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Gram's Equation – a Probabilistic Proof

If β_v are the angles at the vertices v, $v \in V$, of a convex n-gon in the plane, then $\sum_{v\in V} \beta_v = (n-2)\pi$ – known as Euclid's angle-sum equation. We normalize the full angle to 1, i.e. we set $\alpha_v = \beta_v/(2\pi)$. Then the equation reads as

$$\sum_{v\in V} \alpha_v = n/2 - 1 \tag{1}$$

This identity has a generalization to higher dimensions, called Gram's equation. We want to derive this identity by a probabilistic argument. Let us immediately start with the proof, the identity will emerge quite naturally.

Consider a convex polytope $\mathcal{P}$ in 3-space, with vertex set V, edge set E and face set F. For a vertex v, let α_v denote the fraction of an infinitesimally small sphere centered at v that is contained in $\mathcal{P}$. Similarily, for an edge e, let α_e denote the fraction of an infinitesimally small sphere centered at a point in the relative interior of e that is contained in $\mathcal{P}$.

Let us now investigate a random parallel orthogonal projection of $\mathcal{P}$, i.e. we choose a random point uniformly distributed on S^2, and we project $\mathcal{P}$ in the direction specified by this point to a plane orthogonal to the direction. The projection is a convex polygon. What is the expected number of vertices we get? The probability that a vertex v will not project to a vertex in the projected polygon is $2\alpha_v$. Thus the expected number of vertices is

$$\sum_{v\in V} (1 - 2\alpha_v) \ . \tag{2}$$

Similarly, the expected number of edges in the projection is

$$\sum_{e\in E} (1 - 2\alpha_e) \ . \tag{3}$$

Now, since the number of vertices equals the number of edges, (2) equals (3) and we have the equation

$$\sum_{v\in V} \alpha_v - \sum_{e\in E} \alpha_e = (|V| - |E|)/2 = -|F|/2 + 1 \ . \tag{4}$$

The formula was known to de Gua (1783) for the case of a tetrahedron.[1] Hopf attributes the result for 3-polytopes to Brianchon (1837), while Grünbaum refers to Gram [3] for a first proof of the result. Grünbaum reports that Gram's paper was forgotten, and that meanwhile Dehn [1] and Poincaré [7] contributed to the subject.

Hopf gave another simple proof of the identity. He uses the angle-sum equation for spherical triangles (which we somehow replaced by the probabilistic argument).

The formula generalizes to arbitrary dimensions (see e.g. [4]). For a d-polytope, let α_i, $i = 0, 1, \ldots d$, be the sum of all solid angles at i-faces (defined as above for 0- and 1-faces). In particular, $\alpha_d = 1$, and α_{d-1} is half the number f_{d-1} of $d-1$-faces. Then

$$\sum_{i=0}^{d} (-1)^i \alpha_i = 0 \tag{5}$$

which looks similar to Euler's formula, or

$$\sum_{i=0}^{d-2} (-1)^i \alpha_i = (-1)^d (f_{d-1}/2 - 1) \tag{6}$$

[1] Hopf [6] refers to Abbé de Gua (1783), Grünbaum [4] to the paper [5] below by J. P. de Gua de Malves. Probably, Grünbaum mistakenly assigned Gram's initials to de Gua.

to show the similarities to the formulas (1) and (4) in 2- and 3-space, respectively. The probabilistic argument given above readily generalizes to a proof of the higher dimensional result. A proof is also indicated by Edelsbrunner in [2] (Remark 3, Section 6).

References

[1] M. Dehn, Die Eulersche Formel in Zusammenhang mit dem Inhalt in der nicht-Euklidischen Geometrie, *Math. Ann.* **61** (1905) 561–586

[2] H. Edelsbrunner, The union of balls and its dual shape, *Proc.* 9^{th} *Annual ACM Symposium on Computational Geometry* (1993) 218–231

[3] J. P. Gram, Om Rumvinklerne i et Polyeder, *Tidsskr. Math.* **4** (1874), 161–163

[4] B. Grünbaum, *Convex Polytopes*, John Wiley & Sons, Interscience, London (1967)

[5] J. P. de Gua de Malves, Propositions neuves, et non moins utiles que curieuses, sur le teétraère, *Hist. Acad. R. des Sci.*, Paris (1783)

[6] H. Hopf, Über Zusammenhänge zwischen Topologie und Metrik im Rahmen der elemetaren Geometrie, *Math. Physik. Sem. Ber.* **3** (1953) 16–29

[7] H. Poincaré, Sur la generalization d'un theoreme élémentaire de Geometrie, *Compt. Rend. Acad. Sci. Paris* **140** (1905) 113–117