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In between k-Sets, j-Facets, and i-Faces: (i, j)-Partitions*

Artur Andrzejak
Hewlett-Packard Laboratories
1501 Page Mill Road, Palo Alto
CA 94304, USA
artur_andrzejak@hp.com

Emo Welzl
Inst. Theoretische Informatik
ETH Zürich
CH-8092 Zürich, Switzerland
emo@inf.ethz.ch

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Abstract

Let S be a finite set of points in general position in $\mathbb{R}^d$. We call a pair (A, B) of subsets of S an (i, j)-*partition of* S, if $|A| = i$, $|B| = j$ and there is an oriented hyperplane h with $S \cap h = A$ and with B the set of points from S on the positive side of h. (i, j)-Partitions generalize the notions of k-sets (these are $(0, k)$-partitions) and j-facets ((d, j)-partitions) of point sets as well as the notion of i-faces of the convex hull of S ($(i + 1, 0)$-partitions). In oriented matroid terminology, (i, j)-partitions are covectors where the number of 0's is i and the numbers of $+$'s is j.

We obtain linear relations among the numbers of (i, j)-partitions, mainly by means of a correspondence between $(i - 1)$-faces of so-called k-set polytopes on the one side and (i, j)-partitions for certain j's on the other side. We also describe the changes of the numbers of (i, j)-partitions during continuous motion of the underlying point set. This allows us to demonstrate that in dimensions exceeding 3, the vector of the numbers of k-sets does not determine the vector of the numbers of j-facets – nor vice versa. Finally, we provide formulas for the numbers of (i, j)-partitions of points on the moment curve in $\mathbb{R}^d$.

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1 Introduction and Prerequisites

We denote by S a finite set of points in $\mathbb{R}^d$, $n := |S|$. We will frequently assume *general position*: no $i+1$ points lie in a common $(i-1)$-flat, for $i = 1, \ldots, d$. Without further mention, throughout the paper i, j, k, ℓ and m denote integers ($\mathbb{Z}$), while n stands for a natural number (non-negative integer, $\mathbb{N}_0$) and d for a natural number or[1] -1.

(i, j)-Partitions. We assume general position of S. A pair (A, B) of subsets of S is called *(i, j)-partition (of S)*, if $|A| = i$, $|B| = j$ and there is an oriented hyperplane h with $S \cap h = A$ and with B the set of points from S on the positive side of h; we say that h *induces* the (i, j)-partition (A, B). (A, B) is also called a *hyperplane partition of S* if the indices (i, j) do not matter. $D_{i,j} = D_{i,j}(S)$ denotes the number of (i, j)-partitions of S.

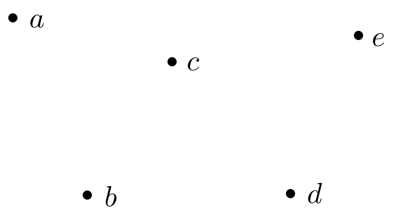
It is easy to see that

$$D_{i,j} \neq 0 \qquad \text{iff } 0 \leq i \leq d \text{ and } 0 \leq j \leq n-i. \tag{1}$$

For example, the planar point set displayed in Figure 1 has the following $(1, 1)$-partitions.

$$\begin{array}{ccccc} (\{a\},\{b\}) & (\{a\},\{e\}) & (\{b\},\{a\}) & (\{b\},\{d\}) & (\{d\},\{b\}) \\ (\{d\},\{e\}) & (\{e\},\{d\}) & (\{e\},\{a\}) & (\{c\},\{a\}) & (\{c\},\{e\}) \end{array}$$

The table in Figure 1 lists all non-zero values $D_{i,j}$ of this point set.[2]



$D_{i,j}$	$j=0$	1	2	3	4	5	
$i=0$	1	4	6	6	4	1	$\leftarrow \overline{a}$
1	4	10	12	10	4		
2	4	6	6	4	$\leftarrow \overline{e}$		
	$\uparrow f$						

Figure 1: A planar point set and the numbers of its (i, j)-partitions.

