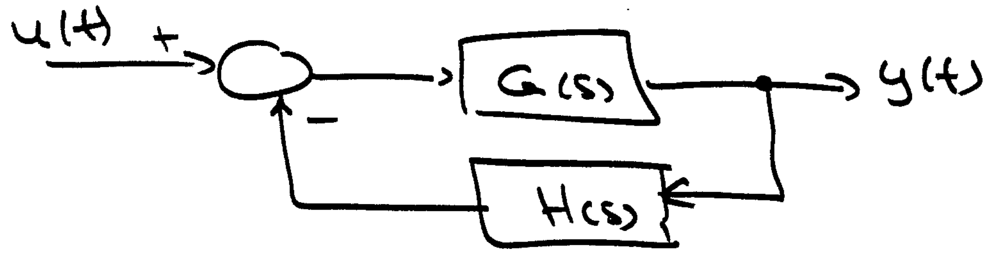
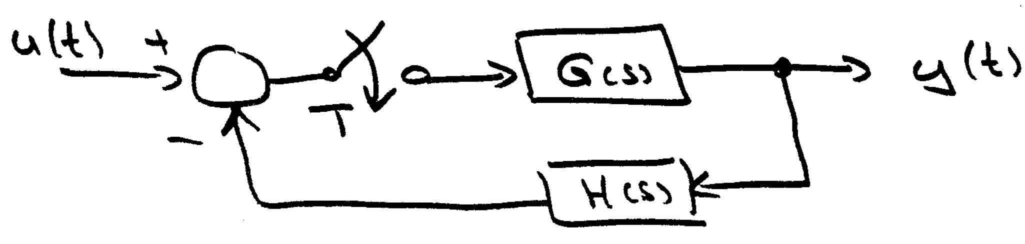
Steady-State Behavior:

Problem: In the continuous case, we were able to study the steady-state behavior of the system:



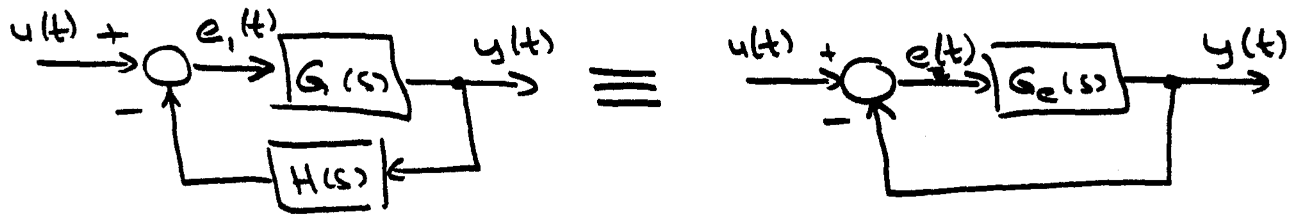
in very general terms as a function of the type of the system and the type of the input signal.

Question: Can we do something similar for the system:



?

In the continuous case, we started by calculating an equivalent system with unity feedback:



$$G_{tot}(s) = \frac{G(s)}{1 + G(s)H(s)} \equiv \frac{G_e(s)}{1 + G_e(s)}$$

$$\Rightarrow G_{tot}(s)\left(1 + G_e(s)\right) = G_e(s)$$

$$\Rightarrow G_e(s)\left(1 - G_{tot}(s)\right) = G_{tot}(s)$$

$$\Rightarrow \quad G_e(s) = \frac{G_{tot}(s)}{1 - G_{tot}(s)}$$

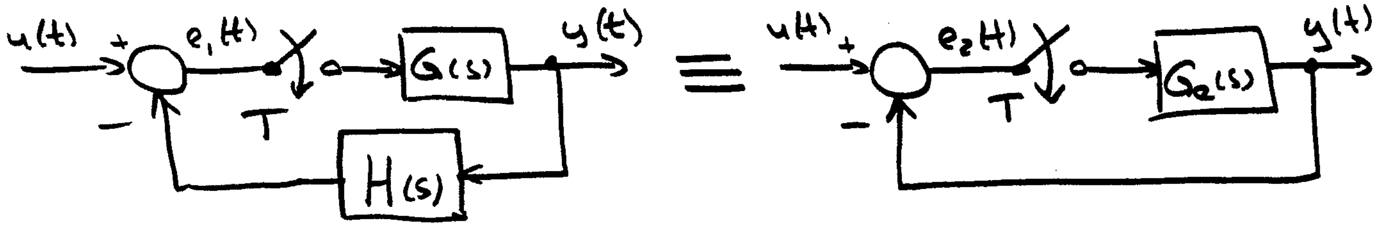
Notice, however, that $e_1(t) \neq e_2(t)$

- Although the signal $e_1(t)$ is available as a physical signal in the system while $e_2(t)$ is a theoretical construct, it makes much more sense to analyze the "steady-state error" as the steady-state value of $e_2(t)$, because:

if $e_2(\infty) \equiv 0 \implies y(\infty) \equiv u(\infty)$

whereas: $e_1(\infty) \equiv 0 \implies$ has no significance.

