Compensator Design

In ECE 441, we have seen that LEAD- and or LAG-compensators can be used to achieve the following properties:

→ steady-state error < ...% to .... input

→ ω_{3dB} > ... s^{-1}

→ $\varphi_r(\omega_{3dB})$ = ...° ± ...°

- The phase reserve (φ_r) was usually chosen around 45°, as this relates approximately to 5% overshoot. For higher order systems, this works even better than the corresponding ±45° rule for the root locus, as the effects of neighboring poles / zeroes are better taken into account.

- The bandwidth (ω_{3dB}) relates to the settling time:

$$T_S \approx \frac{2\pi}{\omega_{3dB}}$$

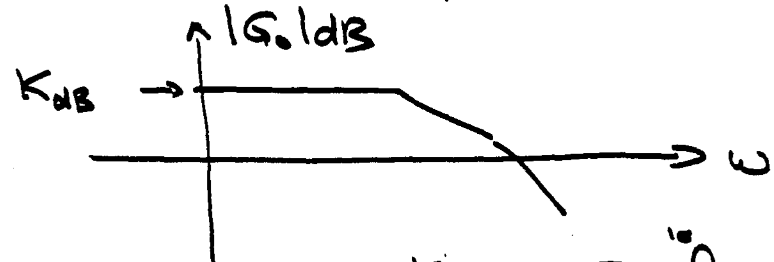
(for a critically damped system, that is: a system with 5% overshoot).

The process of design was as follows:

(1) Satisfy the steady-state error condition by either modifying the type of the system (if stability allows) <u>or</u> by setting the open-loop gain accordingly.

Examples:

a) System is type 0, steady-state error to step $< 2\%$:



$$K_{dB} = 20\,{}^{10}\log K_0$$

assume: $K_{dB} = 20\,dB$

$$\Rightarrow {}^{10}\log K_0 = 1 \Rightarrow \underline{\underline{K_0 = 10}}$$

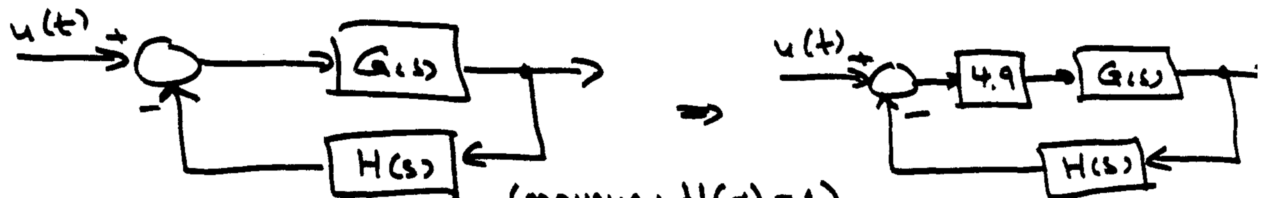
$$\Rightarrow e(\infty) = \frac{1}{1+K_0} = \frac{1}{11} = 9.09\,\% > 2$$

$\Rightarrow$ Either we raise the system type to 1 (if stability allows) $\Longrightarrow e(\infty) \equiv 0$, or we increase the open loop gain:

We want: $\dfrac{1}{1+K_0^*} \leq 2\% = 0.02 = \dfrac{1}{50}$

$$\Rightarrow \underline{\underline{K_0^* \geq 49}} \; ; \; \underline{\underline{K_0^* = K_0 \cdot K_c}}$$

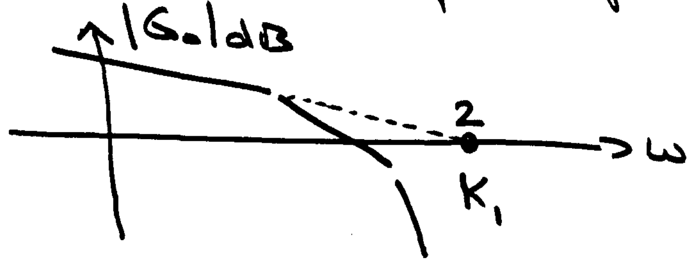
$\Rightarrow$ We need an extra gain factor of $K_c = 4.9$, thus:



(assume: $H(0) = 1$)

$e(\infty) = 9\%$ $\qquad\qquad$ $e(\infty) = 2\%$

b) System is type 1, steady-state error to ramp input $< 5\%$:



$$\Rightarrow e(\infty) = \frac{1}{K_1} = \frac{1}{2} = 50\%$$

We want: $e(\infty) = \dfrac{1}{K_1^*} \leq 5\% = \dfrac{1}{20}$

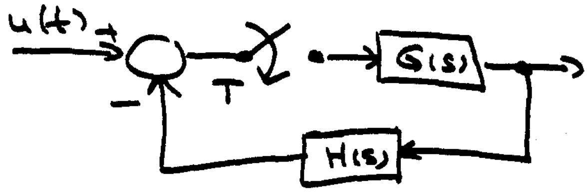
$\Rightarrow \underline{\underline{K_1^* \geq 20}}$ $\qquad$ $K_1^* = K_1 \cdot K_c$

⇒ We need an additional gain factor of $\underline{K_c = 10}$.

⇒ A P-controller can solve this problem.

How does this relate to the discrete case?

Given:



$$\Rightarrow \mathcal{G}(z) = \mathcal{Z}\{G(s)\}\ ;\quad \mathcal{G}_0(z) = \mathcal{Z}\{G(s)\cdot H(s)\}$$

$$\Rightarrow \mathcal{G}_{tot}(z) = \frac{\mathcal{G}(z)}{1+\mathcal{G}_0(z)}$$

$$\Rightarrow \mathcal{G}_e(z) = \frac{\mathcal{G}_{tot}(z)}{1-\mathcal{G}_{tot}(z)} \qquad \text{(assume: Type } \emptyset\text{)}$$

$$\rightarrow \quad K_\emptyset = \lim_{z\to 1} \mathcal{G}_e(z)$$

If the system is stable:

$$e(\infty) = \frac{1}{1+K_\emptyset}$$

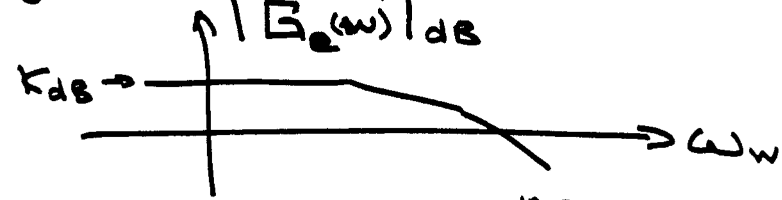
Bilinear transformation:

$$w = \frac{z-1}{z+1} \Rightarrow (z\to 1) \Leftrightarrow (w \to \emptyset)$$

$$\Rightarrow K_\phi = \lim_{W \to \phi} G_e(W)$$

Same algebraic properties as $G(s)$:

$\Rightarrow$



$K_{dB} = 20\,{}^{10}\log K_\phi$

Now, assume: $\mathcal{G}_e(z)$ is Type 1:

$$\Rightarrow K_1 = \frac{1}{T} \cdot \lim_{z \to 1} (z-1)\, \mathcal{G}_e(z)$$

If the system is stable:

$$e(\infty) = \frac{1}{K_1} \quad \text{--- } \underline{\text{etc}}.$$

Bilinear transformation:

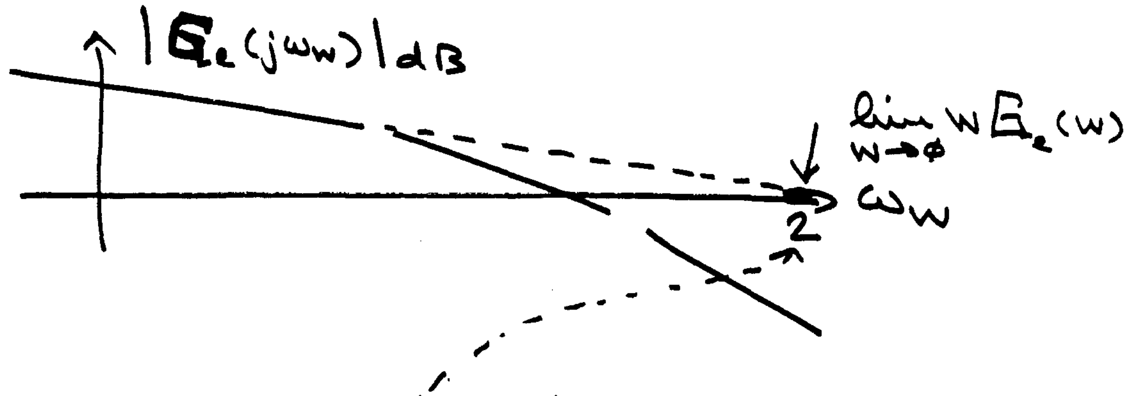
$$z = \frac{1+W}{1-W} \Rightarrow (z-1) = \frac{1+W}{1-W} - 1 = \frac{2W}{1-W}$$

$$\Rightarrow K_1 = \frac{1}{T} \cdot \lim_{W \to \phi} \frac{2W}{1-W} G_e(W) = \frac{2}{T} \cdot \lim_{W \to \phi} W\, G_e(W)$$

Comparison with the continuous system:

$$K_1 = \lim_{s \to 0} s G_e(s)$$

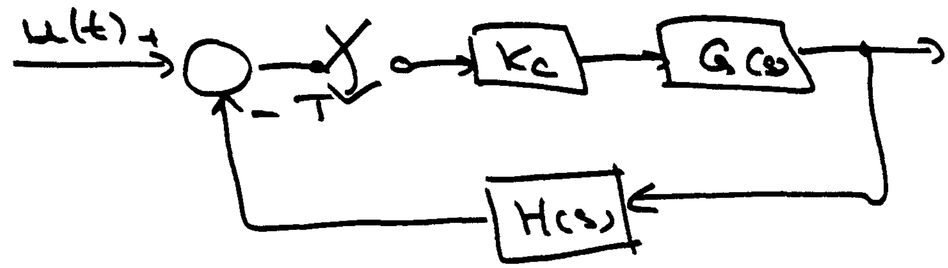
$\Rightarrow$



$$\Rightarrow K_1 = \frac{2}{T} \cdot 2 = \frac{4}{T}$$

$$\Rightarrow e(\infty) = \frac{1}{K_1} \quad \ldots \text{ \underline{ok}.}$$

Can we find a P-controller that will help us to get our steady-state error down.



<u>Problem</u>: We must find how to realize this P-controller for $G_e(w)$ in our true system.

In the continuous case:

$$G_{tot}(s) = \frac{G(s)}{1 + G(s)H(s)}$$

$$\Rightarrow G_e(s) = \frac{G(s)}{1 + G(s)H(s) - G(s)}$$

<u>If</u>: $\lim_{s \to \emptyset} H(s) = 1$

$$\Rightarrow G_e(\emptyset) = \lim_{s \to \emptyset} G_e(s) = \frac{G(\emptyset)}{1 + G(\emptyset)\underbrace{H(\emptyset)}_{=1} - G(\emptyset)}$$

$$\equiv G(\emptyset)$$

$$\Rightarrow k_c \cdot G_e(\emptyset) \equiv k_c \cdot G(\emptyset)$$

⟹ If the open-loop gain of the feedback transfer function $H(s)$ was one

⟹ the gain factor of $G_e(s)$ translates into a gain factor of $G(s)$

In the discrete case, this no longer works as:

$$\mathcal{G}_e(z) = \frac{\mathcal{G}(z)}{1 + \mathcal{GH}(z) - \mathcal{G}(z)}$$

$$\lim_{z \to 1} \mathcal{H}(z) = \mathcal{H}(1) = 1$$

does not help as:

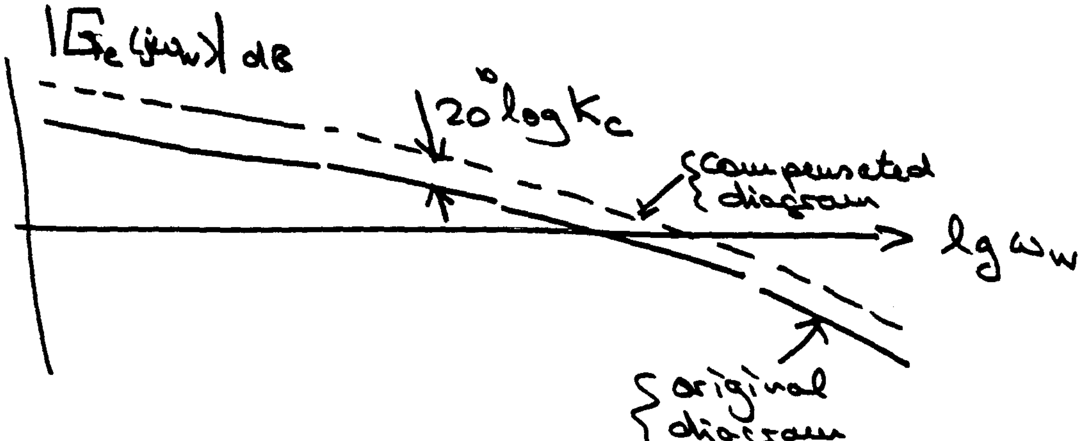
$$\mathcal{GH}(z) \neq \mathcal{G}(z)\,\mathcal{H}(z)$$

and $\lim_{z \to 1} \mathcal{GH}(z) \neq \lim_{z \to 1} \mathcal{G}(z) \cdot \mathcal{H}(z)$

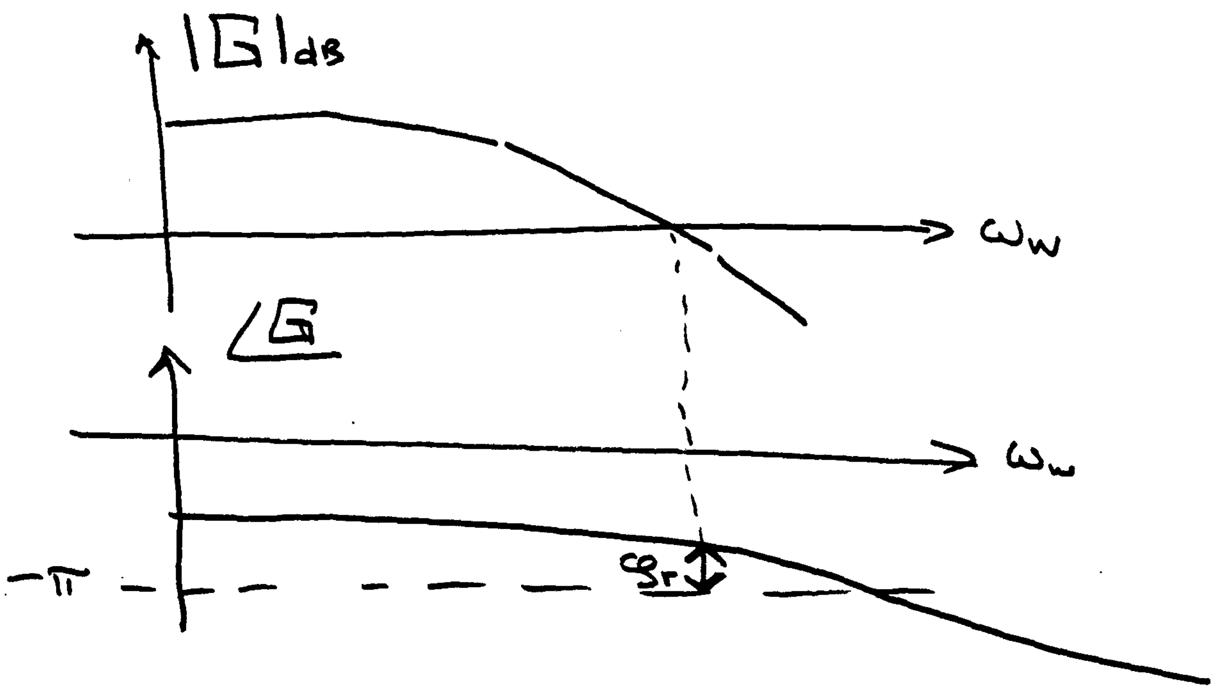
⇒ For the discrete case, we really should request a true unity feedback.

