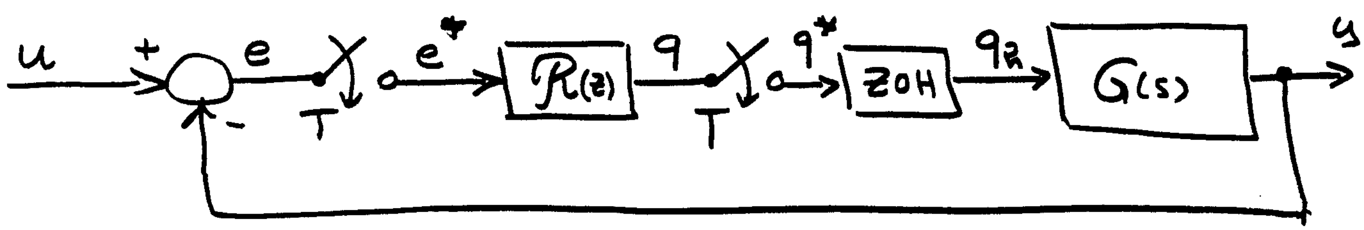
$$y(k) = \begin{bmatrix} 0.0008 & -0.0354 & 0.1561 & -0.2115 & 0.0904 \end{bmatrix} \underline{\xi}(k)$$

$$\Rightarrow \mathcal{G}(z) = \frac{0.0904z^4 - 0.2115z^3 + 0.1561z^2 - 0.0354z + 0.0008}{z^5 - 4.1716z^4 + 6.8241z^3 - 5.4337z^2 + 2.0806z - 0.3001}$$

Example:

Given the system:



$$G(s) = \frac{1}{s(s+1)}$$

$$\mathcal{R}(z) = \frac{a_2 + a_1 z^{-1}}{1 + b_1 z^{-1}} = \frac{a_2 z + a_1}{z + b_1}$$

Find a state-space representation as a function of the parameters $\{a_1, a_2, b_1, T\}$.

(a) Continuous Subsystem:

$$\underline{u} = \begin{bmatrix} u \\ q_h \end{bmatrix} \quad ; \quad \underline{y} = \begin{bmatrix} y \\ e \end{bmatrix}$$

$$\ddot{y} + \dot{y} = q_h \quad \Rightarrow \quad x_1 = y \; ; \; x_2 = \dot{y}$$

$$\Rightarrow \left| \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 + u_2 \end{array} \right|$$

$$\left| \begin{array}{l} y = x_1 \\ e = u - y = u_1 - x_1 \end{array} \right|$$

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \underline{u} \\ \\ \underline{y} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \underline{u} \end{array} \right|$$

is a continuous state-space representation of this system.

$$A = \begin{bmatrix} \emptyset & 1 \\ \emptyset & -1 \end{bmatrix} \Rightarrow A \cdot T = \begin{bmatrix} \emptyset & T \\ \emptyset & -T \end{bmatrix}$$

$$\Rightarrow e^{AT} = \mathcal{L}^{-1}\left\{ (sI - AT)^{-1} \right\}$$

$$(sI - AT) = \begin{bmatrix} s & -T \\ \emptyset & (s+T) \end{bmatrix}$$

$$\Rightarrow (sI - AT)^{-1} = \frac{1}{s(s+T)} \cdot \begin{bmatrix} (s+T) & T \\ \emptyset & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{T}{s(s+T)} \\ \emptyset & \frac{1}{(s+T)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s} & \left(\frac{1}{s} - \frac{1}{(s+T)} \right) \\ \emptyset & \frac{1}{(s+T)} \end{bmatrix}$$

$$\Rightarrow e^{AT} = \begin{bmatrix} 1 & (1 - e^{-T}) \\ \emptyset & e^{-T} \end{bmatrix} = F$$

Inputs: u_1 has no ZOH
u_2 has a ZOH

$$\Rightarrow \quad \underline{\dot{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \underline{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\underline{b}_1} u_1 + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\underline{b}_2} u_2$$

$$\Rightarrow \quad \underline{g}_1 = e^{AT} \cdot \underline{b}_1 = F \cdot \underline{b}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{g}_2 = e^{AT} \int_0^T e^{-A\sigma} d\sigma \cdot \underline{b}_2$$

We must solve this integral as A^{-1} does not exist.