We will do exacty the same in the discrete case:



$$G_o(s) = G(s) \cdot H(s) \longrightarrow \mathcal{G}_o(z) = \mathcal{Z}\{G_o(s)\}$$

$$\Rightarrow \mathcal{G}_{tot}(z) = \frac{\mathcal{G}(z)}{1+\mathcal{G}_o(z)} \equiv \frac{\mathcal{G}_e(z)}{1+\mathcal{G}_e(z)}$$

$$\Rightarrow \boxed{\mathcal{G}_e(z) = \frac{\mathcal{G}_{tot}(z)}{1-\mathcal{G}_{tot}(z)}}$$

$$\Rightarrow \mathcal{E}_2(z) = \frac{1}{1+\mathcal{G}_e(z)} \cdot \mathcal{U}(z)$$

$$\Rightarrow e_2(t\to\infty) = \lim_{z\to 1} (1-z^{-1})\, \mathcal{E}_2(z)$$

Assume: Input = Type ϕ = $\varepsilon(t)$

$$\Rightarrow \quad \mathcal{U}(z) = \frac{z}{z-1}$$

$$\Rightarrow \quad e_{2_{ss}} = e_2(t \to \infty) = \lim_{z \to 1} \frac{z-1}{z} \cdot \frac{1}{1+G_e(z)} \cdot \frac{z}{z-1}$$

Let $K_0 = \lim_{z \to 1} G_e(z)$

$$\Rightarrow \quad e_{2ss} = \frac{1}{1+K_0}$$

If system is Type ϕ ($\Longleftrightarrow$ no pole at $z=1$) $\Longrightarrow$ K_0 is finite $\to$ e_{2ss} = finite

If system is Type $> \phi$

$\Rightarrow K_0 = \infty \quad \Longrightarrow \quad e_{2ss} \equiv \phi$.

Assume: Input = Type 1 = r(t)

$$\Rightarrow \mathcal{U}(z) = \frac{Tz}{(z-1)^2}$$

$$\Rightarrow e_{zss} = \lim_{z \to 1} \left\{ \frac{z-1}{z} \cdot \frac{1}{1+G_e(z)} \cdot \frac{Tz}{(z-1)^2} \right\}$$

$$= \lim_{z \to 1} \left\{ \frac{T}{(z-1)(1+G_e(z))} \right\}$$

$$= \lim_{z \to 1} \left\{ \frac{T}{(z-1) + (z-1)G_e(z)} \right\}$$

$$= \lim_{z \to 1} \frac{T}{(z-1)G_e(z)}$$

$$= \frac{1}{\frac{1}{T} \lim_{z \to 1} (z-1) G_e(z)}$$

Let: $$K_1 = \frac{1}{T} \lim_{z \to 1} (z-1) G_e(z)$$

$$\Rightarrow e_{zss} = \frac{1}{K_1}$$

If the system is of Type 0 (no pole at $z=1$) $\Longrightarrow$

$$\underline{\underline{K_1 = 0}} \Longrightarrow \underline{\underline{e_{2ss} \rightarrow \infty}}$$

If the system is of Type 1 (one pole at $z=1$) $\Longrightarrow$

$$\underline{\underline{K_1 = \text{finite}}} \Longrightarrow e_{2ss} = \underline{\underline{\text{finite}}}$$

If the type of the system is ≥ 2

$$\Longrightarrow \underline{\underline{K_1 \rightarrow \infty}} \Longrightarrow \underline{\underline{e_{2ss} = 0}}$$

etc.

Input is Type 2

$$\Longrightarrow \underline{\underline{K_2 = \frac{1}{T^2} \lim_{z \to 1} (z-1)^2 G_e(z)}}$$

$$\Longrightarrow \underline{\underline{e_{2ss} = \frac{1}{K_2}}}$$

Type of Input \ Type of System	ϕ	1	2	...
ϕ	$\frac{1}{1+K_\phi}$	ϕ	ϕ	...
1	∞	$\frac{1}{K_1}$	ϕ	...
2	∞	∞	$\frac{1}{K_2}$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$\Longrightarrow$ We use a different formulae to compute K_ϕ, K_1, K_2, ... , but then, the results are completely symmetric to the continuous case.

