Controllability & Observability:

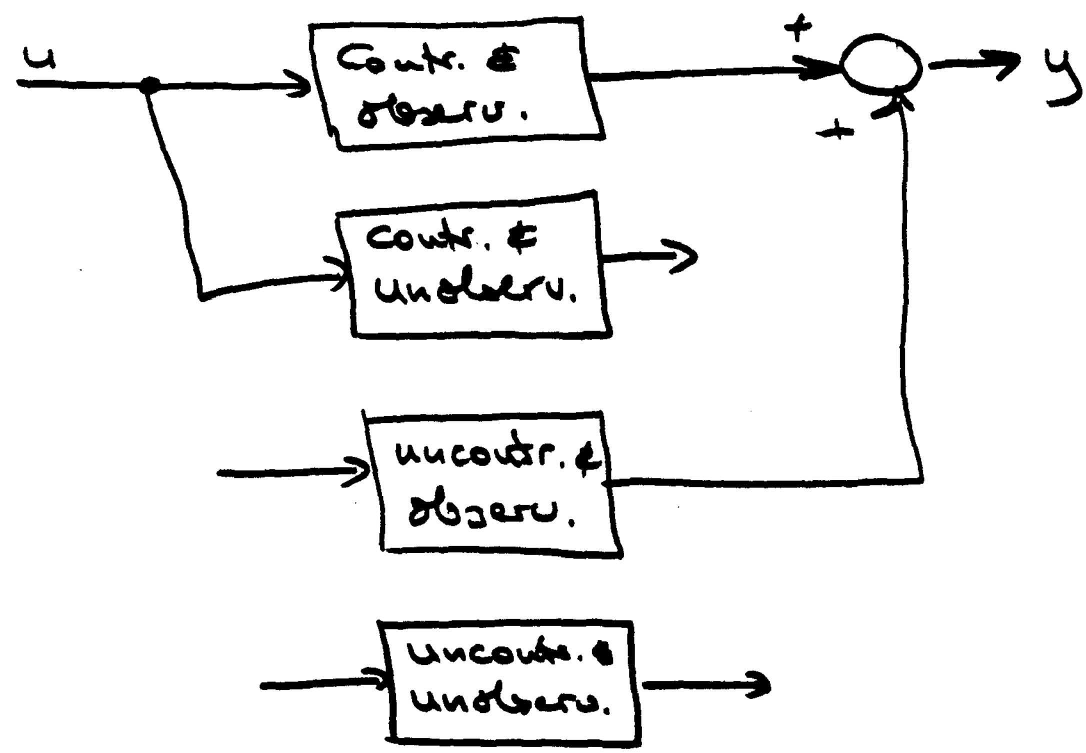
In ECE 441, we have introduced the concepts of controllability and observability of linear continuous systems.

We have seen that:

(1) Problems with either of the two properties lead to pole/zero-cancellations in the frequency domain ⟹ the "true" system order (its minimal representation) has fewer numbers of states than the currently used representation suggests.

(2) The transfer function represents the completely controllable and observable subsystem

(3) Each system can be decomposed into upto 4 subsystems:



This is the so-called Kalman decomposition of the system.

(4) A MIMO-system is controllable iff its controllability matrix

$$Q_c = \left[B \,\vdots\, A \cdot B \,\vdots\, A^2 \cdot B \,\vdots\, \cdots \,\vdots\, A^{n-1} \cdot B \right]$$

has the full rank.

Algorithm to determine the rank of a matrix:

$$\text{Build: } M_1 = Q_c \cdot Q_c^{*} \quad \leftarrow \text{conjugate-complex transpose}$$

$$\text{or: } M_2 = Q_c^{*} \cdot Q_c$$

whatever is smaller. Then calculate the eigenvalues of M_1 or M_2. The rank of $Q_c \equiv$ (# eigenvalues $\neq \emptyset$).

