 0 \iff \boxed{\frac{\partial H}{\partial \underline{u}} = 0}$$

$$\frac{\partial \Phi}{\partial \underline{x}} = -\underline{\dot{\psi}} \iff \boxed{\frac{\partial H}{\partial \underline{x}} = -\underline{\dot{\psi}}}$$

$$\frac{\partial \Phi}{\partial \underline{\psi}} = \phi = \frac{\partial H}{\partial \underline{\psi}} - \underline{\dot{x}}$$

$$\Rightarrow \boxed{\frac{\partial H}{\partial \underline{\psi}} = \underline{\dot{x}}}$$

is an alternate description that is often used.

<u>Modification by Pontrjagin</u>:

(Понтрягин)

$\underline{u}(t)$ is no longer assumed to be a continuous function

[e.g. $\underline{u}(t)$ is <u>limited</u>.]

⇒ Simply replace:

$$\frac{\partial H}{\partial \underline{u}} = \phi$$

by the equation:

$$\boxed{PI_H = H(\underline{x}, \underline{u}, \underline{\Psi}, t) \stackrel{!}{=} \underset{\underline{u}(t)}{\text{Min}}}$$

The other equations remain the same.

⟹ Many more details:

$\boxed{\text{ECE 541}}$

Another technique for the same type of problems is <u>dynamic programming</u>, a technique which is far simpler and therefore can be applied to more complicated systems easily, but is numbercrunching at its very best (!!!)