$$\int_0^T e^{-A\sigma} d\sigma = \int_0^T \begin{bmatrix} 1 & (1 - e^{\sigma}) \\ 0 & e^{\sigma} \end{bmatrix} d\sigma$$

$$= \begin{bmatrix} \sigma & (\sigma - e^{\sigma}) \\ 0 & e^{\sigma} \end{bmatrix} \Bigg|_0^T = \begin{bmatrix} T & (T - e^T + 1) \\ 0 & (e^T - 1) \end{bmatrix}$$

$$\Rightarrow \int_0^T e^{-A\sigma} \cdot d\sigma \cdot \underline{b}_2 = \begin{bmatrix} T & (T + 1 - e^T) \\ 0 & (e^T - 1) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} (T+1-e^{T}) \\ (e^{T}-1) \end{bmatrix}$$

$$\Rightarrow \quad \underline{g}_2 = e^{AT} \begin{bmatrix} (T+1-e^{T}) \\ (e^{T}-1) \end{bmatrix} = F \cdot \begin{bmatrix} (T+1-e^{T}) \\ (e^{T}-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (1-e^{-T}) \\ \phi & e^{-T} \end{bmatrix} \cdot \begin{bmatrix} (T+1-e^{T}) \\ (e^{T}-1) \end{bmatrix} = \begin{bmatrix} (T+1-e^{T}+e^{T}-1-1+e^{-T}) \\ (1-e^{-T}) \end{bmatrix}$$

$$\Rightarrow \quad \underline{g}_2 = \begin{bmatrix} (T-1+e^{-T}) \\ (1-e^{-T}) \end{bmatrix}$$

$$\Rightarrow \quad \underline{x}(k+1) = \begin{bmatrix} 1 & (1-e^{-T}) \\ \phi & e^{-T} \end{bmatrix} \underline{x}(k) + \begin{bmatrix} \phi & (T-1+e^{-T}) \\ \phi & (1-e^{-T}) \end{bmatrix} \underline{u}(k)$$

$$\underline{y}(k) = \begin{bmatrix} 1 & \phi \\ -1 & \phi \end{bmatrix} \underline{x}(k) + \begin{bmatrix} \phi & \phi \\ 1 & \phi \end{bmatrix} \underline{u}(k)$$

is the discretized version.

(b) Discrete Subsystem:

$\underline{u} = [e^*]$ $\quad\quad$ $\underline{y} = [q]$

$$Q(z) = \mathcal{R}(z) \cdot \mathcal{E}(z) = \frac{a_2 z + a_1}{z + b_1} \cdot \mathcal{E}(z)$$

$$= \left[a_2 + \frac{a_1 - a_2 b_1}{z + b_1} \right] \cdot \mathcal{E}(z)$$

$$= a_2 \mathcal{E}(z) + (a_1 - a_2 b_1) \cdot Q_1(z)$$

where: $\quad Q_1(z) = \dfrac{1}{z + b_1} \cdot \mathcal{E}(z)$

$$\Rightarrow z Q_1(z) = -b_1 Q_1(z) + \mathcal{E}(z)$$

$$\Rightarrow \quad q_1(k+1) = -b_1 q_1(k) + e(k)$$

Let $x_3 = q_1$

$$\Rightarrow \left| \begin{array}{l} x_3(k+1) = -b_1 x_3(k) + e(k) \\ \underline{y}_3(k) = (a_1 - a_2 b_1) x_3(k) + a_2 e(k) \end{array} \right|$$

(c) Combine the subsystems:

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad ; \quad \underline{u} = \begin{bmatrix} u \\ e \\ q \end{bmatrix} \qquad \underline{y} = \begin{bmatrix} y \\ e \\ q \end{bmatrix}$$

$$\underline{x}(k+1) = \begin{bmatrix} 1 & (1-e^{-T}) & \phi \\ \phi & e^{-T} & \phi \\ \phi & \phi & -b_1 \end{bmatrix} \underline{x}(k) + \left[\begin{array}{c|cc} \phi & \phi & (T-1+e^{-T}) \\ \phi & \phi & (1-e^{-T}) \\ \phi & 1 & \phi \end{array}\right] \underline{u}(k)$$

$$\underline{y}(k) = \left[\begin{array}{ccc} 1 & \phi & \phi \\ \hline -1 & \phi & \phi \\ \phi & \phi & (a_1-a_2b_1) \end{array}\right] \underline{x}(k) + \left[\begin{array}{c|cc} \phi & \phi & \phi \\ \hline 1 & \phi & \phi \\ \phi & a_2 & \phi \end{array}\right] \underline{u}(k)$$

$$\underline{g}_1 = \begin{bmatrix} \phi \\ \phi \\ \phi \end{bmatrix} \quad ; \quad G_2 = \begin{bmatrix} \phi & (T-1+e^{-T}) \\ \phi & (1-e^{-T}) \\ 1 & \phi \end{bmatrix}$$

$$\underline{h}_1' = \begin{bmatrix} 1 & \phi & \phi \end{bmatrix} \quad ; \quad H_2 = \begin{bmatrix} -1 & \phi & \phi \\ \phi & \phi & (a_1-a_2b_1) \end{bmatrix}$$

