Compensator - Design in the z-domain

Given: $$\mathcal{G}(z) = \frac{\mathcal{N}(z)}{\mathcal{D}(z)}$$

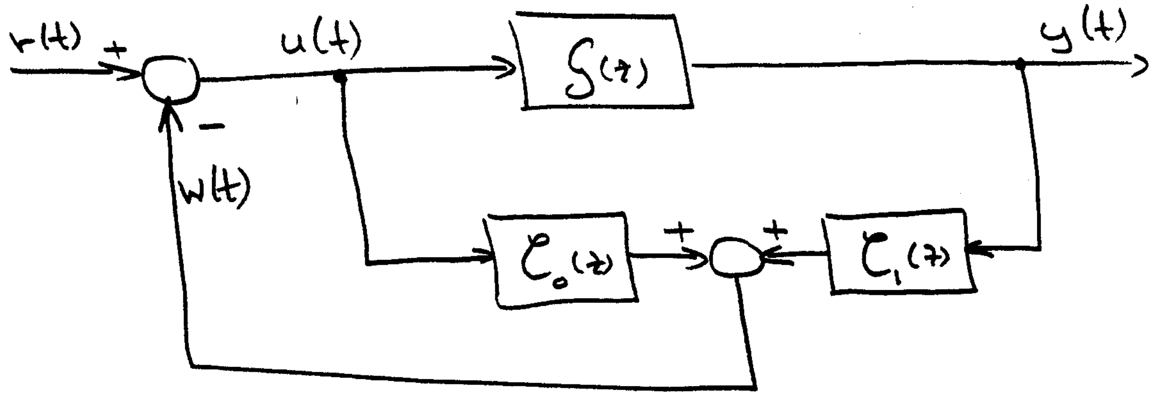
We want to design a dynamic compensator with output feedback such that:

$$\mathcal{G}_{tot}(z) = \frac{\mathcal{N}(z)}{\mathcal{D}_{CL}(z)}$$

(Remember: With state-feedback design, the zeroes don't change.)

- $\mathcal{D}_{CL}(z)$ are our "controller poles" that can be placed at will.

For this purpose, we choose the following structure of our control system:

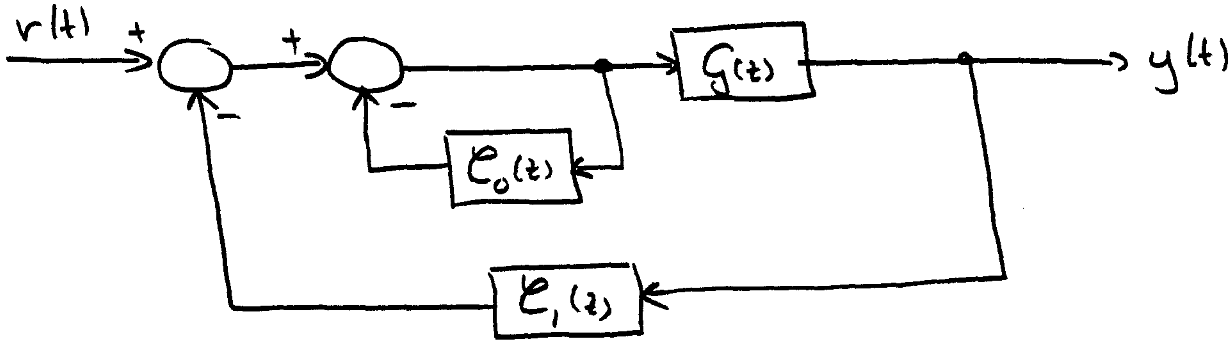


where:

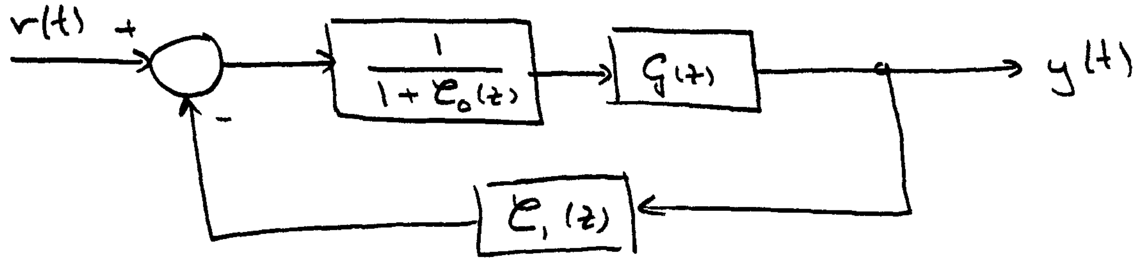
$$\mathcal{C}_0(z) = \frac{N_0(z)}{\mathcal{D}_{OB}(z)} \quad ; \quad \mathcal{C}_1(z) = \frac{N_1(z)}{\mathcal{D}_{OB}(z)}$$

Thus, the control system uses two compensators with the same denominator $\mathcal{D}_{OB}(z)$ which represents our "observer poles" that do not show on the total transfer function as they are all uncontrollable.

