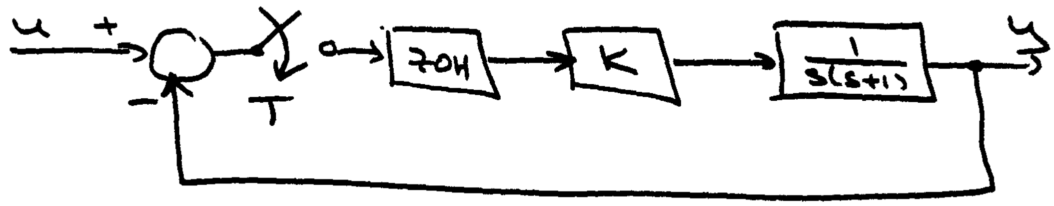
Root Locus Analysis:

As a matter of principle, we could analyze sampled data systems just the same way, we do analyze continuous systems.

Example:



$$G_{tot}(z) = \frac{GG_R(z)}{1 + GG_R(z)}$$

$$GG_R(z) = (1 - z^{-1}) \left[\frac{K}{s^2(s+1)} \right]^*$$

$$= (1 - z^{-1}) \left[\frac{K}{s^2} - \frac{K}{s} + \frac{K}{(s+1)} \right]^*$$

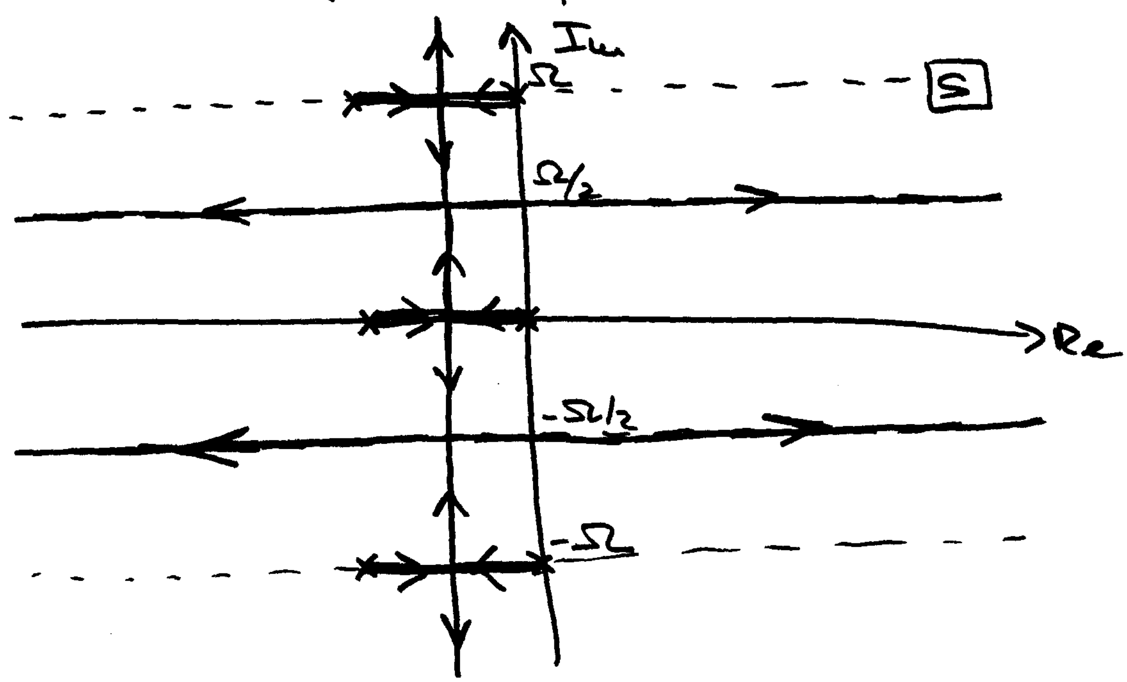
$$= (1 - z^{-1}) \left[\frac{KTz}{(z-1)^2} - \frac{Kz}{(z-1)} + \frac{Kz}{(z - e^{-T})} \right]$$

$$= (1-z^{-1}) \frac{Kz\left((T+e^{-T}-1)z - (Te^{-T}+e^{-T}-1)\right)}{(z-1)^2 (z-e^{-T})}$$

$$= \frac{K\left[(T+e^{-T}-1)z - (Te^{-T}+e^{-T}-1)\right]}{(z-1)(z-e^{-T})} = G_0(z)$$

