Homomorphisms of Geometric Graphs

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A (simple) abstract graph $G$ is given by a finite set $V = V(G)$ of vertices and a set $E = E(G) \subseteq \binom{V}{2}$ of edges. A geometric graph is a straight-edge drawing of an abstract graph in the plane, i.e., the each vertex $v \in V$ is represented by a point $p_v \in \mathbb{R}^2$, and each edge $\{u, v\} \in E$ by the straight-line segment joining $p_u$ and $p_v$. It is assumed that no segment passes through any point $p_w$ apart from its endpoints.

A homomorphism between abstract graphs $G$ and $H$ is an adjacency-preserving map between the vertex sets, i.e., a map $f : V(G) \rightarrow V(H)$ such that $\{u, v\} \in E(G)$ implies $\{f(u), f(v)\} \in E(H)$. There is a rich and powerful theory of such homomorphisms, see for instance the recent textbook [1]. As an example for the expressive power of graph homomorphisms, observe that a graph $G$ has a proper coloring with $r$ colors if and only if there is a homomorphism $G \rightarrow K_r$ into the complete graph on $r$ vertices.

The goal of this project is to define and study an analogous notion of homomorphisms between geometric graphs. There are of course several possible ways how to define a homomorphism of geometric graphs. For example, one could define a homomorphism between geometric graphs $G$ and $H$ as a map $f$ between the vertices (points) that is a homomorphism in the previous sense when $G$ and $H$ are considered as abstract graphs (i.e., $f$ preserves adjacency), and additionally $f$ should preserve geometric intersections between edges: if the edges (segments) $ab$ and $cd$ of $G$ intersect, then also the edges $f(a)f(b)$ and $f(c)f(d)$ should intersect.

One way of finding out how “good” such a definition is is by investigating which parts of the theory of abstract graph homomorphisms carry over to the geometric case. For instance, one of the fundamental theorems about graph homomorphisms is the so-called density theorem, which, roughly speaking says the following. If one defines a partial ordering of abstract graphs by $G \leq H$ if there is a homomorphism $G \rightarrow H$, then the resulting ordering is dense, i.e., for any two graphs $G < H$, there is another graph $X$ such that $G < X < H$ (the precise statement of the density theorem is a little bit more complicated because one has to avoid certain technical difficulties). As an example for a different partially ordered set with such a density property, consider the set of rational numbers with the usual ordering: it is dense in this sense (on the other hand, the set of integers is not, as there is no integer properly between $n$ and $n + 1$).

References