From the above block diagram, we find:



$\equiv$



$$\Rightarrow \quad G_{tot}(z) = \frac{\dfrac{G(z)}{1+C_0(z)}}{1+\dfrac{G(z)\cdot C_1(z)}{1+C_0(z)}}$$

$$\Rightarrow \quad G_{tot}(z) = \frac{G(z)}{1+C_0(z)+G(z)\cdot C_1(z)}$$

$$= \frac{\dfrac{N(z)}{D(z)}}{1+\dfrac{N_0(z)}{D_{0R}(z)}+\dfrac{N(z)\cdot N_1(z)}{D(z)\cdot D_{0R}(z)}}$$

$$\Rightarrow \quad G_{tot}(z) = \frac{\mathcal{N}(z) \cdot \mathcal{D}_{OB}(z)}{\mathcal{D}(z) \cdot \mathcal{D}_{OB}(z) + \mathcal{N}_0(z) \cdot \mathcal{D}(z) + \mathcal{N}(z) \cdot \mathcal{N}_1(z)}$$

$$\stackrel{!}{\equiv} \quad \frac{\mathcal{N}(z)}{\mathcal{D}_{CL}(z)} = \frac{\mathcal{N}(z) \cdot \mathcal{D}_{OB}(z)}{\mathcal{D}_{CL}(z) \cdot \mathcal{D}_{OB}(z)}$$

As can be seen, the "observer poles" will again cancel away in this design.

$$\Rightarrow \mathcal{D}_{CL} \cdot \mathcal{D}_{OB}(z) - \mathcal{D}(z) \cdot \mathcal{D}_{OB}(z) \stackrel{!}{\equiv} \mathcal{N}_0(z) \cdot \mathcal{D}(z) + \mathcal{N}_1(z) \cdot \mathcal{N}(z)$$

<u>or</u>:

$$\mathcal{D}^*(z) = \mathcal{D}_{OB}(z) \cdot \left[\mathcal{D}_{CL}(z) - \mathcal{D}(z)\right] \stackrel{!}{\equiv} \mathcal{N}_0(z) \cdot \mathcal{D}(z) + \mathcal{N}_1(z) \cdot \mathcal{N}(z)$$

<u>where</u>: $\mathcal{N}_0(z)$ and $\mathcal{N}_1(z)$ are the unknowns.

[<u>Remember</u>: The observer poles can be chosen freely.]

As was explained earlier (but never given an algorithm for), the lowest possible order of the observer $\mathcal{D}_{OB}(z) = n-p$. (To observe n state variables with p outputs, we require at least $(n-p)$ observer poles. For the case of the single-output system:

$$\text{ord}\{\mathcal{D}_{OB}(z)\} = (n-1)$$

$\Longrightarrow$ For reasons of realizability:

$$\text{ord}\{\mathcal{N}_0(z)\} \leq (n-1)$$
$$\text{ord}\{\mathcal{N}_1(z)\} \leq (n-1)$$

We can write our polynomials thus as follows:

$$\mathcal{D}(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

$$\mathcal{N}(z) = b_m z^m + \cdots + b_1 z + b_0$$

$$\mathcal{D}_{CL}(z) = z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0$$

$$\mathcal{D}_{OB}(z) = z^{n-1} + \beta_{n-2} z^{n-2} + \cdots + \beta_1 z + \beta_0$$

$$\mathcal{N}_0(z) = c_{0\,n-1} z^{n-1} + \cdots + c_{01} z + c_{00}$$

$$\mathcal{N}_1(z) = c_{1\,n-1} z^{n-1} + \cdots + c_{11} z + c_{10}$$

$$\rightarrow \mathcal{D}^*(z) = \mathcal{D}_{OB}(z) \left[\mathcal{D}_{CL}(z) - \mathcal{D}(z) \right] =$$

$$= d_{2n-1} z^{2n-1} + \cdots + d_1 z + d_0$$

$$\boxed{\mathcal{D}^*(z) \overset{!}{\equiv} \mathcal{N}_0(z) \cdot \mathcal{D}(z) + \mathcal{N}_1(z) \cdot \mathcal{N}(z)}$$

