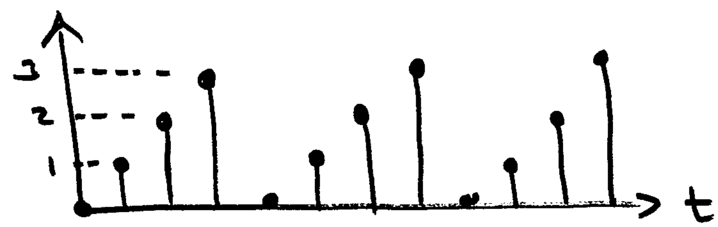
(10) $\boxed{g^*(t)} \longrightarrow \boxed{\mathcal{G}(z)}$

$$g^*(t) = \{ g_0, g_1, g_2, \dots \}$$

$$\mathcal{G}(z) = \sum_{k=0}^{\infty} g_k z^{-k}$$

Example:



$$g^*(t) = \{ 0, 1, 2, 3, 0, 1, 2, 3, \dots \}$$

$$\mathcal{G}(z) = \sum_{k=0}^{\infty} g_k z^{-k} = z^{-1} + 2z^{-2} + 3z^{-3} + z^{-5} + 2z^{-6} + 3z^{-7} + \cdots$$

$$= \left[z^{-1} + 2z^{-2} + 3z^{-3} \right] \cdot \underbrace{\left(1 + z^{-4} + z^{-8} + z^{-12} + \cdots \right)}_{\mathcal{H}(z)}$$

$$\mathcal{H}(z) = 1 + z^{-4} + z^{-8} + z^{-12} + \cdots$$

$$z^{-4}\mathcal{H}(z) = z^{-4} + z^{-8} + z^{-12} + \cdots$$

$$\mathcal{H}(z)\left(1 - z^{-4}\right) = 1 \quad \Longrightarrow \quad \mathcal{H}(z) = \frac{1}{1 - z^{-4}}$$

$$\Rightarrow \mathcal{H}(z) = \frac{z^4}{z^4-1}$$

$$\Rightarrow \underline{\underline{G(z)}} = \frac{z^2+2z+3}{z^3} \cdot \frac{z^4}{z^4-1} = \underline{\underline{\frac{z(z^2+2z+3)}{(z^4-1)}}}$$

(11) $\boxed{G(z)} \longrightarrow \boxed{g^*(t)}$

$$G(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \dots + b_0}{z^n + a_{n-1} z^{n-1} + \dots + a_0} = \sum_{k=0}^{\infty} g_k \cdot z^{-k}$$

Comparison of coefficients leads to the recursive formulae:

$$g_k = b_{n-k} - a_{n-1} g_{k-1} - a_{n-2} g_{k-2} - \dots - a_0 g_{k-n}$$

where: $b_k, g_k \equiv 0 \; ; \; \forall k < 0$

Example: $G(z) = \dfrac{z(z^2+2z+3)}{(z^4-1)} = \dfrac{z^3+2z^2+3z}{z^4-1}$ $\qquad \underline{\underline{n=4}}$

$$g_0 = b_4 - a_3 g_{-1} \dots = b_4 \equiv \underline{\underline{0}}$$

$$g_1 = b_3 - a_3 g_0 = 1 - 0 = \underline{\underline{1}}$$

$$g_2 = b_2 - a_3 g_1 - a_2 g_0 = 2 - 0 - 0 = \underline{\underline{2}}$$

$$g_3 = b_1 - a_3 g_2 - a_2 g_1 - a_1 g_0 = 3 - 0 - 0 - 0 = \underline{\underline{3}}$$

$$g_4 = b_0 - a_3 g_3 - a_2 g_2 - a_1 g_1 - a_0 g_0 = 0 - 0 - 0 - 0 + 1 \cdot 0 = \underline{\underline{1}}$$

$$g_5 = b_{-1} - a_3 g_4 - a_2 g_3 - a_1 g_2 - a_0 g_1 = 0 - 0 - 0 - 0 + 1 \cdot 1 = \underline{\underline{1}}$$

$$g_6 = b_{-2} - a_3 g_5 - a_2 g_4 - a_1 g_3 - a_0 g_2 = 0 - 0 - 0 - 0 + 1 \cdot 2 = \underline{\underline{2}}$$

etc.

✓

(There is a more convenient way to achieve the same result. ⇒ See later!)