$$i_{11} = \phi \quad ; \quad \underline{i}_{12}' = \begin{bmatrix} \phi & \phi \end{bmatrix} ; \quad \underline{i}_{21} = \begin{bmatrix} 1 \\ \phi \end{bmatrix} ; \quad I_{22} = \begin{bmatrix} \phi & \phi \\ a_2 & \phi \end{bmatrix}$$

$$(I^{(2)} - I_{22}) = \begin{bmatrix} 1 & \emptyset \\ -a_2 & 1 \end{bmatrix} \Rightarrow \left[I^{(2)} - I_{22}\right]^{-1} = \begin{bmatrix} 1 & \emptyset \\ a_2 & 1 \end{bmatrix}$$

$$\Rightarrow F_{new} = F + G_2 \underbrace{\left[I^{(2)} - I_{22}\right]^{-1} H_2}$$

$$\begin{bmatrix} 1 & \emptyset \\ a_2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & \emptyset & \emptyset \\ \emptyset & \emptyset & (a_1 - a_2 b_1) \end{bmatrix} = \begin{bmatrix} -1 & \emptyset & \emptyset \\ -a_2 & \emptyset & (a_1 - a_2 b_1) \end{bmatrix}$$

$$\Rightarrow G_2 \left[I^{(2)} - I_{22}\right]^{-1} H_2 =$$

$$= \begin{bmatrix} \emptyset & (T - 1 + e^{-T}) \\ \emptyset & (1 - e^{-T}) \\ 1 & \emptyset \end{bmatrix} \cdot \begin{bmatrix} -1 & \emptyset & \emptyset \\ -a_2 & \emptyset & (a_1 - a_2 b_1) \end{bmatrix}$$

$$= \begin{bmatrix} (a_2 - a_2 T - a_2 e^{-T}) & \emptyset & (a_1 T - a_2 b_1 T - a_1 + a_2 b_1 + a_1 e^{-T} - a_2 b_1 e^{-T}) \\ (a_2 e^{-T} - a_2) & \emptyset & (a_1 - a_2 b_1 - a_1 e^{-T} + a_2 b_1 e^{-T}) \\ -1 & \emptyset & \emptyset \end{bmatrix}$$

$$\Rightarrow F_{new} = \begin{bmatrix} (1 + a_2 - a_2 T - a_2 e^{-T}) & (1 - e^{-T}) & (a_1 T - a_2 b_1 T - a_1 + a_2 b_1 + a_1 e^{-T} - a_2 b_1 e^{-T}) \\ (a_2 e^{-T} - a_2) & e^{-T} & (a_1 - a_2 b_1 - a_1 e^{-T} + a_2 b_1 e^{-T}) \\ -1 & \emptyset & -b_1 \end{bmatrix}$$

$$\underline{g}_{new} = \underline{g}_1 + G_2 \cdot \underbrace{\left[I^{(2)} - I_{22} \right]^{-1} \cdot \underline{i}_{21}}_{\begin{bmatrix} 1 & \phi \\ a_2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \phi \end{bmatrix} = \begin{bmatrix} 1 \\ a_2 \end{bmatrix}}$$

$$\Rightarrow G_2 \left[I^{(2)} - I_{22} \right]^{-1} \cdot \underline{i}_{21} = \begin{bmatrix} \phi & (T-1+e^{-T}) \\ \phi & (1-e^{-T}) \\ 1 & \phi \end{bmatrix} \cdot \begin{bmatrix} 1 \\ a_2 \end{bmatrix} = \begin{bmatrix} (a_2T - a_2 + a_2e^{-T}) \\ (a_2 - a_2e^{-T}) \\ 1 \end{bmatrix}$$

$$\Rightarrow \underline{g}_{new} = \begin{bmatrix} (a_2T - a_2 + a_2e^{-T}) \\ (a_2 - a_2e^{-T}) \\ 1 \end{bmatrix}$$

$$\underline{h}'_{new} = \underline{h}'_1 + \underbrace{\underline{i}'_{12}}_{[\phi\ \phi]} \cdot \left[I^{(2)} - I_{22} \right]^{-1} \cdot H_2$$

$$\Rightarrow \underline{h}'_{new} = \underline{h}'_1 = \begin{bmatrix} 1 & \phi & \phi \end{bmatrix}$$

$$\underline{i}_{new} = \underbrace{i_{11}}_{\phi} + \underbrace{\underline{i}'_{12}}_{[\phi\ \phi]} \cdot \left[I^{(2)} - I_{22} \right]^{-1} \cdot \underline{i}_{21} = \phi$$