(5) A MIMO-system is observable iff its observability matrix

$$Q_o = \begin{bmatrix} C \\ \hline C \cdot A \\ \hline C \cdot A^2 \\ \hline \vdots \\ \hline C \cdot A^{n-1} \end{bmatrix}$$

has full rank.

As we have exactly the same algebraic structure:

$$\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} + D\underline{u} \end{array} \right| \Longleftrightarrow G(s) = C(sI^{(n)} - A)^{-1}B + D$$

$$\left| \begin{array}{l} \underline{x}(k+1) = F\underline{x}(k) + G\underline{u}(k) \\ \underline{y}(k) = H\,\underline{x}(k) + I\underline{u}(k) \end{array} \right| \Longleftrightarrow \mathcal{G}(z) = H(zI^{(n)} - A)^{-1}G + I$$

we obviously get the same pole/zero cancellations of $\mathcal{G}(z)$ iff:

$$\mathrm{Rank}(Q_c) < n$$

where: $Q_c = [G \,\vdots\, F \cdot G \,\vdots\, \cdots \,\vdots\, F^{n-1} G]$

or: $$\mathrm{Rank}(Q_o) < n$$

where: $$Q_o = \begin{bmatrix} H \\ \hdashline H \cdot F \\ \hdashline \vdots \\ \hdashline H \cdot F^{n-1} \end{bmatrix}.$$

Algorithm to input-decouple uncontrollable modes:

$$\left| \begin{array}{l} \underline{x}(k+1) = F\cdot\underline{x}(k) + \underline{g}\cdot u(k) \\ y(k) = \underline{h}'\cdot\underline{x}(k) + i\cdot u(k) \end{array} \right| \qquad \underline{x} \in \mathbb{R}^n$$

$$\Rightarrow \quad Q_c = \left[\, \underline{g} \;\vdots\; F\underline{g} \;\vdots\; \cdots \;\vdots\; F^{(n-1)}\underline{g} \,\right]$$

r uncontrollable modes

$$\Longleftrightarrow \quad \text{Rank}(Q_c) = (n-r) < n$$

- Choose $(n-r)$ linearly independent columns of Q_c (any) $\Rightarrow \hat{Q}_{c_1}$

$$n\;\underbrace{\boxed{\;\hat{Q}_{c_1}\;}}_{(n-r)} \quad ; \quad \text{Rank}(\hat{Q}_{c_1}) = (n-r)$$

- Extend this matrix from the right by anything that makes the rank full:

$$\hat{Q}_c = \left[\hat{Q}_{c_1} \,\vdots\, \hat{Q}_{c_2} \right] \quad ; \quad \text{Rank}(\hat{Q}_c) = n$$

- Apply a similarity transformation with $T = \hat{Q}_c^{-1}$

$$\Rightarrow \left| \begin{array}{l} \underline{\underline{\xi}}(k+1) = \hat{F} \cdot \underline{\underline{\xi}}(k) + \underline{\hat{g}}\, u(k) \\ y(k) = \underline{\hat{h}}' \cdot \underline{\underline{\xi}}(k) + i \cdot u(k) \end{array} \right|$$

where:

$$\hat{F} = \underbrace{}_{}\left[\begin{array}{c|c} F_c & F_{12} \\ \hline \phi & F_{\bar{c}} \end{array} \right] \begin{array}{l} \}(n-r) \\ \}r \end{array} \qquad \underline{\hat{g}} = \left[\begin{array}{c} \hat{g}_1 \\ \hline \phi \end{array} \right] \begin{array}{l} \}(n-r) \\ \}r \end{array}$$

$$\begin{array}{cc} \underbrace{\qquad\qquad}_{(n-r)} & \underbrace{\qquad}_{r} \end{array}$$

$$\Rightarrow \underline{\underline{\xi}} = \left[\begin{array}{c} \underline{\underline{\xi}}_1 \\ \hline \underline{\underline{\xi}}_2 \end{array} \right] \begin{array}{l} \}(n-r) \\ \}r \end{array}$$

$$\boxed{\underline{\xi}_2(k+1) = F_{\bar{c}} \cdot \underline{\xi}_2(k)} \qquad \underline{\xi}_2 \in \mathbb{R}^r$$

does not depend on any input
$\Rightarrow$ uncontrollable modes.