[1] $\mathbb{R}^{-1} := \emptyset$ and $\mathbb{R}^0$ is a singleton.

[2] As a result of our findings in this paper, all $D_{i,j}$'s of a planar 5-point set are completely determined by entry $D_{0,1}$ (or by $D_{1,2}$), while $D_{1,1}$ equals 10 – independently from the configuration.

The 'boundary values' $D_{0,k}$, $D_{d,j}$ and $D_{i,0}$ specialize to the established notions of k-sets, j-facets of point sets and $(i-1)$-faces of a simplicial polytope, respectively. These notions will be recapitulated below, also since they play a key role in discussions and proofs of this paper.

The goal of these investigations is to establish the 'missing link' between k-sets and j-facets, similar to the situation for convex polytopes, where the faces of various dimensions interpolate between vertices and facets. These 'in-between' objects are indispensable for the understanding of the structure (face-lattice) of a polytope. Even if one is only interested in the vertices versus facets aspects of the Upper Bound Theorem ([18], cf. [27]), consideration of the whole f-vector is essential for any proof known.

Results. Theorem 2.2 in Section 2 exhibits some linear relations among the numbers of (i,j)-partitions. Section 3 describes the changes of the $D_{i,j}$'s during continuous motion of the underlying point set. While the linear relations in Section 2 reveal certain redundancies in the $D_{i,j}$'s, Section 4 shows that in dimensions exceeding 3, the vector of the numbers of k-sets does not determine the vector of the numbers of j-facets – nor vice versa. Finally, in Section 5 we derive formulas for the numbers of (i,j)-partitions for points on the moment curve.

For our proofs we analyze k-set polytopes and we employ oriented matroids terminology (see definitions and discussion of these notions later in this section).

A notion related to (i,j)-partitions has been introduced by Mulmuley [19] in the dual setting, where he generalizes h-vectors and derives equivalents of the Dehn-Sommerville Relations. For a simple hyperplane arrangement in $\mathbb{R}^d$, he considers i-faces of the arrangement at level j (relative to $\mathbf{0}$), where the level of a face is the number of hyperplanes in the arrangement that separate the relative interior of the face from the origin $\mathbf{0} \in \mathbb{R}^d$.

For comparison to our setting, we briefly translate (dualize by polarity)[3] Mulmuley's to the equivalent problem for point configurations S, where $\mathbf{0} \notin S$ and $S \dot\cup \{\mathbf{0}\}$ in general position is assumed. Let us call a pair (A,B) of subsets of S an (i,j)*-level pair* (relative to the origin $\mathbf{0}$), if there is an oriented hyperplane h *with the origin* $\mathbf{0}$ *on its negative side* such that $S \cap h = A$ and B is the set of points from S on the positive side of h. Mulmuley considers the numbers[4] $M_{i,j}$ of (i,j)-level pairs. The main result in [19] establishes relations among the $M_{i,j}$'s, $j \leq k$, for k fixed, under the assumption that every hyperplane through $\mathbf{0}$ contains at least $k+1$ points in both of its halfspaces.

Note that $M_{i,j} \leq D_{i,j}$, and if every open halfspace (defined by a hyperplane) containing $\mathbf{0}$ has at least $j+1$ points, then $D_{i,j} = M_{i,j}$. This property can be

[3]Alternatively, consider Mulmuley's definitions for linear arrangements instead of affine arrangements.

