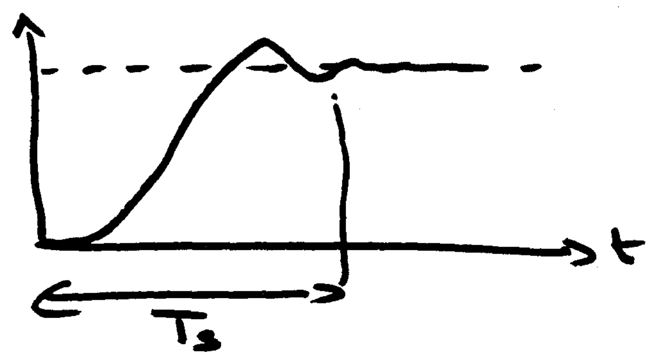
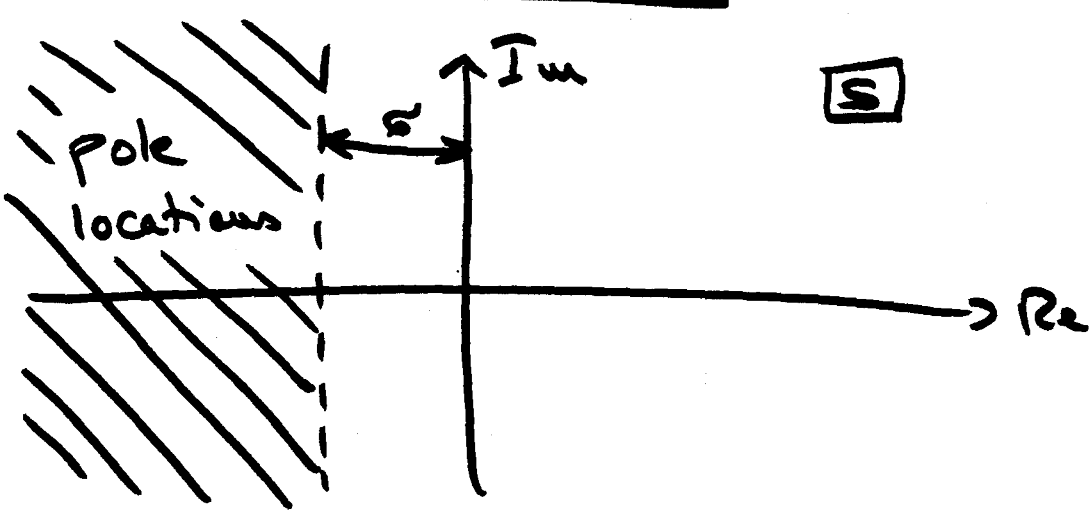
Design Considerations:

(1) Settling Time:



Remember from ECE 441: to guarantee a certain settling time not to be exceeded, we need to make the damping sufficiently large:

$$\boxed{\sigma \approx \frac{4}{T_s}}$$



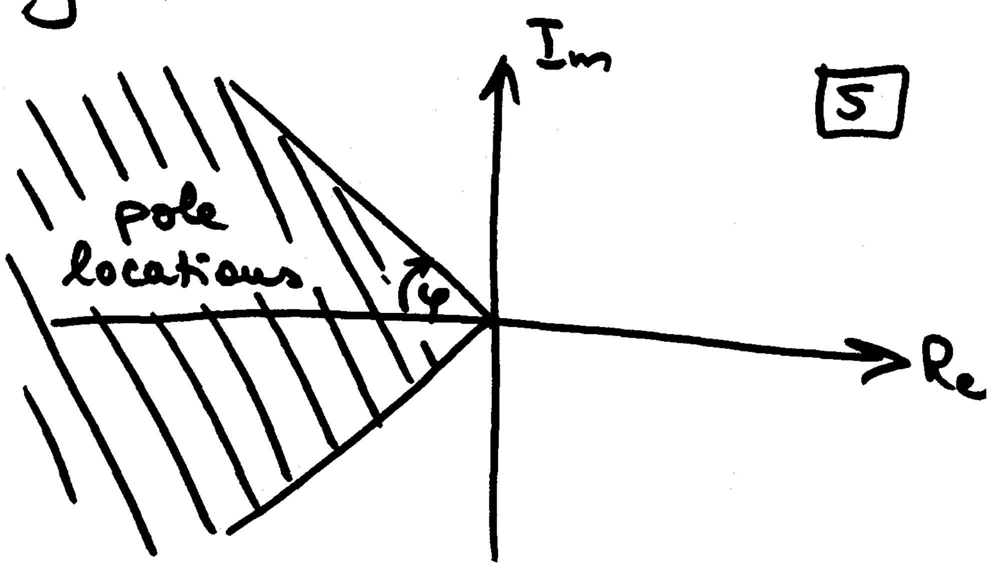
(2) Overshoot:

To avoid too large an overshoot, we must limit the damping ratio:

$$\zeta = \frac{\sigma}{\omega_0} = \cos(\varphi)$$

For 5% overshoot, we can tolerate $\varphi \approx 45^\circ$ (phase margin):

$\boxed{5}$



(3) Feedback Gains:

In order not to make the system too sensitive to noise, we want to limit the feedback gains ($\underline{k}$-vector and $\underline{h}$-vector) to $\underline{\underline{\approx 100}}$.