Question: Is there a relation between the continuous steady-state error and its discrete counterpart ?

E.g.: $$K_1^C = \lim_{s \to \emptyset} s\, G_e(s)$$

Assume: G_e is of Type 1

$$\Rightarrow \quad G_e(s) = \frac{K}{s} \cdot \frac{(1+T_a s)(1+T_b s)\cdots(1+T_m s)}{(1+T_1 s)(1+T_2 s)\cdots(1+T_n s)}$$

$$\Rightarrow \quad \underline{\underline{K_1^C \equiv K}}$$

$$K_1^D = \frac{1}{T} \lim_{z \to 1} (z-1)\, \mathcal{G}_e(z)$$

$$G_e(s) = \frac{K}{s} \cdot \frac{(1+T_a s)(1+T_b s)\cdots(1+T_m s)}{(1+T_1 s)(1+T_2 s)\cdots(1+T_n s)}$$

$$= \frac{K}{s} + \frac{R_1}{1+T_1 s} + \frac{R_2}{1+T_2 s} + \cdots + \frac{R_n}{1+T_n s}$$

$$\Rightarrow \mathcal{G}_e(z) = \frac{Kz}{z-1} + \frac{(R_1/T_1)\cdot z}{z - e^{-T/T_1}} + \cdots + \frac{(R_n/T_n)\cdot z}{z - e^{-T/T_n}}$$

$$\Rightarrow K_1^D = \frac{1}{T} \cdot \left(K + \emptyset + \emptyset + \cdots + \emptyset \right)$$

$$\Rightarrow \quad K_1^D = \frac{K}{T} \qquad \Rightarrow \quad \boxed{K_1^D = \frac{K_1^C}{T}}$$

Assume: G_e is of Type 2:

$$G_e(s) = \frac{K}{s^2} \cdot \frac{(1+T_a s)(1+T_b s)\cdots(1+T_m s)}{(1+T_1 s)(1+T_2 s)\cdots(1+T_n s)}$$

$$\Rightarrow \quad K_2^C = \lim_{s \to 0} s^2 G_e(s) = K$$

$$K_2^D = \frac{1}{T^2} \lim_{z \to 1} (z-1)^2 \mathcal{G}_e(z)$$

$$\Rightarrow \quad G_e(s) = \frac{K}{s^2} + \frac{K_1}{s} + \frac{R_1}{1+T_1 s} + \cdots + \frac{R_n}{1+T_n s}$$

$$\Rightarrow \quad \mathcal{G}_e(z) = \frac{KTz}{(z-1)^2} + \frac{K_1 z}{(z-1)} + \frac{(R_1/T_1)z}{z-e^{-T/T_1}} + \cdots + \frac{(R_n/T_n)z}{z-e^{-T/T_n}}$$

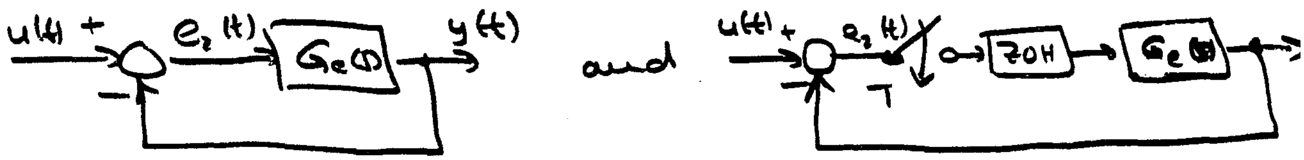
$$\Rightarrow \quad K_2^D = \frac{1}{T^2}\left(KT + \emptyset + \cdots + \emptyset\right) = \frac{K}{T}$$

$$\Rightarrow \quad \boxed{K_2^D = \frac{K_2^C}{T}}$$

etc.

The <u>type 0</u> case is not so convenient.

<u>Question</u>: Is there a convenient relation between the coefficient of:



with a ZOH ?