- Another way, of course, is to go through the (modified) partial fraction expansion of $G(z)$, and remember which $g^*(t)$ produce the individual terms, thus:

$$G(z) = \frac{Az}{z+a} + \frac{Bz}{z+b} + \frac{C_1 z}{z+c} + \frac{C_2 z}{(z+c)^2}$$

$$\Rightarrow g^*(t) = g_a^*(t) + g_b^*(t) + g_{c_1}^*(t) + g_{c_2}^*(t)$$

Difference Equations:

Remember: $Y^*(s) = G^*(s) \cdot U^*(s)$

$$\Longleftrightarrow \quad \mathcal{Y}(z) = \mathcal{G}(z) \cdot \mathcal{U}(z)$$

$$\Rightarrow \quad \mathcal{G}(z) = \frac{\mathcal{Y}(z)}{\mathcal{U}(z)}$$

Example: $\mathcal{G}(z) = \dfrac{z^3 + 2z^2 + 3z}{z^4 - 1} = \dfrac{\mathcal{Y}(z)}{\mathcal{U}(z)}$

$$\Rightarrow \quad z^4 \cdot \mathcal{Y}(z) - \mathcal{Y}(z) = z^3 \cdot \mathcal{U}(z) + 2z^2 \cdot \mathcal{U}(z) + 3z \cdot \mathcal{U}($$

Now, remember the shifting property.

$$\Rightarrow \quad y^*(t+4T) - y^*(t) = u^*(t+3T) + 2u^*(t+2T) + 3u^*(t+T)$$

or: $y^*(t+T) = y^*(t-3T) + u^*(t) + 2u^*(t-T) + 3u^*(t-2T$

$\Rightarrow$ Given a system with the z-Transfer function $\mathcal{G}(z)$, we can "simulate" the behavior of the output out of measurements of previous inputs and outputs.

In future, we shall simplify the notation by writing:

$$y(k+1) = y(k-3) + u(k) + 2u(k-1) + 3u(k-2)$$

(We omit the "*", and replace the time value by the relative index.)

Another approach:

$$G(z) = \frac{z^3 + 2z^2 + 3z}{z^4 - 1} = \frac{\mathcal{Y}(z)}{\mathcal{U}(z)} = \frac{\mathcal{Y}(z)}{\mathcal{X}(z)} \cdot \frac{\mathcal{X}(z)}{\mathcal{U}(z)}$$

Assume: order (numerator) < order (denominator)

where: $\dfrac{\mathcal{X}(z)}{\mathcal{U}(z)} = \dfrac{z^3}{z^4 - 1}$; $\dfrac{\mathcal{Y}(z)}{\mathcal{X}(z)} = 1 + 2z^{-1} + 3z^{-2}$

$$\Rightarrow z^4 \mathcal{X}(z) - \mathcal{X}(z) = z^3 \mathcal{U}(z)$$

$$\Rightarrow x(k+4) - x(k) = u(k+3)$$

$$\Rightarrow x(k+1) = x(k-3) + u(k)$$

Let us call:

$$x_1(k) = x(k-3)$$
$$x_2(k) = x(k-2)$$
$$x_3(k) = x(k-1)$$
$$x_4(k) = x(k)$$

$$\Rightarrow \quad \begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= x_3(k) \\ x_3(k+1) &= x_4(k) \\ x_4(k+1) &= x_1(k) + u(k) \end{aligned} \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

<u>or</u>: $$\underline{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$\mathcal{Y}(z) = \mathcal{X}(z) + 2z^{-1}\mathcal{X}(z) + 3z^{-2}\mathcal{X}(z)$$

$$\Rightarrow \quad y(k) = x(k) + 2x(k-1) + 3x(k-2)$$

$$\Rightarrow \quad y(k) = x_4(k) + 2x_3(k) + 3x_2(k)$$

<u>or</u>: $$y(k) = \begin{bmatrix} 0 & 3 & 2 & 1 \end{bmatrix} \cdot \underline{x}(k)$$

<u>where</u>: $$\underline{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}$$

Generalize:

Given $$g(z) = \frac{b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \cdots + b_1 z + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0}$$

$$\Rightarrow \left| \begin{aligned} \underline{x}(k+1) &= F \cdot \underline{x}(k) + \underline{g} \cdot u(k) \\ y(k) &= \underline{h}' \cdot \underline{x}(k) + i \cdot u(k) \end{aligned} \right|$$