$$\Rightarrow \quad \mathcal{G}_0(z) \equiv \mathcal{G}(z) \equiv \mathcal{G}_e(z), \qquad \mathcal{H}(z) \equiv 1$$

In this case, we can evaluate K_c, calculate its in dB, redraw the Bode-diagram just like in the continuous case:



(2) Now, we try to adjust the phase margin :

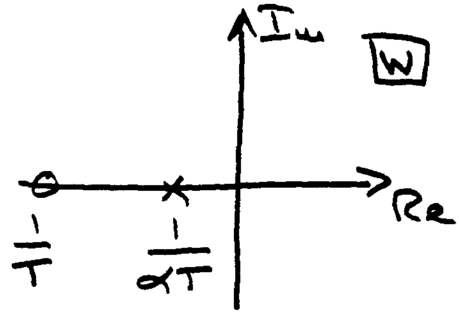


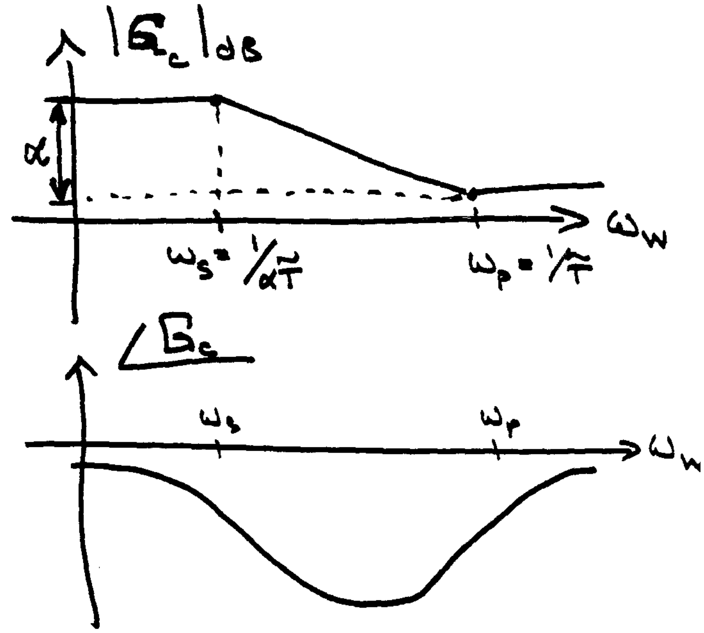
We can use two approaches :

(a) <u>LAG - compensator :</u>

The idea is simple. A LAG - compensator :

$$G_c(w) = \frac{1 + w\tilde{T}}{1 + w\alpha\tilde{T}} \; ; \quad \alpha > 1$$





$$\lim_{W \to \phi} G_c(W) = G_c(\phi) = 1$$

⇒ The gain factor does not change

⇒ The steady-state error is not affected.

⇒ Thus, if we design the LAG-compensat sufficiently to the left of the 3dB point, the influence of the phase will be minimal, but the amplitude is reduced by α.