- We noticed that, while the <u>system</u> representation does not really matter (we measure only inputs and outputs), the <u>model</u> representation does influence the values of $\underline{k}$ and $\underline{h}$.

- By selecting an appropriate representation, we can probably keep $\underline{k}$ small, but this will go at the expense of a large $\underline{h}$-vector, and vice-versa.

  $\Rightarrow$ We need to <u>balance</u> the two feedback vectors.

<u>Solution</u>: Make:

$$|\tilde{k}_i| \equiv |\tilde{h}_i|$$

Let us select the representation:

$$\underline{\mu} = \tilde{T} \cdot \underline{x} \iff \underline{x} = \tilde{T}^{-1} \cdot \underline{\mu}$$

$$\Rightarrow u = r - \underline{k}' \underline{x} = r - \underbrace{\underline{k}' \tilde{T}^{-1}}_{\underline{\tilde{k}}'} \cdot \underline{\mu}$$

Let $\underline{\nu} = \tilde{T} \cdot \underline{z} \iff \underline{\dot{z}} = \tilde{T}^{-1} \cdot \underline{\dot{\nu}}$

$$\Rightarrow \quad \underline{\dot{z}} = A\underline{z} + \underline{h}(y - \hat{y}) + \underline{b}u$$

$$\Rightarrow \tilde{T}^{-1}\underline{\dot{\nu}} = \tilde{T}^{-1}\tilde{A}\underbrace{\tilde{T} \cdot \tilde{T}^{-1}}_{I}\underline{\nu} + \underline{h}(y - \hat{y}) + \tilde{T}^{-1}\underline{\tilde{b}}u$$

$$\Rightarrow \quad \underline{\dot{\nu}} = \tilde{A}\underline{\nu} + \underbrace{\tilde{T}\underline{h}}_{\underline{\tilde{h}}}(y - \hat{y}) + \underline{\tilde{b}}u$$

$\Rightarrow$ In the new representation, we find that:

$$\boxed{\begin{aligned} \underline{\tilde{k}}' &= \underline{k}' \cdot \tilde{T}^{-1} \\ \underline{\tilde{h}} &= \tilde{T} \cdot \underline{h} \end{aligned}}$$

Let us choose

$$\tilde{T} = \operatorname{diag}\{t_i\}$$

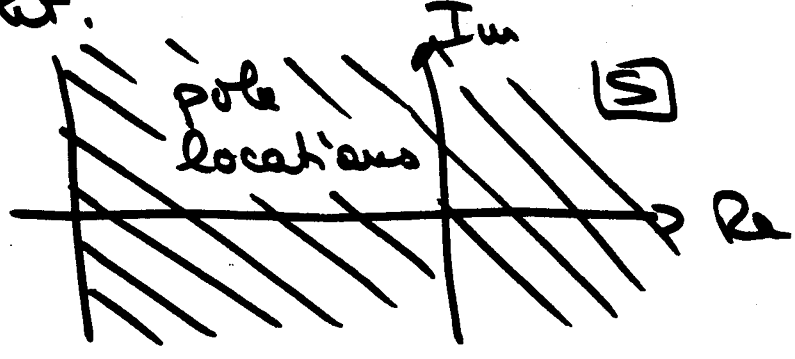
$$\Rightarrow |\tilde{k}_i| = \left| k_i \cdot \left(\frac{1}{t_i}\right) \right| \stackrel{!}{\equiv} |\tilde{h}_i| = |t_i \cdot h_i|$$

$$\Rightarrow t_i^2 = \left| \frac{k_i}{h_i} \right|$$

$$\Rightarrow \boxed{t_i = \sqrt{|k_i / h_i|}}$$

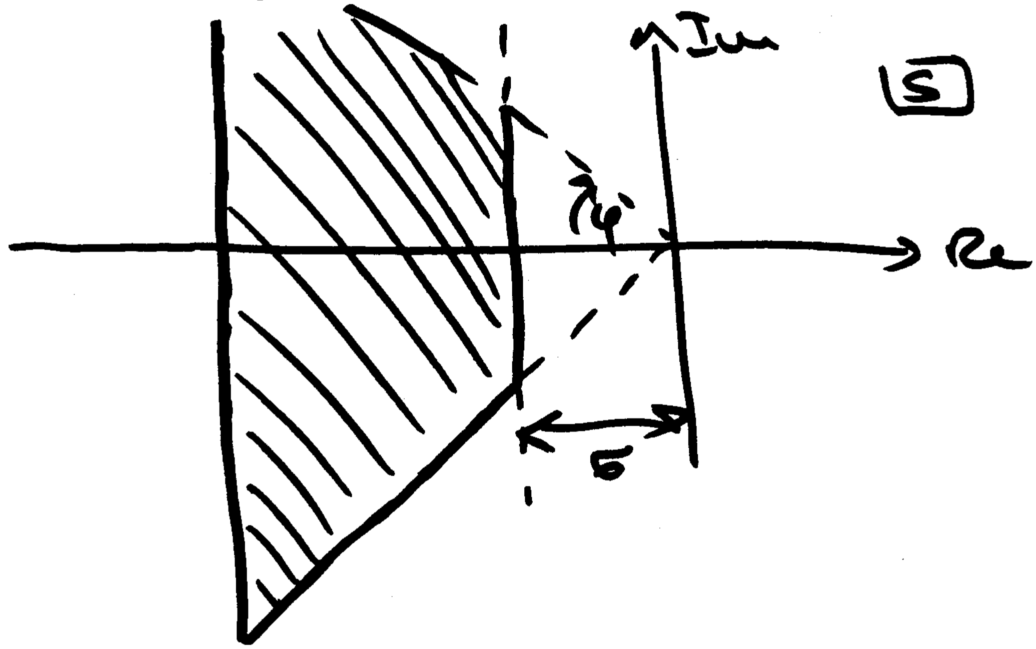
will <u>balance</u> the <u>k</u>- and the <u>h</u>-vector.