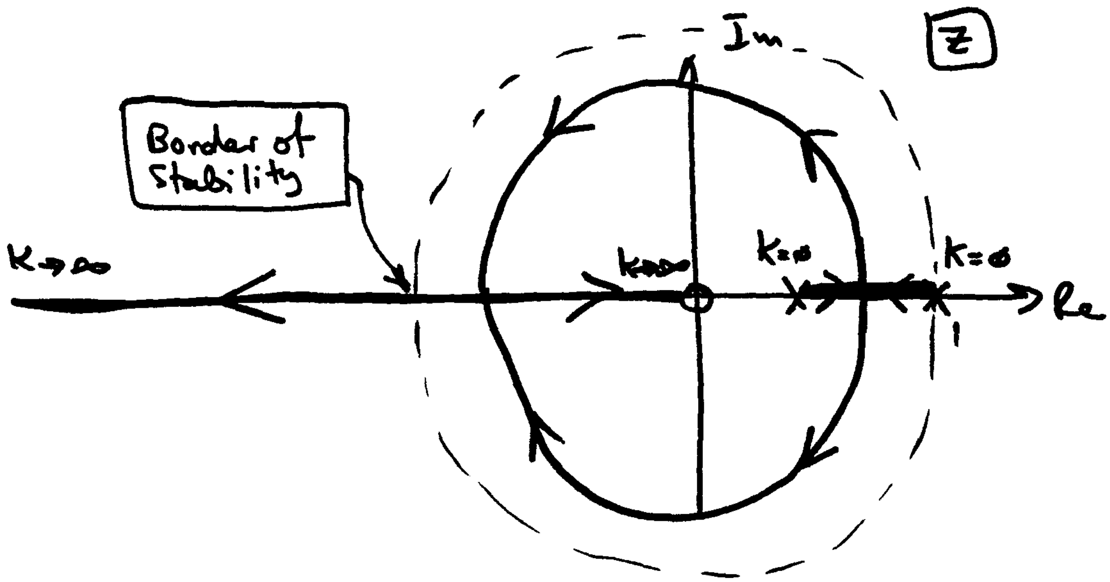
We have now two possibilities:

(α) Analyze the system in the Laplace domain ($G^*(s)$). This is possible but impractical, as we have to deal with an infinity of poles:



(β) Analyze the system in the z-domain ($G(z)$).

- Here, we can apply exactly the same rules again as in the Laplace-Domain, as the algebraic structure of $G(z)$ is the same as that of $G(s)$ (rational function).



This is more practical. However, it will happen more frequently than in the continuous case that the parameter is <u>not</u> in the wanted location (e.g. if the parameter is T (like in our first example).

⇒ We then have to use the generalized Root-Locus (as introduced in ECE 441).

Differences to the RL for Continuous Systems

A) Although the rule to find the intersection with the imaginary axis is still valid, its significance is largely reduced.

→ We would prefer to have a rule for the intersection with the unity circle.

⇒ Such a rule exists, but it is less handy to calculate.

RL: $1 + G_o(z) = 0 = 1 + \frac{K\,P(z)}{Q(z)}$

$$\rightarrow \boxed{Q(z) + K\,P(z) = 0}$$

Any value of z that satisfies this equation belongs to the root locus.

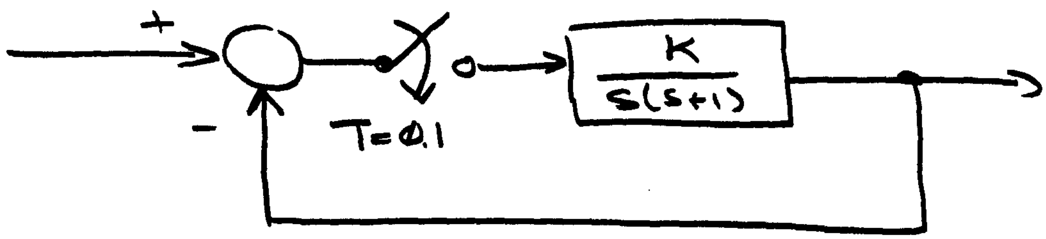
$\Longrightarrow$ On the unity circle:

$$|z| = 1 \iff z = e^{j\varphi} = \cos\varphi + j\sin\varphi$$

We can plug this into the above equation, and obtain 2 equations (unfortunately highly nonlinear) for K and φ.