Recipe:

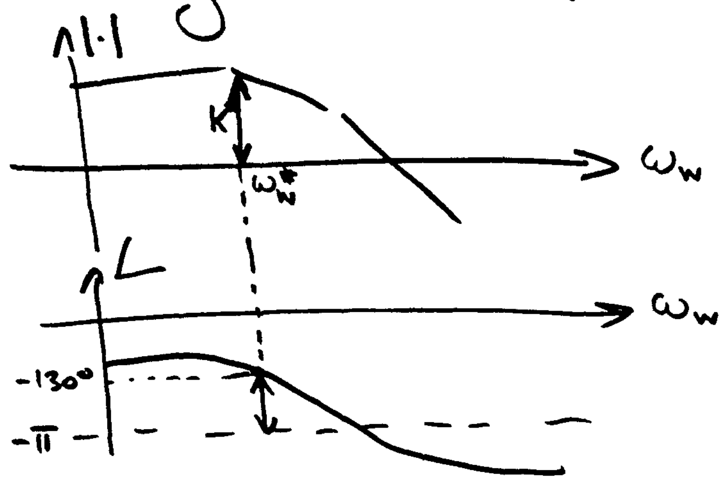
((1)) Find the point where the phase is $-180° + \varphi_{r_{desired}} + 5°$.

Eg.: $\varphi_{r_{desired}} = 45° \Rightarrow$ Find ω_W where $\angle G_c$ is $-130°$. $\Rightarrow \omega_W^*$

((2)) Put the zero one decade to the left of this point: $\omega_p = \omega_W^* / 10$

$\Rightarrow \bar{T} = \frac{1}{\omega_p}$

((3)) Read the gain at $\omega_W^* \Rightarrow K^*_{(dB)}$



$\Rightarrow K^* = 20 \log_{10} \alpha \quad \Rightarrow \alpha$

((4)) Now, everything is determined, and we find:

$$G_c(w) = \frac{1 + w\tilde{T}}{1 + w\alpha\tilde{T}}$$

is the compensator we want.

((5)) This needs to be translated back into something that is realizable:

$$w = \frac{z-1}{z+1}$$

$$\Rightarrow \mathcal{G}_c(z) = \frac{1 + \frac{z-1}{z+1}\tilde{T}}{1 + \frac{z-1}{z+1}\alpha\tilde{T}}$$

$$\Rightarrow \mathcal{G}_c(z) = \frac{(1+\tilde{T})z + (1-\tilde{T})}{(1+\alpha\tilde{T})z + (1-\alpha\tilde{T})}$$

Notice that this should either

be realized as a discrete controller, or if backtransformation into the s-plane is preferred, that a sampler between the compensator and the plant is required.

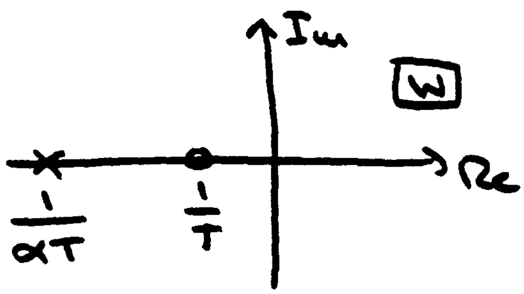
→ A practical problem may still be that $G(j\omega_w)$ needs to be evaluated as a function of ω_w whereas, what we can measure, is $G^*(j\omega)$ as a function of ω, but we know that

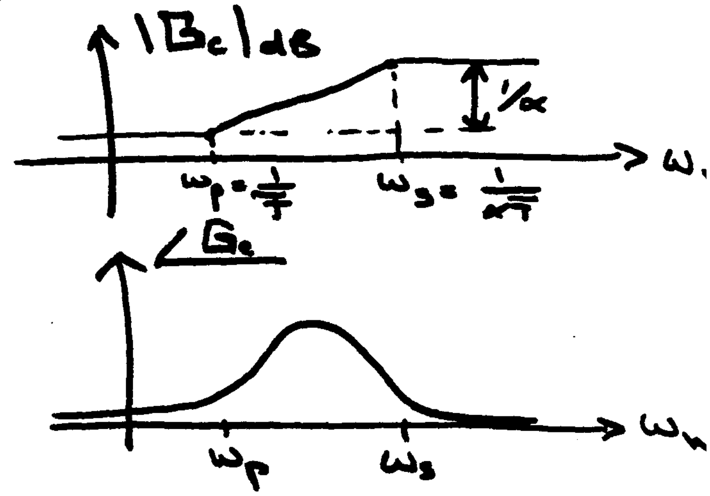
$$G(j\omega_w)\Big|_{\omega_w} \equiv G(j\omega)\Big|_{\omega_w = tg\left(\frac{T}{2}\omega\right)}$$