- As F is in a block-triangular form:

$$\mathrm{eig}(F) = \{ \underbrace{\mathrm{eig}(F_c)}_{\text{controllable modes}}, \underbrace{\mathrm{eig}(F_{\bar{c}})}_{\text{uncontrollable modes}} \}$$

- Make sure that all the uncontrollable modes are <u>stable</u>, otherwise the system is no good at all.

Example:

$$\underline{x}(k+1) = \begin{bmatrix} -2 & -2 & 2 & 3 \\ 0 & -2 & 0 & 0 \\ 1 & 1 & -3 & -3 \\ -2 & 0 & 4 & 5 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 5 & -4 & -8 & -12 \end{bmatrix} \underline{x}(k)$$

$$Q_c = \begin{bmatrix} 2 & -5 & 11 & -23 \\ 1 & -2 & 4 & -8 \\ -1 & 3 & -7 & 15 \\ 1 & -3 & 7 & -15 \end{bmatrix}$$

$$\Rightarrow Q_c \cdot Q_c^* = \begin{bmatrix} 679 & 240 & -439 & 439 \\ 240 & 85 & -155 & 155 \\ -439 & -155 & 284 & -284 \\ 439 & 155 & -284 & 284 \end{bmatrix}$$

$$\Rightarrow \operatorname{eig}(Q_c \cdot Q_c^*) = \{ 1331.56 \quad 0.4318 \quad 0 \quad 0 \}$$

$\Rightarrow$ 2 eigenvalues at origin

$$\Rightarrow \operatorname{Rank}(Q_c) = 2$$

$\Rightarrow$ 2 controllable modes

2 uncontrollable modes

$$\hat{Q}_{c_1} = \underbrace{\begin{bmatrix} 2 & -5 \\ 1 & -2 \\ -1 & 3 \\ 1 & -3 \end{bmatrix}}_{\text{linearly independent}}$$

We choose e.g. $\hat{Q}_{c_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \hat{Q}_c = \begin{bmatrix} 2 & -5 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ -1 & 3 & 1 & 0 \\ 1 & -3 & 0 & 1 \end{bmatrix} \; ; \quad \det(\hat{Q}_c) \equiv 1$$

$$\Rightarrow \operatorname{Rank}(\hat{Q}_c) = 4$$

$$\Rightarrow T = \hat{Q}_c^{-1} = \begin{bmatrix} -2 & 5 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \hat{F} = T \cdot F \cdot T^{-1} = \left[\begin{array}{cc|cc} 0 & -2 & -4 & -6 \\ 1 & -3 & -2 & -3 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

$$\hat{\underline{g}} = T \cdot \underline{g} = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] \qquad \hat{\underline{h}}' = \left[\begin{array}{cc|cc} 2 & -5 & -8 & -12 \end{array}\right]$$

$\Rightarrow$ Controllable subsystem:

$$\underline{\xi}_1(k+1) = \begin{bmatrix} \phi & -2 \\ 1 & -3 \end{bmatrix} \underline{\xi}_1(k) + \begin{bmatrix} 1 \\ \phi \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 2 & -5 \end{bmatrix} \underline{\xi}_1(k)$$

appears in controllability canonical form (always in this transformation) $\Rightarrow$ This is input decoupled

Uncontrollable modes:

$$\underline{\xi}_2(k+1) = \begin{bmatrix} -1 & \phi \\ 2 & 2 \end{bmatrix} \underline{\xi}_2(k)$$

$$\mathrm{eig}(F_{\bar{c}}) = \{-1 \quad 2\}$$

$\Rightarrow |\lambda_2| > 1 \quad \Rightarrow$ instable

$\Rightarrow$ The system is no good. No controller can make it any good.