- If now the $|k_i| \equiv |h_i| > 100$
  $\Rightarrow$ if have asked too much from our system. Move the poles further to the right.



(4) Sensitivity:

In order to avoid un-necessary sensitivity, **spread** the poles in the remaining domain:



Avoid placing poles in the vicinity of each other.

(5) Make the observer poles about twice as fast as the controller poles. Controller- and observer-pole may coincide without

increasing the sensitivity.

Example:

Given the system:

$$\dot{\underline{x}} = \begin{bmatrix} -150 & 192 & 12 & 165 \\ 143 & -181 & -15 & -154 \\ -142 & 179 & 15 & 153 \\ -291 & 370 & 28 & 316 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} -27 & 51 & -18 & 48 \end{bmatrix} \underline{x}$$

Analysis:

$$\mathrm{EIG}(a) \Longrightarrow \underline{\lambda} = \begin{bmatrix} -2 \\ 3 \\ 4 \\ -5 \end{bmatrix}$$

$\Rightarrow$ two instable & two stable modes.

$\mathrm{RANK}(qc) \Longrightarrow 4$

$\Rightarrow$ system is controllable.

$\mathrm{RANK}(qo) \Longrightarrow 3$

$\Rightarrow$ one unobservable mode.

$$[p, q] = \mathrm{SS2TF}(a, b, c, d, 1)$$

$$\Rightarrow \quad p = [\emptyset \;\; \emptyset \;\; 3 \;\; 18 \;\; 15]$$

$$q = [1 \;\; \emptyset \;\; -27 \;\; 14 \;\; 12\emptyset]$$

$$p = p(3:5)$$

$$\Rightarrow \quad p = [3 \;\; 18 \;\; 15]$$

$$\mathrm{ROOTS}(p) \Rightarrow \begin{bmatrix} -1 \\ -5 \end{bmatrix}$$

$$\mathrm{ROOTS}(q) \Rightarrow \begin{bmatrix} -2 \\ 3 \\ 4 \\ -5 \end{bmatrix}$$

unobser-
vable
mode

Fortunately, the unobservable mode is <u>stable</u>.

- We build the polynomial

$$r(s) = s + 5$$

$$\Rightarrow r = [1 \;\; 5]$$

then we divide numerator and denominator:

p2 = DECONV (p, r)

$\Rightarrow$ p2 = [3 3]

q2 = DECONV (q, r)

$\Rightarrow$ q2 = [1 -5 -2 24]

that is:

$$G(s) = \frac{3(s+1)}{(s+2)(s-3)(s-4)}$$

$$= \frac{3s+3}{s^3 - 5s^2 - 2s + 24}$$

Now, we transform this back into the time domain.

[an, bn, cn, dn] = TF2SS (p2, q2)

$$\Rightarrow \left| \begin{aligned} \dot{\underline{\xi}} &= \begin{bmatrix} 5 & 2 & -24 \\ 1 & \phi & \phi \\ \phi & 1 & \phi \end{bmatrix} \underline{\xi} + \begin{bmatrix} 1 \\ \phi \\ \phi \end{bmatrix} u \\ y &= [\phi \quad 3 \quad 3] \underline{\xi} \end{aligned} \right|$$

This is similar to our controller-canonical form, but the state variables are numbered the other way through. To get into our "normal" representation, we choose the transformation:

$$\underline{\eta} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{T} \underline{\xi}$$

$$\Rightarrow \quad \begin{aligned} \hat{A} &= T \cdot A / T \\ \hat{\underline{b}} &= T \cdot \underline{b} \\ \hat{\underline{c}}' &= \underline{c}' / T \\ \hat{d} &= d \end{aligned}$$

will give us:

$$\begin{vmatrix} \dot{\underline{\eta}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & 2 & 5 \end{bmatrix} \underline{\eta} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 3 & 5 & 0 \end{bmatrix} \underline{\eta} \end{vmatrix}$$

This will serve as our <u>model</u>. It does not matter that the system is of 4th order while the model is of 3rd order, as we only use inputs and outputs of the system. It does not matter either that the model uses different state variables (for the same reason).