Example:

Let us analyze the stability of the system:



(a) by use of Jury's scheme:

$$G(s) = \frac{K}{s(s+1)} = \frac{K}{s} - \frac{K}{(s+1)}$$

$$\Rightarrow \mathcal{G}(z) = \frac{Kz}{z-1} - \frac{Kz}{z-e^{-T}} = \mathcal{G}_0(z)$$

$$\Rightarrow \mathcal{G}_0(z) = \frac{Kz(z-e^{-T}) - Kz(z-1)}{(z-1)(z-e^{-T})} = \frac{Kz(1-e^{-T})}{(z-1)(z-e^{-T})}$$

$$\Rightarrow \mathcal{G}_{tot}(z) = \frac{\mathcal{G}(z)}{1+\mathcal{G}_0(z)} = \frac{Kz(1-e^{-T})}{Kz(1-e^{-T}) + (z-1)(z-e^{-T})}$$

$$\Rightarrow \quad Q(z) = z^2 + (K - Ke^{-T} - 1 - e^{-T})z + e^{-T} = 0$$

z^2	e^{-T}	$(K - Ke^{-T} - 1 - e^{-T})$	1
z^2	1	$(K - Ke^{-T} - 1 - e^{-T})$	e^{-T}

(1) $$Q(1) = 1 + K - Ke^{-T} - 1 - e^{-T} + e^{-T}$$

$$= K(1 - e^{-T}) \in [0, 4)$$

$$\Rightarrow \underline{\underline{K > 0}} \quad ; \quad \underline{\underline{K < \frac{4}{1 - e^{-T}} = 42.0333}}$$

(2) $$Q(-1) = 1 - K + Ke^{-T} + 1 + e^{-T} + e^{-T}$$

$$= K(e^{-T} - 1) + 2(e^{-T} + 1) \in [0, 4)$$

$$\Rightarrow \underline{\underline{K < \underbrace{\frac{-2(e^{-T}+1)}{(e^{-T}-1)}}_{40.0333}}} \quad ; \quad \underline{\underline{K > \underbrace{\frac{4 - 2(e^{-T}+1)}{(e^{-T}-1)}}_{-2}}}$$

(3) $|a_0| < |a_n|$

$$\Rightarrow |e^{-T}| < 1$$

$\Rightarrow$ always guaranteed.

(4) Condition falls away.

$\Rightarrow$ System stable for

$$\underline{\underline{K \in (0, 40.0333)}}$$

(6) by use of the Root Locus Technique:

$$G_o(z) = \frac{Kz(1-e^{-T})}{(z-1)(z-e^{-T})} = \frac{K\,P(z)}{Q(z)}$$

$$\Rightarrow Q(z) + K\,P(z) = z^2 + \left[K - Ke^{-T} - 1 - e^{-T}\right]z + e^{-T} = 0$$

$$z = \cos\varphi + j\sin\varphi = e^{j\varphi}$$

$$\Rightarrow e^{j2\varphi} + \left[K - Ke^{-T} - 1 - e^{-T}\right] e^{j\varphi} + e^{-T} = \phi$$

$$\Rightarrow \left(\cos(2\varphi) + j\sin(2\varphi)\right) + \left[K - Ke^{-T} - 1 - e^{-T}\right]\left(\cos\varphi + j\sin\varphi\right) + e^{-T} = \phi$$

$$\Rightarrow \left|\begin{array}{l} \cos(2\varphi) + \left[K - Ke^{-T} - 1 - e^{-T}\right]\cos\varphi + e^{-T} = \phi \\ \sin(2\varphi) + \left[K - Ke^{-T} - 1 - e^{-T}\right]\sin\varphi = \phi \end{array}\right|$$