F : System matrix

$\underline{g}$: Input (column) vector

$\underline{h}'$: Output (row) vector

i : direct coupling

$$F = \begin{bmatrix} & 1 & & & 0 \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{bmatrix} \qquad \underline{g} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\underline{h}' = [b_0 \; b_1 \; \cdots \; b_{n-1}] \qquad i \equiv 0$$

- If the order of the numerator polynomial is the same as the order of the denominator polynomial, we start by dividing through:

$$\mathcal{G}(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}$$

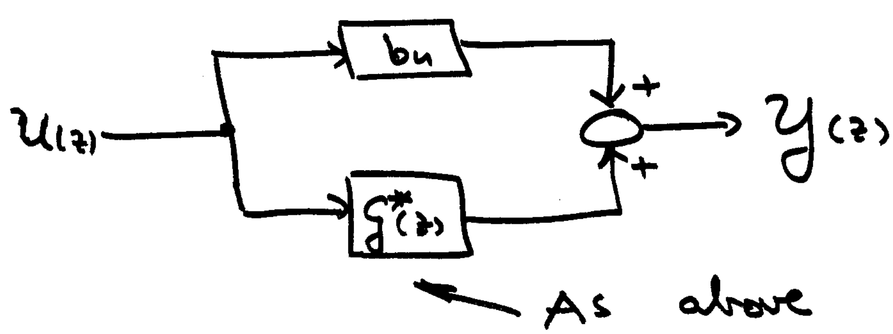
$$\equiv b_n + \frac{b_{n-1}^* z^{n-1} + \cdots + b_1^* z + b_0^*}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}$$

$$\equiv b_n + \mathcal{G}^*(z) = \frac{\mathcal{Y}(z)}{\mathcal{U}(z)}$$

$$\Rightarrow \mathcal{Y}(z) = \mathcal{Y}_1(z) + \mathcal{Y}_2(z)$$

<u>where</u>: $\mathcal{Y}_1(z) = b_n \cdot \mathcal{U}(z)$

$\mathcal{Y}_2(z) = \mathcal{G}^*(z) \cdot \mathcal{U}(z)$



$\Rightarrow$ F, $\underline{g}$, $\underline{h}'$ as above, but now:

$i \equiv b_n$

- These are called discrete state-space representations.
- Like their continuous counterparts, they are not unique:

$$\underline{\xi}(k) = T \cdot \underline{x}(k) \qquad ; \; T^{-1} \text{ exists}$$

$$\Rightarrow \quad \begin{aligned} \underline{x}(k) &= T^{-1} \underline{\xi}(k) \\ \underline{x}(k+1) &= T^{-1} \underline{\xi}(k+1) \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} T^{-1} \underline{\xi}(k+1) &= F \cdot T^{-1} \underline{\xi}(k) + \underline{g}\, u(k) \\ y(k) &= \underline{h}' T^{-1} \underline{\xi}(k) + i\, u(k) \end{aligned}$$

$$\Rightarrow \left| \begin{aligned} \underline{\xi}(k+1) &= \hat{F} \underline{\xi}(k) + \hat{\underline{g}}\, u(k) \\ y(k) &= \hat{\underline{h}}' \underline{\xi}(k) + \hat{i}\, u k) \end{aligned} \right|$$

is another state-space representation

where:

$$\left|\begin{array}{l} \hat{F} = T \cdot F \cdot T^{-1} \\ \underline{\hat{g}} = T \cdot \underline{g} \\ \underline{\hat{h}}' = \underline{h}' \cdot T^{-1} \\ \hat{i} = i \end{array}\right|$$

$\Rightarrow$ As the similarity transformations are just the same, all canonical forms are the same in both cases, and the transformation algorithms are the same also.

Given:

$$\left|\begin{array}{l} \underline{x}(k+1) = F\underline{x}(k) + \underline{g}\,u(k) \\ y(k) = \underline{h}'\underline{x}(k) + i\,u(k) \end{array}\right|$$

$$\underline{x}(k+1) \circ\!\!-\!\!\bullet\; z \cdot \underline{x}(k)$$

$$\Rightarrow \quad z\,\underline{x}(k) = F\underline{x}(k) + \underline{g}\,u(k)$$

$$\Rightarrow \quad (zI - F)\underline{x}(k) = \underline{g}\,u(k)$$

$$\Rightarrow \quad \underline{x}(k) = (zI-F)^{-1} \cdot \underline{g}\,u(k)$$

$$\Rightarrow \quad y(k) = \Big[\underbrace{\underline{h}'(zI-F)^{-1} \cdot \underline{g} + i}_{G(z)}\Big]\,u(k)$$