(b) LEAD – compensator :

The idea is again very simple. A LEAD - compensator :

$$G_c(w) = \frac{1 + w\tilde{T}}{1 + w\alpha\tilde{T}} \quad ; \quad \alpha < 1$$





$$\lim_{w \to 0} G_c(w) = G_c(0) = 1$$

⇒ Thus, we can design the LEAD-compensator at the place where we want the 3dB-point to be, and calculate how much additional phase we require to get the desired phase margin there.

We will probably design a LAG-compensator thereafter around it to let the 3dB-point be there.

Recipe:

((1)) Determine where you want the 3-dB-point to be, and measure the current phase there: φ_{3dB}

((2)) Design a LEAD-compensator with:

$$\varphi_{max} = -180° + \varphi_{r_{desired}} + 5° - \varphi_{3dB}$$

IF $\varphi_{max} > 60°$ ⇒ split into several portions of $\leq 60°$ (not desirable due to sensitivity!)

((3)) For each of these:

$$\alpha_i = \frac{1 - \sin(\varphi_{max})}{1 + \sin(\varphi_{max})}$$

((4)) Determine the zero at:

$$\omega_p = \frac{1}{T} = \omega_{3dB} \cdot \sqrt{\alpha}$$

$$\omega_s = \frac{1}{\alpha T} = \omega_{3dB} \cdot \frac{1}{\sqrt{\alpha}}$$

$$\Rightarrow T_i = \frac{1}{\omega_{3dB} \cdot \sqrt{\alpha_i}}$$

((5)) This determines one (or several) LEAD-compensators that can be mapped back into the z-plane as before.

((6)) Add the newly found compensator to the plant, and then design a LAG-compensator that brings the 3dB-point down as desired.

(c) <u>General Rules</u> :

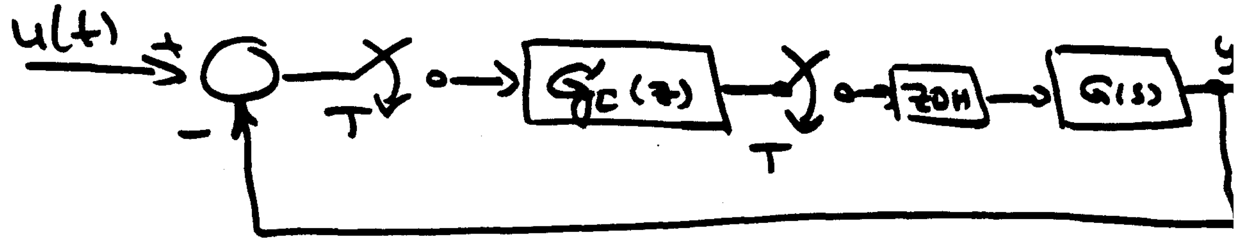
→ The LAG-compensator makes the system <u>slower</u> (that is: reduces the bandwidth), while the LEAD-compensator makes the system <u>faster</u> (increases its bandwidth).

→ Design a LAG-compensator only if the system without compensator is too fast (too sensitive to noise).