- Now, we want to design our state feedback.

$$q_{OL}(s) = s^3 - 5s^2 - 2s + 24$$

$$\begin{aligned} q_{CL}(s) &= (s+8)(s+4+4j)(s+4-4j) \\ &= (s+8)(s^2+8s+32) \\ &= s^3 + 16s^2 + 96s + 256 \end{aligned}$$

$$\Rightarrow a_0 = 24 \quad ; \quad a_0 + k_0 = 256 \Rightarrow k_0 = 23$$

$$a_1 = -2 \quad ; \quad a_1 + k_1 = 96 \Rightarrow k_1 = 98$$

$$a_2 = -5 \quad ; \quad a_2 + k_2 = 16 \Rightarrow k_2 = 21$$

$$\Rightarrow \underline{k}' = \begin{bmatrix} 232 & 98 & 21 \end{bmatrix}$$

We can get this result at once by:

$$lc = [-8\,;-4+4*j\,;-4-4*j]$$

$$k = \text{PLACE}(a2, b2, lc)$$

where: $A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & 2 & 5 \end{bmatrix}$; $\underline{b}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

- Last, we want to design our observer. Let the observer poles be twice as fast as the controller poles:

$$lo = 2 * lc$$

- We need the transformation matrix to go into observer-canonical form:

$$Q_o = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 3 & 3 \\ -72 & 6 & 18 \end{bmatrix}$$

$$\Rightarrow \quad Q_o^{-1} = \begin{bmatrix} -0.0667 & 0.1 & -0.0167 \\ 0.4 & -0.1 & 0.0167 \\ -0.4 & 0.4333 & -0.0167 \end{bmatrix}$$

$$\Rightarrow \underline{q} = \begin{bmatrix} -0.0167 \\ 0.0167 \\ -0.0167 \end{bmatrix} \Rightarrow P = [\underline{q}, A\underline{q}, A^2\underline{q}]$$

$$\Rightarrow P = \begin{bmatrix} -0.0167 & 0.0167 & -0.0167 \\ 0.0167 & -0.0167 & 0.35 \\ -0.0167 & 0.35 & 1.3167 \end{bmatrix}$$

$$\Rightarrow \underline{\underline{T_{OCF} = P^{-1}}} = \underline{\underline{\begin{bmatrix} -78 & -15 & 3 \\ -15 & -12 & 3 \\ 3 & 3 & 0 \end{bmatrix}}}$$

$$q_{obs}(s) = (s+16)(s+8+8j)(s+8-8j)$$
$$= s^3 + 32s^2 + 384s + 2048$$

$$\Rightarrow a_0 = 24 \quad ; \quad a_0 + \hat{h}_0 = 2048 \quad \Rightarrow \quad \hat{h}_0 = 2024$$
$$a_1 = -2 \quad ; \quad a_1 + \hat{h}_1 = 384 \quad \Rightarrow \quad \hat{h}_1 = 386$$
$$a_2 = -5 \quad ; \quad a_2 + \hat{h}_2 = 32 \quad \Rightarrow \quad \hat{h}_2 = 37$$

$$\Rightarrow \underline{\hat{h}} = \begin{bmatrix} 2024 \\ 386 \\ 37 \end{bmatrix}$$

$$\Rightarrow \underline{h} = T_{ocf}^{-1} \cdot \underline{\hat{h}} = P \cdot \underline{\hat{h}} = \begin{bmatrix} -27.9167 \\ 40.25 \\ 150.0833 \end{bmatrix}$$

The same result would have been found quickly with:

h = PLACE(a2', c2', l0);
h = h'

Unfortunately, some components are a little too large. Let us try our balancing algorithm:

T= SQRT(ABS(K'./ħ))

$$\Rightarrow \quad T = \begin{bmatrix} 2.8828 \\ 1.5604 \\ 0.3741 \end{bmatrix}$$

T=DIAG(T)

$$\Rightarrow \quad T = \begin{bmatrix} 2.8828 & 0 & 0 \\ 0 & 1.5604 & 0 \\ 0 & 0 & 0.3741 \end{bmatrix}$$

a3 = t * a2 / t
b3 = t * b2
c3 = c2 / t

$$\Rightarrow \quad \left| \begin{aligned} \underline{\dot{v}} &= \begin{bmatrix} 0 & 1.8475 & 0 \\ 0 & 0 & 4.1714 \\ -3.1142 & 0.4794 & 5 \end{bmatrix} \underline{v} + \begin{bmatrix} 0 \\ 0 \\ 0.3741 \end{bmatrix} u \\ y &= \begin{bmatrix} 1.0407 & 1.9226 & 0 \end{bmatrix} \underline{v} \end{aligned} \right|$$

is a modified modal represen=
tation.

Repeating the design, we find:

$$k = PLACE(a3, b3, lc)$$

$$\Rightarrow \underline{k}' = [80.4777 \quad 62.8053 \quad 56.1404]$$

$$h = PLACE(a3', c3', lo);$$
$$h = h'$$

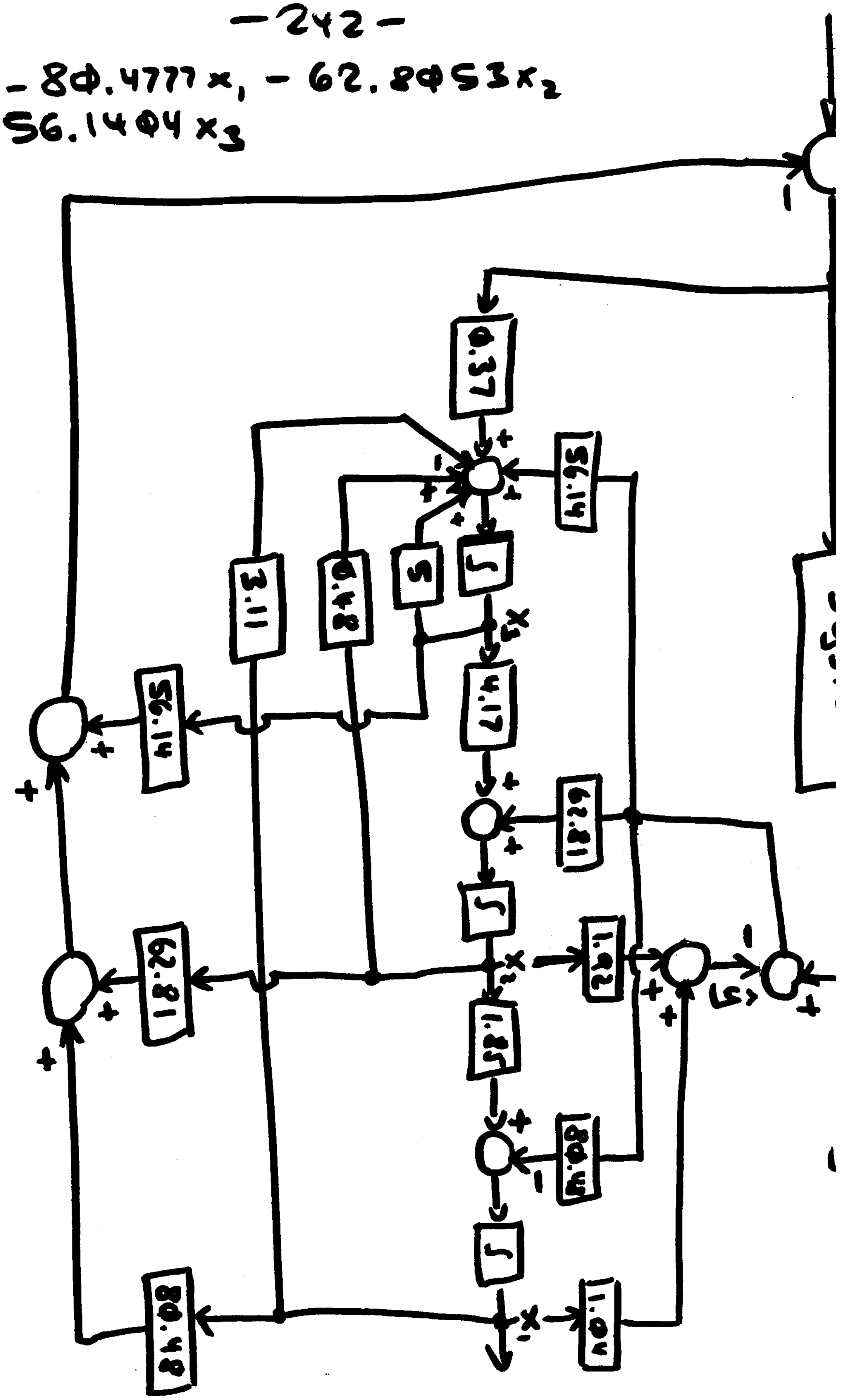
$$\Rightarrow \underline{h} = \begin{bmatrix} -80.4777 \\ 62.8053 \\ 56.1404 \end{bmatrix}$$

This design looks acceptable.

With this design, the controller equations look as follows:

$$\dot{x}_1 = 1.8475\,x_2 - 80.4777(y - \hat{y})$$
$$\dot{x}_2 = 4.1714\,x_3 + 62.8053\,(y - \hat{y})$$
$$\dot{x}_3 = -3.1142\,x_1 + 0.4794\,x_2 + 5x_3 + 0.3741\,u + 56.1404\,(y - \hat{y})$$
$$\hat{y} = 1.0407x_1 + 1.9276\,x_2$$

$$u = r - 80.4777\,x_1 - 62.8053\,x_2 - 56.1404\,x_3$$



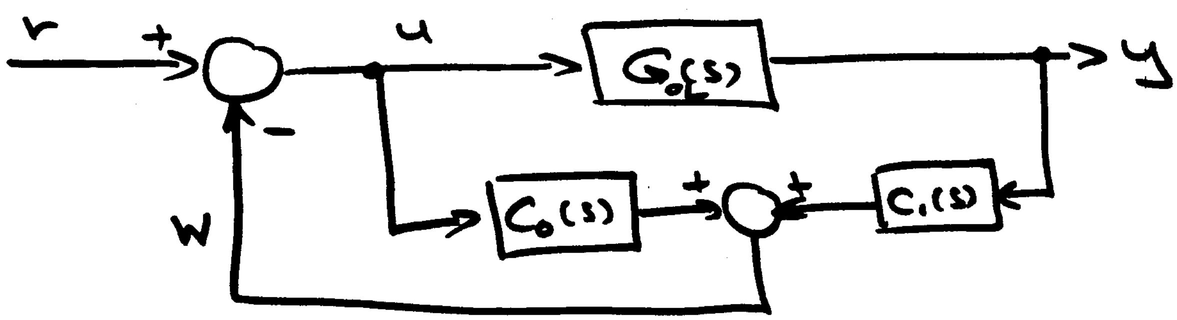
Controller-Design in the Frequency Domain:

