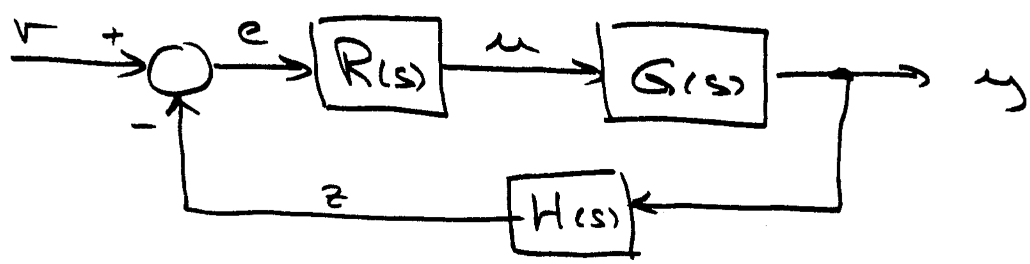
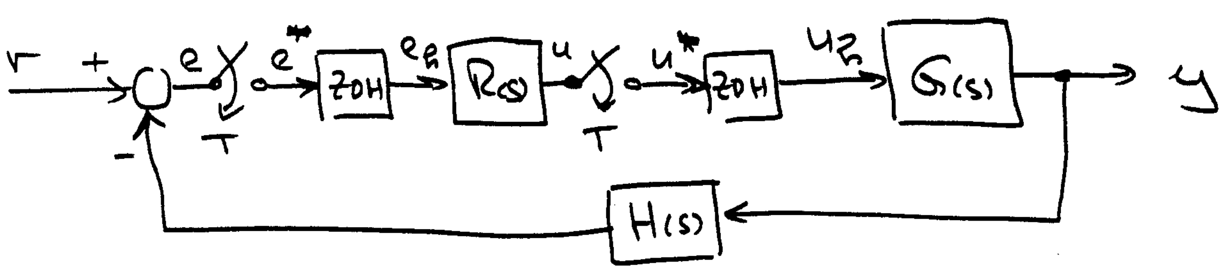
Discrete Implementation of Continuous Controllers

Assume we have designed a controller for a continuous-time system using any of the techniques offered in ECE 441:



As long as the sampling rate is chosen sufficiently fast, the following system exhibits almost identical behavior:



The advantage of the sampled version is that $u(t)$ isn't really needed in between samplings. We only need the correct value at the sampling instants.

A) Implementation using numerical integration

Example:

$$R(s) = \frac{10}{s+2}$$

$$\Rightarrow U(s) = \frac{10}{s+2} \cdot E(s)$$

$$\Rightarrow (s+2) \cdot U(s) = 10 \cdot E(s)$$

$$\Rightarrow \dot{u} + 2u = 10 \cdot e$$

$$\Rightarrow \dot{u} = -2u + 10e$$

Let us use forward Euler (FE):

$$u(k+1) = u(k) + h \cdot \dot{u}(k)$$

$$h \equiv T$$

$$\Rightarrow u(k+1) = u(k) + T \cdot \left[-2 \cdot u(k) + 10 \cdot e(k) \right]$$

$$\Rightarrow u(k+1) = (1-2T)\cdot u(k) + 10T\cdot e(k)$$

$$\Rightarrow z\cdot U(z) = (1-2T)\cdot U(z) + 10T\cdot E(z)$$

$$\Rightarrow (z+2T-1)\cdot U(z) = 10T\cdot E(z)$$

$$\Rightarrow U(z) = \frac{10T}{z+2T-1}\cdot E(z)$$

$$\Rightarrow R(z) = \frac{10T}{z+2T-1}$$

$$\Rightarrow R(z) = \frac{10}{\frac{z-1}{T}+2}$$

It looks like "s" is simply being replaced by "$\frac{z-1}{T}$".

Example:

$$R(s) = \frac{(s+5)}{s(s+10)}$$

— NS —

$$\Rightarrow U(s) = \frac{s+5}{s^2+10s} \cdot E(s)$$

$$\bullet\!\!-\!\!\circ$$

$$\left| \begin{aligned} \dot{\underline{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & -10 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e \\ u &= \begin{bmatrix} 5 & 1 \end{bmatrix} \underline{x} \end{aligned} \right|$$

$$\left| \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10\, x_2 + e \\ u &= 5x_1 + x_2 \end{aligned} \right|$$