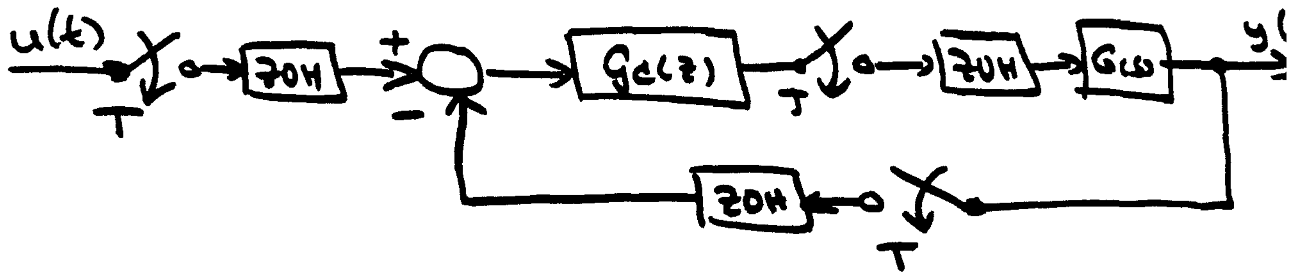
→ If the system is too slow, design a LEAD-compensator first, then a LAG-compensator thereafter.

Practical implementation :

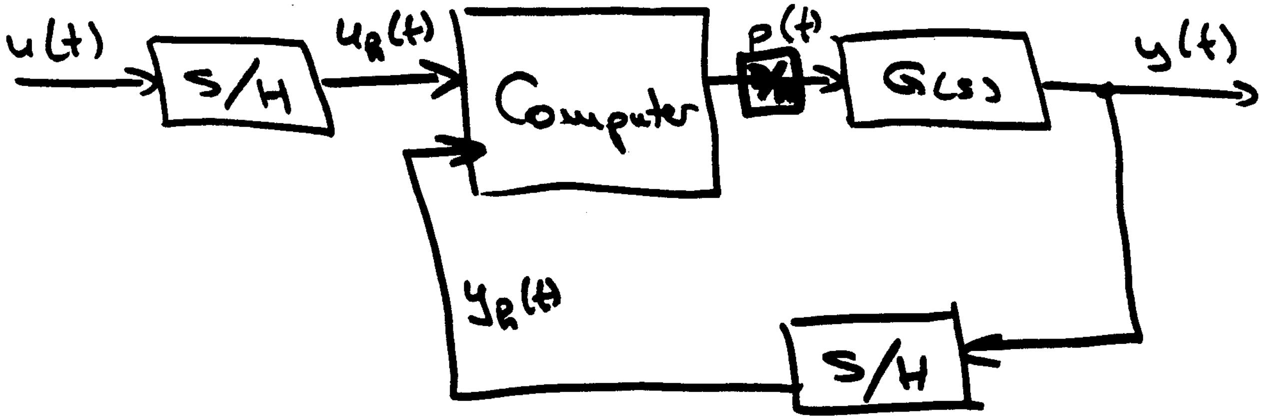
After all compensators (P-controller, LEAD-compensator, LAG-compensator) have been designed, we have the following situation:



which is equivalent to:



or :



The computer contains the summer and the z-transfer function.

E.g.: We may have designed the following P-controller + LAG-compensator:

$$G_c(w) = 10\,\frac{1+10w}{1+150w}\ ; \quad \tilde{T}=10,\ \alpha=15,\ K_p=10$$

$$\Rightarrow\ G_c(z) = 10\,\frac{11z-9}{151z-149}$$

$$= \frac{110z-90}{151z-149} = \frac{\frac{110}{151}z-\frac{90}{151}}{z-\frac{149}{151}}$$

$$\Rightarrow \quad G_c(z) = 0.7285 \cdot \frac{z - 0.8182}{z - 0.9868}$$

For a practical realization, it is more practical to express this in terms of z^{-1} instead of z (delay instead of advance):

$$G_c(z^{-1}) = 0.7285 \cdot \frac{1 - 0.8182 z^{-1}}{1 - 0.9868 z^{-1}} = \frac{P(z)}{E(z)}$$

$$\Rightarrow \quad 0.7285\, E(z) - 0.5961\, z^{-1} E(z) = P(z) - 0.9868\, z^{-1} P(z)$$

where: $E(z) = U_R(z) - Y_R(z)$

⇒ Algorithm:

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E = U - Y
P = 0.9868*PL + 0.7285*E
      - 0.5961*EL
EL = E
PL = P
```

is executed once per sampling.