Comparison of coefficients:

z^0 :: $d_0 \overset{!}{=} c_{00} a_0 + c_{10} b_0$

z^1 :: $d_1 \overset{!}{=} c_{01} a_0 + c_{00} a_1 + c_{11} b_0 + c_{10} b_1$

<u>etc.</u>

This can be written in a matrix form:

$$\left[\begin{array}{cccccc|cccccc}
a_0 & & & & & & b_0 & & & & & \\
a_1 & a_0 & & \emptyset & & & b_1 & b_0 & & \emptyset & & \\
a_2 & a_1 & a_0 & & & & \vdots & & \ddots & & & \\
\vdots & & \ddots & \ddots & & & b_m & \cdots & \cdots & b_0 & & \\
a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 & & & b_m & \cdots & b_1 & b_0 & \\
1 & a_{n-1} & \cdots & \cdots & a_1 & & & & \ddots & \vdots & \ddots & b_0 \\
& 1 & a_{n-1} & \cdots & a_2 & & & & & b_m & \cdots & \vdots \\
& & \ddots & & \vdots & & & \emptyset & & & \ddots & \vdots \\
& \emptyset & & \ddots & 1 & & & & & & & b_m
\end{array}\right] \cdot \begin{bmatrix} c_{00} \\ c_{01} \\ \vdots \\ c_{0_{n-1}} \\ c_{10} \\ c_{11} \\ \vdots \\ c_{1_{n-1}} \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ \vdots \\ d_{2n-2} \\ d_{2n-1} \end{bmatrix}$$

$\Rightarrow$ This is a linear system of $2n$ equations for the $2n$ unknowns $c_{00} \cdots c_{0_{n-1}}$ and $c_{10} \cdots c_{1_{n-1}}$.

Example:
$$G(z) = \frac{1}{(z+1)(z-2)} = \frac{1}{z^2 - z - 2}$$

is instable.

We choose our controller poles both at zero (deadbeat response):

$$G_{tot}(z) = \frac{1}{z^2}$$

$$\Rightarrow \quad \begin{aligned} \underline{x}(k+1) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}(k) \end{aligned}$$

is a realization of the closed-loop system.

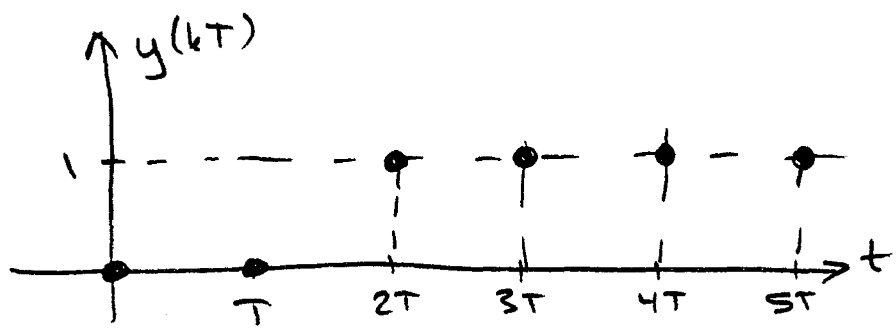
$\Rightarrow$ step response:

$u(k) \equiv 1$; $\underline{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow y(0) = 0$

$\underline{x}(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Longrightarrow y(1) = 0$

$\underline{x}(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Longrightarrow y(2) = 1$

$\underline{x}(3) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Longrightarrow y(3) = 1$ etc.



$$G_{tot}(z) = \frac{1}{z^2}$$

We choose the one observer pole at

$$D_{OB}(z) = z - 0.5$$

$$\Rightarrow D^*(z) = D_{OB}(z)\left[D_{CL}(z) - D(z)\right]$$

$$= (z - 0.5) \cdot \left[z^2 - z^2 + z + 2\right]$$

$$= (z - 0.5)(z+2) = z^2 + 1.5z - 1$$

$$\Rightarrow \left[\begin{array}{cc|cc} -2 & 0 & 1 & 0 \\ -1 & -2 & 0 & 1 \\ \hline 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \cdot \begin{bmatrix} c_{00} \\ c_{01} \\ c_{10} \\ c_{11} \end{bmatrix} = \begin{bmatrix} -1 \\ 1.5 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \underline{\underline{c_{01} = 0}} \; ;$$

$$c_{00} - c_{01} = 1 \Rightarrow \underline{\underline{c_{00} = 1}} \; ;$$

$$-c_{00} - 2c_{01} + c_{11} = 1.5 \Rightarrow \underline{\underline{c_{11} = 2.5}}$$

$$-2c_{00} + c_{10} = -1 \Rightarrow \underline{\underline{c_{10} = 1}}$$

$$\Rightarrow \mathcal{C}_0(z) = \frac{1}{z - 0.5} \; ; \; \mathcal{C}_1(z) = \frac{2.5z + 1}{z - 0.5}$$

<u>Realization:</u>

(α) $\mathcal{C}_0(z) = \dfrac{1}{z - 0.5}$:

$$\Rightarrow \left| \begin{array}{l} x_0(k+1) = [0.5]\, x_0(k) + [1]\, u(k) \\ y_0(k) = [\,1\,]\, x_0(k) \end{array} \right|$$