Using FE algorithm:

$$\left| \begin{aligned} x_1(k+1) &= x_1(k) + T \cdot x_2(k) \\ x_2(k+1) &= x_2(k) + T \cdot \left[-10 \cdot x_2(k) + e(k)\right] \\ u(k) &= 5 \cdot x_1(k) + x_2(k) \end{aligned} \right|$$

$$\circ\!\!-\!\!\bullet$$

$$\left| \begin{aligned} z \cdot \mathcal{X}_1(z) &= \mathcal{X}_1(z) + T \cdot \mathcal{X}_2(z) \\ z \cdot \mathcal{X}_2(z) &= \mathcal{X}_2(z) - 10T \cdot \mathcal{X}_2(z) + T \cdot E(z) \\ \mathcal{U}(z) &= 5 \cdot \mathcal{X}_1(z) + \mathcal{X}_2(z) \end{aligned} \right|$$

$$\Rightarrow (z + 10T - 1)\,\mathcal{X}_2(z) = T \cdot \mathcal{E}(z)$$

$$\Rightarrow \mathcal{X}_2(z) = \frac{T}{z + 10T - 1} \cdot \mathcal{E}(z)$$

$$(z-1)\cdot \mathcal{X}_1(z) = T \cdot \mathcal{X}_2(z)$$

$$\Rightarrow \mathcal{X}_1(z) = \frac{T}{z-1} \cdot \mathcal{X}_2(z)$$

$$= \frac{T^2}{(z-1)(z-1+10T)} \cdot \mathcal{E}(z)$$

$$\Rightarrow \mathcal{U}(z) = \frac{5T^2}{(z-1)(z-1+10T)} \cdot \mathcal{E}(z) + \frac{T}{(z-1+10T)} \cdot \mathcal{E}(z)$$

$$\Rightarrow \mathcal{U}(z) = \frac{T(z-1) + 5T^2}{(z-1)(z-1+10T)} \cdot \mathcal{E}(z)$$

$$\Rightarrow \mathcal{U}(z) = \frac{\left(\frac{z-1}{T}\right) + 5}{\left(\frac{z-1}{T}\right)\left[\left(\frac{z-1}{T}\right) + 10\right]} \cdot \mathcal{E}(z)$$

Again, $s \longrightarrow \dfrac{z-1}{T}$.

This is generally true.

FE: $\quad x(k+1) \cong x(k) + T \cdot \dot{x}(k)$

$$\Rightarrow \quad \dot{x}(k) \cong \frac{x(k+1) - x(k)}{T}$$

$$s \cdot X(s) \cong \left(\frac{z-1}{T}\right) \cdot \mathcal{X}(z)$$

Let us try another algorithm, the backward Euler method (BE):

BE: $\quad x(k+1) \cong x(k) + T \cdot \dot{x}(k+1)$

$$\Rightarrow \quad x(k) \cong x(k-1) + T \cdot \dot{x}(k)$$

$$\Rightarrow \quad \dot{x}(k) \cong \frac{x(k) - x(k-1)}{T}$$

$$\underbrace{\dot{x}(k)}_{\substack{\circ \\ | \\ \bullet}} \cong \underbrace{\frac{x(k)-x(k-1)}{T}}_{\substack{\circ \\ | \\ \bullet}}$$

$$s \cdot X(s) \cong \left(\frac{1-z^{-1}}{T}\right) \cdot \mathcal{X}(z)$$

$$\Rightarrow \quad \underline{\underline{s \longrightarrow \frac{z-1}{T \cdot z}}}$$

Both FE and BE have the disadvantage of only being 1st-order accurate. A 2nd-order accurate technique is the Trapezoidal rule (TR):

$$\underline{TR}: \quad x(k+1) \cong x(k) + \frac{T}{2} \cdot \left(\dot{x}(k) + \dot{x}(k+1)\right)$$

This is a little harder. We simply replace every $(k+1)$ by z, and every derivative by s.

$$x(k+1) \cong x(k) + \frac{T}{2} \cdot \left[\dot{x}(k) + \dot{x}(k+1) \right]$$

$$\circ\!\!-\!\!\bullet$$

<u>Operator equation</u>:

$$z \cong 1 + \frac{T}{2} \cdot \left[s + s z \right]$$

$$\Rightarrow \quad s \cdot \frac{T}{2} (z+1) \cong z - 1$$

$$\Rightarrow \quad s \cong \frac{2}{T} \cdot \frac{(z-1)}{(z+1)}$$

<u>Warning</u>: "Operator equations" are sloppy math. They sometimes work, and sometimes don't!

This works equally well, when the controller is specified in the time domain:

<u>Controller</u>:

$$\left|\begin{aligned} \dot{\underline{x}} &= A\,\underline{x} + \underline{b}\cdot e \\ u &= \underline{c}'\cdot\underline{x} + d\cdot e \end{aligned}\right|$$

$$\circ\!\!-\!\!\bullet$$

$$\left|\begin{aligned} s\underline{X}(s) &= A\cdot\underline{X}(s) + \underline{b}\cdot E(s) \\ U(s) &= \underline{c}'\cdot\underline{X}(s) + d\cdot E(s) \end{aligned}\right|$$