Given: $G_{OL}(s) = \dfrac{P(s)}{Q_{OL}(s)}$

We know that, if $\operatorname{ord}(P) < \operatorname{ord}(Q_{OL})$ it is possible to design a output feedback such that

$$G_{CL}(s) = \frac{P(s)}{Q_{CL}(s)}$$

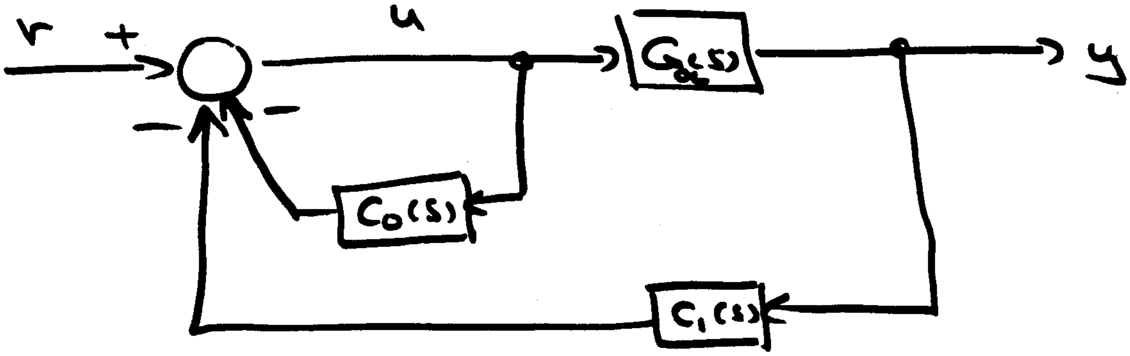
We try the following approach:



with so far unknown transfer functions $C_0(s)$ and $C_1(s)$.

Let: $\quad C_0(s) = \dfrac{P_0(s)}{Q_{OBS}(s)}$; $C_1(s) = \dfrac{P_1(s)}{Q_{OBS}(s)}$

$Q_{OBS}(s)$ are the "observer poles" (as before).



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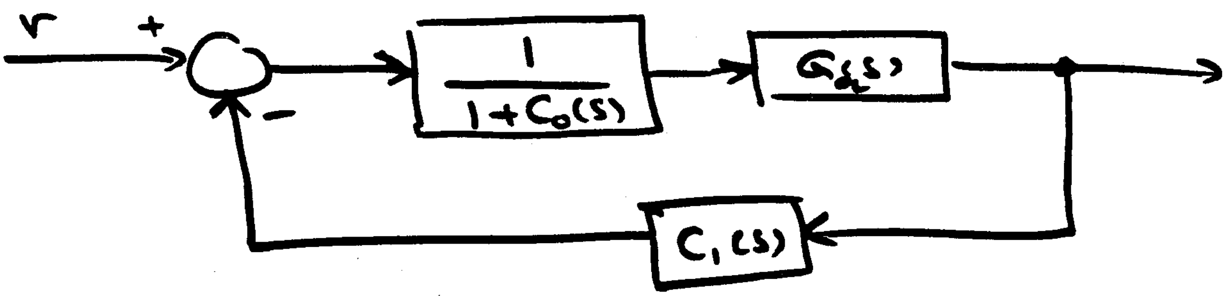

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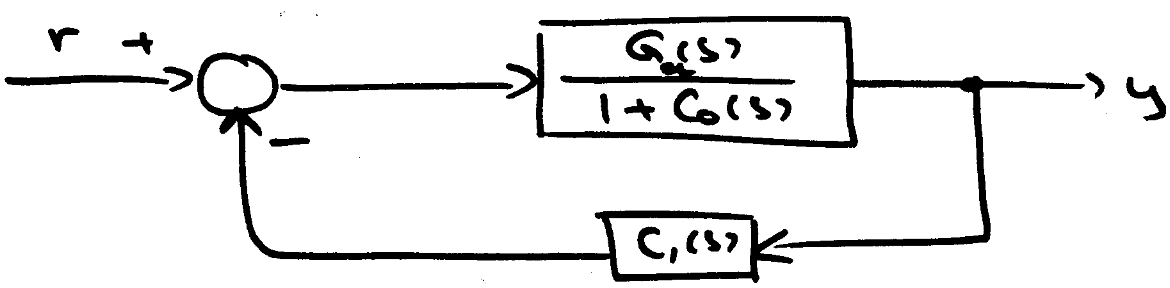


$$\Rightarrow G_{CL}(s) = \frac{\dfrac{G_{OL}(s)}{1+C_0(s)}}{1 + \dfrac{G_{OL}(s)\,C_1(s)}{1+C_0(s)}}$$

$$\Rightarrow G_{CL}(s) = \frac{G_{OL}(s)}{1 + C_0(s) + G_{OL}(s)\,C_1(s)}$$

$$\Rightarrow G_{CL}(s) = \frac{P(s)/Q_{OL}(s)}{1 + \dfrac{P_0(s)}{Q_{OBS}(s)} + \dfrac{P(s)\cdot P_1(s)}{Q_{OL}(s)\cdot Q_{OBS}(s)}}$$

$$\Rightarrow G_{CL}(s) = \frac{P(s)\cdot Q_{OBS}(s)}{Q_{OL}(s)\cdot Q_{OBS}(s) + P_0(s)\cdot Q_{OL}(s) + P(s)\cdot P}$$

$$\stackrel{!}{=} \frac{P(s)}{Q_{CL}(s)} \equiv \frac{P(s)\cdot Q_{OBS}(s)}{Q_{CL}(s)\cdot Q_{OBS}}$$

the observer poles are uncontrollab (as desired)

$$\Rightarrow \boxed{Q_{CL}(s)\cdot Q_{OBS}(s) \equiv Q_{OL}(s)\cdot Q_{OBS}(s) + P_0(s)\cdot Q_{OL}(s) + P(s)\cdot}$$