(β) $\mathcal{C}_1(z) = \dfrac{2.5z+1}{z - 0.5} = 2.5 + \dfrac{2.25}{z - 0.5}$

$$\Rightarrow \left| \begin{array}{l} x_1(k+1) = [0.5]\, x_1(k) + [1]\, y(k) \\ y_1(k) = [2.25]\, x_1(k) + [2.5]\, y(k) \end{array} \right|$$

(γ) $\left| w(k) = y_0(k) + y_1(k) \right|$

Connect to one system:

$$\text{"}\underline{u}\text{"} = \begin{bmatrix} u \\ y \end{bmatrix} \; ; \; \text{"}\underline{x}\text{"} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \; ; \; \text{"}y\text{"} = [w]$$

$$\Rightarrow \left| \begin{array}{l} \underline{x}(k+1) \left[\begin{array}{c|c} 0.5 & 0 \\ \hline 0 & 0.5 \end{array}\right] \underline{x}(k) + \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right] \underline{u}(k) \\ y(k) = \left[\begin{array}{c|c} 1 & 2.25 \end{array}\right] \underline{x}(k) + \left[\begin{array}{c|c} 0 & 2.5 \end{array}\right] \underline{u}(k) \end{array} \right|$$

Check whether this is a minimal realization:

$$Q_c = \left[\begin{array}{cc|cc} 1 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0.5 \end{array}\right]$$

$\text{Rank}(Q_c) = 2 \Rightarrow$ controllable. ✓

$$Q_o = \left[\begin{array}{cc} 1 & 2.25 \\ \hline 0.5 & 1.125 \end{array}\right]$$

$\det(Q_o) \equiv 0 \Rightarrow \text{Rank}(Q_o) = 1$

$\Rightarrow$ There is one unobservable mode to be output-decoupled.

$$\Rightarrow \underline{\underline{T \equiv \hat{Q}_o}} \; ; \; \hat{Q}_o = \left[\begin{array}{cc} 1 & 2.25 \\ \hline 0 & 1 \end{array}\right]$$

$$\Rightarrow \det(\hat{Q}_o) = 1 \Rightarrow \hat{Q}_o^{-1} = \hat{Q}_o^{+} = T^{-1} = \begin{bmatrix} 1 & -2.25 \\ \phi & 1 \end{bmatrix}$$

$$\Rightarrow \hat{F} = T \cdot F \cdot T^{-1} = \begin{bmatrix} \phi.5 & \phi \\ \phi & \phi.5 \end{bmatrix}$$

$$\hat{G} = T \cdot G = \begin{bmatrix} 1 & 2.25 \\ \phi & 1 \end{bmatrix}$$

$$\underline{\hat{h}}' = \underline{h}' \cdot T^{-1} = \begin{bmatrix} 1 & \phi \end{bmatrix}$$

$$\underline{\hat{i}} = \underline{i} = \begin{bmatrix} \phi & 2.5 \end{bmatrix}$$

$$\Rightarrow \left| \begin{aligned} \underline{\xi}(k+1) &= \left[\begin{array}{c|c} \phi.5 & \phi \\ \hline \phi & \phi.5 \end{array}\right] \underline{\xi}(k) + \left[\begin{array}{c|c} 1 & 2.25 \\ \hline \phi & 1 \end{array}\right] \underline{u}(k) \\ y(k) &= \left[\begin{array}{c|c} 1 & \phi \end{array}\right] \underline{\xi}(k) + \left[\begin{array}{c|c} \phi & 2.5 \end{array}\right] \underline{u}(k) \end{aligned} \right|$$

ξ_2 is unobservable

$$\Rightarrow \left| \begin{aligned} \mu(k+1) &= [\phi.5]\,\mu(k) + \left[\begin{array}{c:c} 1 & 2.25 \end{array}\right] \underline{u}(k) \\ y(k) &= [\,1\,]\,\mu(k) + \left[\begin{array}{c:c} \phi & 2.5 \end{array}\right] \underline{u}(k) \end{aligned} \right|$$

is a minimal realization of this controller.

<u>Program</u>:

```
[Unew] = COMP(R, UOLD, YOLD)
W = X + 2.5*YOLD
X = Ø.5*X + UOLD + 2.25*YOLD
UNEW = R - W
RETURN
```

assuming that X will be zero initially, and maintains its value over consecutive calls of the routine.

Notice :

(1) In this design, we were able to design the state feedback ($D_{CL}(z)$) and the observer ($D_{OB}(z)$) simultaneously.

(2) We obtain directly a minimum order observer.

(3) This design is often referred to as functional observer design.