$$\Big\downarrow \text{FE}$$

$$\left|\begin{aligned} \frac{z-1}{T}\cdot\underline{\mathcal{X}}(z) &\cong A\cdot\underline{\mathcal{X}}(z) + \underline{b}\cdot\mathcal{E}(z) \\ \mathcal{U}(z) &= \underline{c}'\cdot\underline{\mathcal{X}}(z) + d\cdot\mathcal{E}(z) \end{aligned}\right|$$

$$\Rightarrow (z-1)\cdot\underline{\mathcal{X}}(z) \cong (AT)\cdot\underline{\mathcal{X}}(z) + (\underline{b}T)\cdot\mathcal{E}(z)$$

$$\Rightarrow z\cdot\underline{\mathcal{X}}(z) \cong [I+AT]\cdot\underline{\mathcal{X}}(z) + (\underline{b}T)\cdot\mathcal{E}(z)$$

$$z \cdot \underline{\mathcal{X}}(z) \cong [I + AT] \cdot \underline{\mathcal{X}}(z) + (\underline{b}T) \cdot \mathcal{E}(z)$$

$$\left| \begin{array}{l} \underline{x}(k+1) \cong [I + AT] \cdot \underline{x}(k) + (\underline{b}T) \cdot e(k) \\ y(k) = \underline{c}' \cdot \underline{x}(k) + d \cdot e(k) \end{array} \right|$$

$$\Rightarrow \quad \left| \begin{array}{lcl} F & \cong & I + AT \\ \underline{g} & \cong & \underline{b}T \\ \underline{h}' & = & \underline{c}' \\ i & = & d \end{array} \right|$$

Remember: $I + AT$ is a 1^{st}-order accurate approximation of e^{AT}.

Let us use BE instead.

$$\left|\begin{aligned} \frac{z-1}{Tz}\cdot \underline{\mathcal{X}}(z) &\cong A\cdot \underline{\mathcal{X}}(z) + \underline{b}\cdot \mathcal{E}(z) \\ \mathcal{U}(z) &= \underline{c}'\cdot \underline{\mathcal{X}}(z) + d\cdot \mathcal{E}(z) \end{aligned}\right|$$

$$\Rightarrow (z-1)\,\underline{\mathcal{X}}(z) \cong ATz\cdot \underline{\mathcal{X}}(z) + \underline{b}Tz\,\mathcal{E}(z)$$

$$\Rightarrow [I-AT]\,z\,\underline{\mathcal{X}}(z) \cong \underline{\mathcal{X}}(z) + \underline{b}Tz\,\mathcal{E}(z)$$

↑ a problem!

$$\underline{x}(k+1) - AT\cdot \underline{x}(k+1) \cong \underline{x}(k) + \underline{b}T\cdot e(k+1)$$

$$\Rightarrow \underline{x}(k+1) - AT\cdot \underline{x}(k+1) - \underline{b}T\cdot e(k+1) \cong \underline{x}(k)$$

Let us introduce a new state variable:

$$\underline{\xi}(k) = [I-AT]\cdot \underline{x}(k) - \underline{b}T\cdot e(k)$$

$$\Rightarrow \underline{x}(k) = [I-AT]^{-1}\cdot \underline{\xi}(k) + [I-AT]^{-1}\cdot \underline{b}T\cdot e(k)$$

$$\Rightarrow \underline{\xi}(k+1) \cong \underline{x}(k+1) - AT\cdot \underline{x}(k+1) - \underline{b}T\cdot e(k+1)$$

$$\Rightarrow \underline{\xi}(k+1) \cong [I-AT]^{-1} \cdot \underline{\xi}(k) + [I-AT]^{-1} \cdot \underline{b}T \cdot e(k)$$

is in the desired form.

$$u(k) = \underline{c}' \cdot \underline{x}(k) + d \cdot e(k)$$
$$= \underline{c}'[I-AT]^{-1}\underline{\xi}(k) + \underline{c}'[I-AT]^{-1}\underline{b}T \cdot e(k) + d \cdot e(k)$$

$$\Rightarrow u(k) = \underline{c}'[I-AT]^{-1} \cdot \underline{\xi}(k) + \left[\underline{c}'[I-AT]^{-1}\underline{b}T + d\right] \cdot e(k)$$

$$\Rightarrow \left| \begin{array}{l} F \cong [I-AT]^{-1} \\ \underline{g} \cong [I-AT]^{-1} \cdot \underline{b}T \\ \underline{h}' = \underline{c}'[I-AT]^{-1} \\ i = \underline{c}'[I-AT]^{-1}\underline{b}T + d \end{array} \right|$$

- The advantage of these techniques is that s can be locally replaced by an expression in z.

- The disadvantage is that all of these techniques introduce additional sources of error, because the numerical integration is <u>not</u> exact.

- By using the z-transform as introduced earlier:

$$R(z) = (1 - z^{-1}) \, \mathcal{Z}\left\{ \frac{R(s)}{s} \right\}$$

we avoid these additional sources of error, since the solution of the discrete system is now accurate at the sampling points.