$\Rightarrow$ 2 equations for K and φ

$$\Rightarrow \left|\begin{array}{l} \left[K - Ke^{-T} - 1 - e^{-T}\right]\cos\varphi = -e^{-T} - \cos(2\varphi) \\ \left[K - Ke^{-T} - 1 - e^{-T}\right]\sin\varphi = -\sin(2\varphi) \end{array}\right|$$

$$\Rightarrow \frac{\cos\varphi}{\sin\varphi} = \frac{e^{-T} + \cos(2\varphi)}{\sin(2\varphi)}$$

$$\Rightarrow \cos\varphi \cdot \sin(2\varphi) = \sin\varphi\, e^{-T} + \sin\varphi \cdot \cos(2\varphi)$$

$$\Rightarrow 2\sin\varphi\cos^2\varphi = \sin\varphi \cdot e^{-T} + \sin\varphi\left(2\cos^2\varphi - 1\right)$$

$$\Rightarrow \phi = \sin\varphi \cdot e^{-T} - \sin\varphi = \sin\varphi\left(e^{-T} - 1\right)$$

$$\Rightarrow \sin\varphi = \phi \Rightarrow \underline{\underline{\varphi = \phi}} \; ; \; \underline{\underline{\varphi = \pi}}$$

$\underline{\underline{\varphi=0}}$:

$$\Rightarrow \left[K - Ke^{-T} - 1 - e^{-T}\right]\cdot 1 = -e^{-T} - 1$$

$$\Rightarrow K\left(1-e^{-T}\right) = 1 + e^{-T} - e^{-T} - 1 = 0$$

$$\Rightarrow \underline{\underline{K=0}}$$

$\underline{\underline{\varphi=\pi}}$:

$$\Rightarrow \left[K - Ke^{-T} - 1 - e^{-T}\right](-1) = -e^{-T} - 1$$

$$\Rightarrow K\left(1-e^{-T}\right) = 2\left(1+e^{-T}\right)$$

$$\Rightarrow \underline{\underline{K}} = \frac{2\left(1+e^{-T}\right)}{\left(1-e^{-T}\right)} = \underline{\underline{40.0333}}$$

⇒ Same answer as above.
Unfortunately, the 2 equations are highly nonlinear, and in a more complicated example, we better use a numerical approximation (Newton Raphson) and a calculator.

B) Very often, we wish to express the RL in terms of a parameter that is <u>not</u> in the normal position.

In ECE 441, we learned a technique (generalized RL) for that purpose:

$$\mathcal{G}_o(z, K, \alpha) = \frac{K \cdot \mathcal{P}(z, \alpha)}{Q(z, \alpha)} = -1$$

$$\Rightarrow \quad Q(z, \alpha) + K \cdot \mathcal{P}(z, \alpha) = \emptyset$$

$$\Rightarrow \quad \tilde{Q}(z, K) + \alpha \cdot \tilde{\mathcal{P}}(z, K) = \emptyset$$

$$\Rightarrow \quad \tilde{\mathcal{G}}_o(z, K, \alpha) = \frac{\alpha \cdot \tilde{\mathcal{P}}(z, K)}{\tilde{Q}(z, K)} = -1$$

This equivalent transfer function gives us the RL $= f(\alpha)$.

Example:

$$\mathcal{G}_0(z) = \frac{Kz\left(1-e^{-T}\right)}{(z-1)\left(z-e^{-T}\right)}$$

Let $\alpha = e^{-T}$

$$\Rightarrow \quad \mathcal{G}_0(z,K,\alpha) = \frac{Kz(1-\alpha)}{(z-1)(z-\alpha)}$$

$$\Rightarrow \quad Q(z) + K\mathcal{P}(z) = (z-1)(z-\alpha) + Kz(1-\alpha) = \phi$$

$$\Rightarrow \quad z^2 + (K - K\alpha - 1 - \alpha)z + \alpha = \phi$$

$$\Rightarrow \underbrace{\left[z^2 + (K-1)z\right]}_{\tilde{Q}(z,K)} + \alpha \cdot \Big\{ \underbrace{-(K+1)z + 1}_{\tilde{\mathcal{P}}(z,K)} \Big\} = \phi$$

$$\Rightarrow \quad \tilde{\mathcal{G}}_0(z,K,\alpha) = \frac{\alpha \cdot \left[-(K+1)z+1\right]}{z \cdot \left[z + (K-1)\right]} = -1$$

With this, we can draw a RL in α which we can easily scale in T.

<u>However</u>: We often want to get the RL $= f(T)$. In more complex cases, the characteristic equation will have terms in T, e^{-T}, e^{-2T}, $\underline{etc.}$, and thus be no longer linear in T (or e^{-T}).

$\Rightarrow$ The RL still exists, but the geometric rules are gone! We have no choice but to use a computer, and iteratively solve for the roots for increasing values of T.