<u>Assume</u>: System is <u>Type ϕ</u> :

$$\Rightarrow \quad G_e(s) = \frac{K(1-T_a s)(1-T_b s)\cdots(1-T_m s)}{(1-T_1 s)(1-T_2 s)\cdots(1-T_n s)}$$

$$\Rightarrow \quad \underline{K_\phi^c} = \lim_{s\to\phi} G_e(s) = \underline{K}$$

$$\Rightarrow \mathcal{Z}\left\{ G_h(s)\cdot G_e(s) \right\} = (1-z^{-1})\, \mathcal{Z}\left\{ \frac{G_e(s)}{s} \right\}$$

$$\frac{G_e(s)}{s} = \frac{K}{s} + \frac{R_1}{1-T_1 s} + \cdots + \frac{R_n}{1-T_n s}$$

$$\Rightarrow \mathcal{Z}\left\{\frac{G_e(s)}{s}\right\} = \frac{Kz}{z-1} + \frac{(R_1/T_1)z}{z-e^{-T/T_1}} + \cdots + \frac{(R_n/T_n)z}{z-e^{-T/T_n}}$$

$$\Rightarrow \underline{\underline{K_\emptyset^D}} = \lim_{z\to 1} (1-z^{-1})\,\mathcal{Z}\left\{\frac{G_e(s)}{s}\right\}$$

$$= \lim_{z\to 1} \frac{z-1}{z}\left[\frac{Kz}{z-1} + \frac{(R_1/T_1)z}{z-e^{-T/T_1}} + \cdots + \frac{(R_n/T_n)z}{z-e^{-T/T_n}}\right]$$

$$= \underline{\underline{K}} = K_\emptyset^C \quad \Rightarrow \quad \boxed{K_\emptyset^D = K_\emptyset^C}$$

Assume: System is Type 1:

$$G_e(s) = \frac{K}{s}\cdot\frac{(1-T_a s)(1-T_b s)\cdots(1-T_m s)}{(1-T_1 s)(1-T_2 s)\cdots(1-T_n s)}$$

$$\Rightarrow \underline{\underline{K_1^C = K}}$$

$$\mathcal{Z}\left\{G_R(s)\cdot G_e(s)\right\} = (1-z^{-1})\,\mathcal{Z}\left\{\frac{G_e(s)}{s}\right\}$$

$$\frac{G_e(s)}{s} = \frac{K}{s^2} + \frac{K_1}{s} + \frac{R_1}{1-T_1 s} + \cdots + \frac{R_n}{1-T_n s}$$

$$\Rightarrow \mathcal{Z}\left\{\frac{G_e(s)}{s}\right\} = \frac{KTz}{(z-1)^2} + \frac{K_1 z}{(z-1)} + \frac{(R_1/T_1)z}{z-e^{-T/T_1}} + \cdots$$

$$\Rightarrow K_1^D = \frac{1}{T}\lim_{z\to 1}(z-1)\cdot\frac{(z-1)}{z}\left[\frac{KTz}{(z-1)^2} + \frac{K_1 z}{(z-1)} + \frac{(R_1/T_1)z}{z-e^{-T/T_1}} + \cdots\right]$$

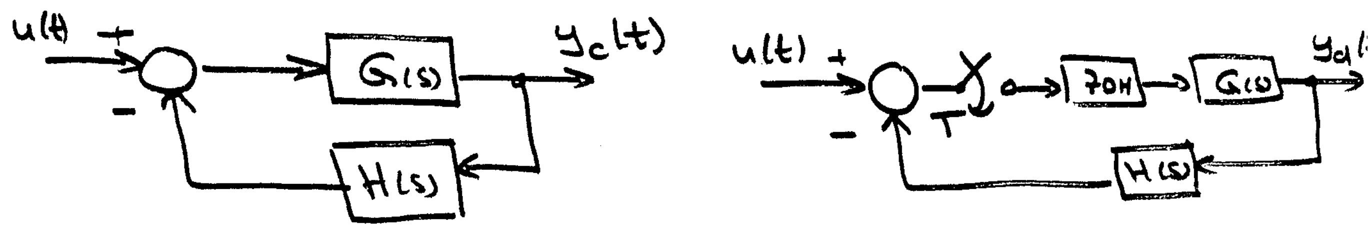
$$\Rightarrow \underline{\underline{K_1^D = K}} \qquad \Rightarrow \boxed{K_1^D \equiv K_1^C}$$

etc.

$\Rightarrow$ If a sampler with ZOH is introduced, stability behavior is modified. However, as long as the sampled data system remains stable, the steady-state behavior does not change.

As the steady-state error depends only on $G_o(t)$, it does not really matter, where exactly the sampler is located.