Duality principle:

Given:

$$\left| \begin{aligned} \underline{x}_1(k+1) &= F \cdot \underline{x}_1(k) + G\,\underline{u}_1(k) \\ \underline{y}_1(k) &= H \cdot \underline{x}_1(k) + I \cdot \underline{u}_1(k) \end{aligned} \right|$$

and:
$$\left| \begin{aligned} \underline{x}_2(k+1) &= F'\,\underline{x}_2(k) + H' \cdot \underline{u}_2(k) \\ \underline{y}_2(k) &= G'\,\underline{x}_2(k) + I' \cdot \underline{u}_2(k) \end{aligned} \right|$$

or:
$$S_1 = \begin{array}{c} \overbrace{}^{n}\ \overbrace{}^{m} \\ \left[\begin{array}{c|c} F & G \\ \hline H & I \end{array}\right] \end{array} \begin{array}{l} \}n \\ \}p \end{array}$$

and:
$$S_2 = \begin{array}{c} \overbrace{}^{n}\ \overbrace{}^{p} \\ \left[\begin{array}{c|c} F' & H' \\ \hline G' & I' \end{array}\right] \end{array} \begin{array}{l} \}n \\ \}m \end{array} \equiv S_1'$$

These two systems are <u>not</u> related by a similarity transforma=

tion (if $m \neq p$, even the # of inputs and outputs is different), <u>but</u> they are <u>dual systems</u>.

<u>Lemma</u>: IF S_1 is controllable $\Leftrightarrow$ S_2 is observable, and vice versa. IF S_1 is observable $\Leftrightarrow$ S_2 is controllable.

$\Rightarrow$ Controllability and observability change their role.

(Proof not given here).

Algorithm to output-decouple unobservable modes:

- $Q_o = \begin{bmatrix} \underline{h}' \\ \hdashline \underline{h}'F \\ \hdashline \vdots \\ \hdashline \underline{h}'F^{(n-1)} \end{bmatrix}$ has $\mathrm{Rank}(Q_o) = (n-r) < n$

$\Rightarrow$ r unobservable modes.

- Choose $(n-r)$ linearly independent rows of $Q_o \Rightarrow \hat{Q}_{o_1}$

$$\begin{array}{c} n \\ \boxed{\quad \hat{Q}_{o_1} \quad} \end{array} (n-r)$$

- Extend from below with anything that makes the rank full:

$$\hat{Q}_o = \begin{bmatrix} \hat{Q}_{o_1} \\ \hdashline \hat{Q}_{o_2} \end{bmatrix} \quad ; \mathrm{Rank}(\hat{Q}_o) \equiv n$$

- Apply the similarity transformation

$$T = \hat{Q}_o$$

$$\Rightarrow \quad \hat{F} = \left[\begin{array}{c|c} F_o & \phi \\ \hline F_{21} & F_{\bar{o}} \end{array}\right] \begin{array}{l} \}(n-r) \\ \}r \end{array} \quad \underline{\hat{g}} = \left[\begin{array}{c} \underline{g}_1 \\ \hline \underline{g}_2 \end{array}\right]$$

$$\qquad \underbrace{\phantom{F_{21}}}_{(n-r)} \quad \underbrace{\phantom{F_{\bar{o}}}}_{r}$$

$$\underline{\hat{h}}' = \left[\begin{array}{c|c} \underline{h}_1' & \phi \end{array}\right]$$

$\Rightarrow$ <u>Observable subsystem</u>:

$$\left|\begin{array}{l} \underline{\xi}_1(k+1) = F_o\,\underline{\xi}_1(k) + \underline{g}_1 u(k) \\ y(k) = \underline{h}_1' \cdot \underline{\xi}_1(k) + i \cdot u(k) \end{array}\right|$$

<u>Unobservable modes</u>:

$$\mathrm{Eig}(F_{\bar{o}})$$

Alternative Algorithm (Duality principle)

(1) Find the dual system.

(2) Input-decouple the dual system

(3) Find the dual system of the resulting reduced-order model.

$\Rightarrow$ This is output-decoupled.