We remember that the observer poles can be chosen freely, e.g. twice as fast as the controller poles. $\Rightarrow$ $P_0(s)$ and $P_1(s)$ are unknown, must satisfy the above condition.

$$Q_{OL}(s) = s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$
$$P(s) = b_{n-1} s^{n-1} + \cdots + b_1 s + b_0$$
$$Q_{OBS}(s) = s^{n-1} + \alpha_{n-2} s^{n-2} + \cdots + \alpha_1 s + \alpha_0$$
$$P_0(s) = c_{0_{n-1}} s^{n-1} + \cdots + c_{0_1} s + c_{00}$$
$$P_1(s) = c_{1_{n-1}} s^{n-1} + \cdots + c_{1_1} s + c_{10}$$

(Notice that, contrary to our previous design experience, the observer was chosen of degree $(n-1)$ rather than of degree n.)

Finally:

$$D(s) = Q_{OBS}(s)\left[Q_{CL}(s) - Q_{OL}(s)\right]$$
$$= d_{2n-1} s^{2n-1} + \cdots + d_1 s + d_0$$

where: a_i, b_i, α_i, d_i are all given while c_{0_i} and c_{1_i} are still unknown.

- The equation to be satisfied can now be written as:

$$\boxed{D(s) = P_0(s) \cdot Q_{OL}(s) + P(s) \cdot P_1(s)}$$

- Comparison of coefficients leads to the following matrix equation:

$$\boxed{M \cdot \underline{c} = \underline{d}}$$

where:

$$M = \left[\begin{array}{cccc|cccc}
a_0 & & & & b_0 & & & \\
a_1 & a_0 & \emptyset & & b_1 & b_0 & \emptyset & \\
\vdots & & \ddots & & \vdots & & \ddots & \\
a_{n-1} & a_{n-2} & \cdots & a_0 & b_{n-1} & b_{n-2} & \cdots & b_0 \\
\hline
1 & a_{n-1} & \cdots & a_1 & \emptyset & b_{n-1} & \cdots & b_1 \\
 & 1 & \cdots & a_2 & & \emptyset & \ddots & \vdots \\
\emptyset & & \ddots & \vdots & \emptyset & & \ddots & b_{n-1} \\
 & & & 1 & & & & \emptyset
\end{array}\right]$$

is a concatenation of two lower Toeplitz matrices:

$$M = \left[\, \mathcal{T}_{\ell_0}(\underline{q}_s) \,,\; \mathcal{T}_{\ell_0}(\underline{p}_s) \,\right]$$

$$\underline{c} = [c_{00}; c_{01}; \cdots; c_{0\,n-1}; c_{10}; c_{11}; \cdots; c_{1\,n-1}$$

$$\underline{d} = [d_0; d_1; \cdots; d_{n-1}; d_n; d_{n+1}; \cdots; d_2$$

This is a linear system of order $(2n)$ to be solved for the unknown vector $\underline{c}$.

Without proof: Iff the system is fully controllable and observable (no pole-zero cancellation the M-matrix is always non-singular.

In CTRL-C:

$$c = M \backslash d$$

will provide all coefficients (controller and observer) at once.

- We shall understand later why we were able to design the controller/observer with an observer of degree (n-1) rather than of degree n as before.

Example:

$$G_{OL}(s) = \frac{1}{(s+1)(s-2)} = \frac{1}{s^2 - s - 2}$$

We want:

$$G_{CL}(s) = \frac{10}{s^2 + 5s + 10}$$

We choose the observer slightly faster:

$$Q_{OBS}(s) = s + 3$$

(first order as the system is second order).

$$\Rightarrow \quad P_0(s) = C_{01}s + C_{00}$$

$$P_1(s) = C_{11}s + C_{10}$$

$$D(s) = Q_{OBS}(s)\left[Q_{CL}(s) - Q_{OL}(s)\right]$$

$$= (s+3)\left[(s^2 + 5s + 10) - (s^2 - s - 2)\right]$$

$$= (s+3)(6s+12)$$

$$= 6s^2 + 30s + 36$$

$$\Rightarrow \begin{bmatrix} -2 & 0 & 1 & 0 \\ -1 & -2 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} C_{00} \\ C_{01} \\ C_{10} \\ C_{11} \end{bmatrix} = \begin{bmatrix} 36 \\ 30 \\ 6 \\ 0 \end{bmatrix}$$