$\Rightarrow$



$$\Rightarrow \quad y_d(kT) \neq y_c(kT)$$

(The dynamic behavior is different)

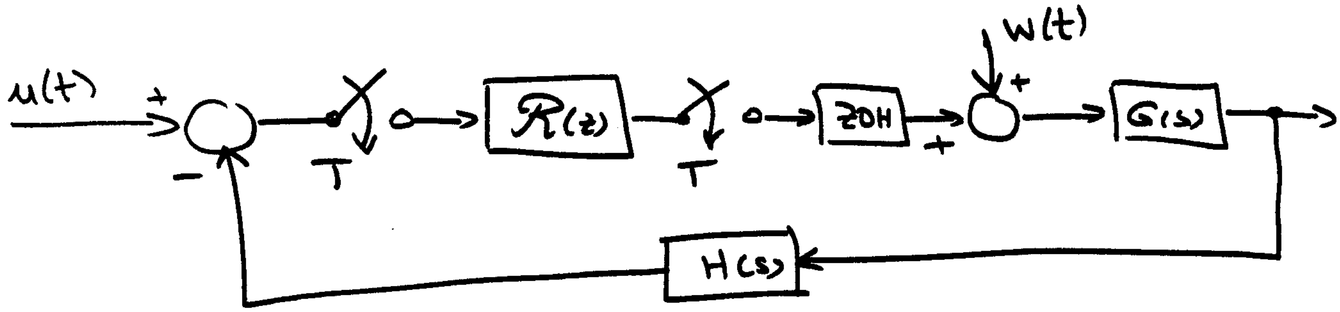
<u>but</u>: $$\lim_{k\to\infty} y_d(kT) \equiv \lim_{k\to\infty} y_c(kT)$$

independent of T, as long as the system remains stable.

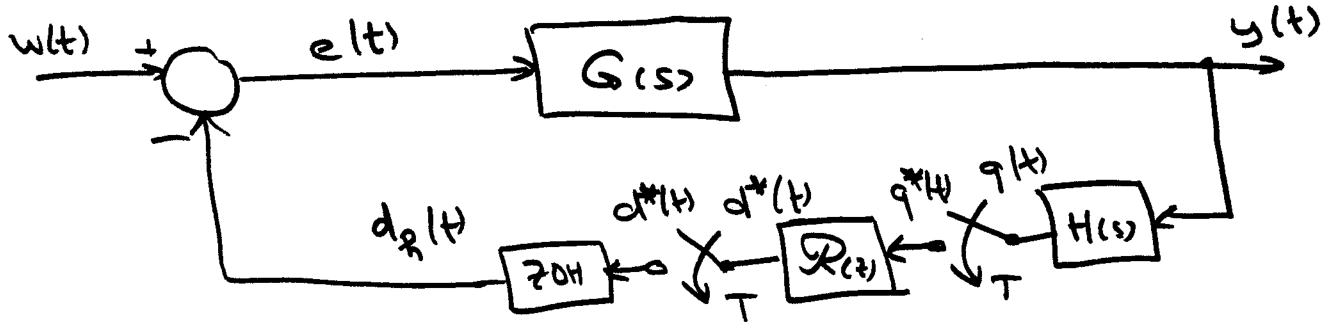
$\Rightarrow$ The steady-state behavior of the two systems is identical.

$\Rightarrow$ only for unit feedback !

Influences of Disturbances:



- Of course, there may exist many different configurations of systems that we may be interested in.
- Let us take the above configuration as an example. Others can be handled using the same methodology.
- As the system is linear, the superposition principle holds, and we can neglect u(t) while analyzing the effect of w(t):



$$E(s) = W(s) - D_H(s)$$
$$= W(s) - G_R(s) \cdot D^*(s)$$

$$Y(s) = G(s) \cdot E(s)$$
$$= G(s) \cdot W(s) - G_R(s) G(s) \cdot D^*(s)$$

$$\Rightarrow Y^*(s) = G^*(s) \cdot W^*(s) - \left[G_R(s) G(s) \right]^* \cdot D^*(s)$$

$$\Rightarrow \mathcal{Y}(z) = \mathcal{G}(z) \cdot \mathcal{W}(z) - (1 - z^{-1}) \mathcal{Z}\left\{ \frac{G(s)}{s} \right\} \cdot \mathcal{D}(z)$$

$$\mathcal{D}(z) = \mathcal{R}(z) \cdot \mathcal{Q}(z)$$

$$Q(s) = H(s) \cdot Y(s) = H(s) G(s) W(s) - G_R(s) G(s) H(s) \cdot D^*(s)$$

$$\Rightarrow Q^*(s) = \left[G(s) H(s) \right]^* \cdot W^*(s) - \left[G_R(s) G(s) H(s) \right]^* \cdot D^*(s)$$

$$\Rightarrow \mathcal{Q}(z) = \mathcal{GH}(z) \cdot \mathcal{W}(z) - (1 - z^{-1}) \mathcal{Z}\left\{ \frac{G(s) H(s)}{s} \right\} \cdot \mathcal{D}(z)$$