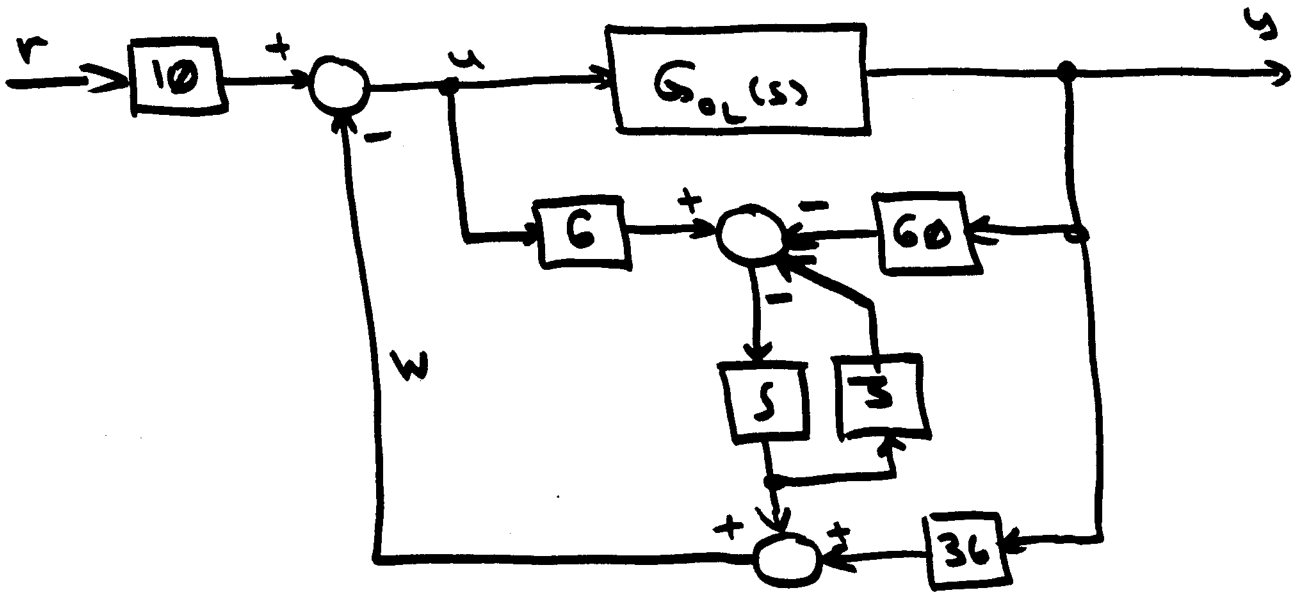
$$\Rightarrow C_{01} = 0 \qquad \Rightarrow C_{00} = 6$$

$$\Rightarrow C_{11} = 36 \qquad \Rightarrow C_{10} = 48$$

$$\Rightarrow C_0(s) = \frac{6}{s+3}$$

$$C_1(s) = \frac{36s+48}{s+3} \equiv 36 + \frac{-60}{s+3}$$

The gain factor of 10 must be realized separately. One solution is:



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(cf. p. 232)

- We start the same way, and find:

$$G_{OL}(s) = \frac{3s+3}{s^3 - 5s^2 - 2s + 24}$$

We select:

$$Q_{CL}(s) = s^3 + 16s^2 + 96s + 256$$

We want:

$$G_{CL}(0) = 1$$

$$\Rightarrow P_{CL}(s) = 256(s+1) = 85.333\ldots$$

We choose the observer poles at:

$$\lambda_{1,2} = -8 \pm 8j$$

$$\Rightarrow Q_{OBS}(s) = (s+8+8j)(s+8-8j)$$

$$= s^2 + 16s + 128$$

$$\Rightarrow D(s) = Q_{OBS}(s)\left[Q_{CL}(s) - Q_{OL}(s)\right]$$

$$= (s^2+16s+128)(21s^2+98s+232)$$

$$= 21s^4 + 434s^3 + 4488s^2 + 16256s + 29696$$

$$\Rightarrow \underbrace{\begin{bmatrix} 24 & 0 & 0 & 3 & 0 & 0 \\ -2 & 24 & 0 & 3 & 3 & 0 \\ -5 & -2 & 24 & 0 & 3 & 3 \\ 1 & -5 & -2 & 0 & 0 & 3 \\ 0 & 1 & -5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}}_{M} \cdot \underbrace{\begin{bmatrix} c_{00} \\ c_{01} \\ c_{02} \\ c_{10} \\ c_{11} \\ c_{12} \end{bmatrix}}_{\underline{c}} = \underbrace{\begin{bmatrix} 29696 \\ 16256 \\ 4488 \\ 434 \\ 21 \\ 0 \end{bmatrix}}_{\underline{d}}$$

$$M \cdot \underline{c} = \underline{d}$$

$$\Rightarrow \quad c = M \backslash d$$

$$\Rightarrow \quad \underline{c} = \begin{bmatrix} 0.8968 \\ 0.021 \\ 0 \\ 2.7247 \\ 3.1238 \\ -0.1192 \end{bmatrix}$$

$$\Rightarrow \quad C_0(s) = \frac{0.021 s + 0.8968}{s^2 + 16 s + 128}$$

$$C_1(s) = \frac{-0.1192 s^2 + 3.1238 s + 2.7247}{s^2 + 16 s + 128}$$

will give the desired closed-loop system behavior.