Let us call:

$$\mathcal{G}(z) = \mathcal{Z}(G(s))$$

$$\mathcal{G}_s(z) = \mathcal{Z}\left\{\frac{G(s)}{s}\right\} \qquad \underline{\text{etc.}}$$

$$\Rightarrow \left|\begin{array}{l} \mathcal{Y}(z) = \mathcal{G}(z)\mathcal{W}(z) - (1-z^{-1})\,\mathcal{G}_s(z)\cdot\mathcal{D}(z) \\ \mathcal{D}(z) = \mathcal{R}(z)\cdot\mathcal{Q}(z) \\ \mathcal{Q}(z) = \mathcal{GH}(z)\cdot\mathcal{W}(z) - (1-z^{-1})\,\mathcal{GH}_s(z)\cdot\mathcal{D}(z) \end{array}\right|$$

$$\Rightarrow \mathcal{D}(z) = \mathcal{R}(z)\cdot\mathcal{GH}(z)\cdot\mathcal{W}(z) - (1-z^{-1})\,\mathcal{R}(z)\,\mathcal{GH}_s(z)\cdot\mathcal{D}(z)$$

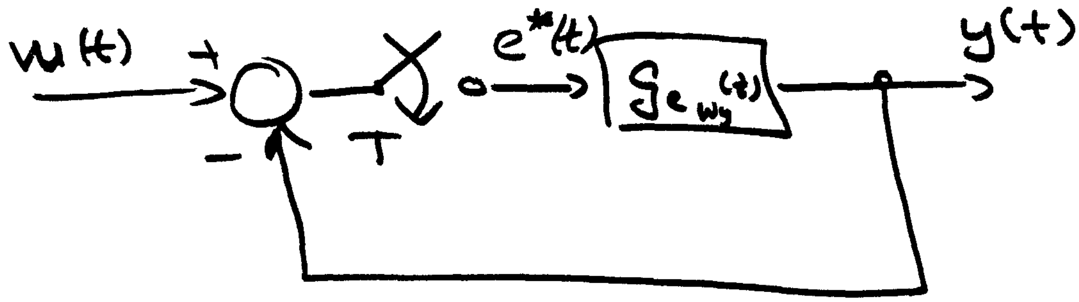
$$\Rightarrow \mathcal{D}(z) = \frac{\mathcal{R}(z)\cdot\mathcal{GH}(z)}{1+(1-z^{-1})\,\mathcal{R}(z)\,\mathcal{GH}_s(z)}\cdot\mathcal{W}(s)$$

$$\Rightarrow \mathcal{Y}(z) = \underbrace{\left[\mathcal{G}(z) - \frac{(1-z^{-1})\,\mathcal{G}_s(z)\cdot\mathcal{R}(z)\cdot\mathcal{GH}(z)}{1+(1-z^{-1})\,\mathcal{R}(z)\,\mathcal{GH}_s(z)}\right]}_{\mathcal{G}_{tot_{wy}}(z)}\mathcal{W}(s)$$

Now, we calculate an equivalent transfer function:

$$\mathcal{G}_{e_{wy}}(z) = \frac{\mathcal{G}_{tot_{wy}}(z)}{1 - \mathcal{G}_{tot_{wy}}(z)}$$

From there:



we calculate the error signal:

$$\mathcal{E}(z) = \frac{1}{1 + \mathcal{G}_e(z)} \cdot \mathcal{W}(z)$$

Assume: Type Ø disturbance:

$$\mathcal{W}(z) = \frac{z}{z-1}$$

$$\Rightarrow e(\infty) = \lim_{z \to 1} (1 - z^{-1}) \mathcal{E}(z)$$

$$= \lim_{z \to 1} \frac{z-1}{z} \cdot \frac{1}{1 + G_e(z)} \cdot \frac{z}{z-1}$$

$$\Rightarrow e(\infty) = \lim_{z \to 1} \frac{1}{1 + G_e(z)} \overset{!}{=} \cancel{1}$$

For steady-state suppression.

$\Rightarrow$ Unfortunately, the results are here not as simple as in the continuous case.