[4]In [19], this is the number of $(d-i)$-faces at level j, denoted by $f^j(d-i)$ there.

achieved for all $j \leq \lfloor \frac{n}{d+1} \rfloor - 1$ by 'placing' $\mathbf{0}$ at or close[5] to a centerpoint of S (by translation of S), cf. [9, Theorem 4.3]. The set of relations from [19] have been further investigated and extended in [1].

j-Facets. For a sequence $(p_1, \ldots, p_{d+1}) \in (\mathbb{R}^d)^{d+1}$ of $d+1$ points in $\mathbb{R}^d$ we define its *sign*[6] $\chi(p_1, \ldots, p_{d+1})$ as the sign (-1, 0, or $+1$) of the determinant $\det\,(p_i\ 1)_{i=1}^{d+1}$ (the matrix has the coordinates of the points as rows, extended by a 1). A sequence $(p_1, \ldots, p_d)$ of d distinct points in general position partitions space into

$$\{p \in \mathbb{R}^d \mid \chi(p_1, \ldots, p_d, p) = s\} \quad \text{for } s = -1, 0, +1.$$

The set for $s = 0$ constitutes the hyperplane containing $\{p_1, \ldots, p_d\}$, while the sets for $s = +1$ and $s = -1$ are called the *positive and negative*, resp., *side of* $(p_1, \ldots, p_d)$; positive and negative side are invariant under even permutations of the defining point sequence. For $d \geq 2$ an ordered d-tuple of points in general position is called an *oriented* $(d-1)$*-simplex*, where we consider even permutations of the same sequence to be equivalent (i.e. every d-point set in general position gives rise to exactly two oriented $(d-1)$-simplices). The case $d = 1$ needs special treatment: here an *oriented* 0*-simplex* is a pair $(p_1, o) \in \mathbb{R} \times \{-1, +1\}$ where positive and negative side are $\{p \in \mathbb{R} \mid s \cdot o \cdot (p - p_1) > 0\}$ for $s = +1$ and $s = -1$, resp.

Assume general position of S. A *j-facet of S* is an oriented $(d-1)$-simplex spanned by d distinct points in S that has exactly j points of S on its positive side. 0-Facets of S are in correspondence to facets of the convex hull of S. We write $e_j = e_j(S)$ for the number of j-facets of S and we call the vector $\overline{e} = \overline{e}(S) := (e_j)_{j \in \mathbb{Z}}$ the *vector of j-facets* (of S). Clearly, $e_j = 0$ for $j \notin \{0, \ldots, n-d\}$.

There is an obvious correspondence between (d, j)-partitions and j-facets which gives

$$e_j = D_{d,j}, \qquad \text{provided } n \neq d.$$

The case $n = d$ is peculiar, since then the unique d-tuple in S gives rise to one $(d, 0)$-partition, while there are two 0-facets, one for each orientation of the simplex spanned by these points; hence, $e_0 = 2$, while $D_{d,0} = 1$, in this case.

Remark 1 *A hyperplane h inducing a hyperplane partition (A, B) can be perturbed so that it induces any of the hyperplane partitions in*

$$\operatorname{pert}(A, B) := \{(A', B \cup B') \mid A' \subseteq A,\ B' \subseteq A \backslash A'\}\ . \tag{2}$$

Moreover, h can be 'moved' until it contains d points, while never moving over a point and while preserving incidence to A; then it induces a (d, k)-partition

[5] The actual centerpoint may be unique and enforce a degeneracy (e.g. be part of S) – the necessary perturbation is accounted for by the use of floor instead of ceiling brackets.

[6] Obviously, transposition of two entries in the sequence switches the sign to its negative value – such a map is called *alternating sign map*.

(A'', B'') with $A \subseteq A''$ and $B \backslash (A'' \backslash A) \subseteq B'' \subseteq B$; thus, $(A, B) \in \operatorname{pert}(A'', B'')$. (The pair (A'', B'') we reach is not unique.)

Therefore, given i and j, the set of all (i, j)-partitions[7] *can be obtained as*[8]

$$\{(A', B \cup B') \mid A' \in \tbinom{A}{i},\ B' \in \tbinom{A \backslash A'}{j-k},\ \text{for some } (d,k)\text{-partition } (A, B)\}\ . \tag{3}$$

In this way the set *of (d, k)-partitions determines the* set *of all hyperplane partitions.*[9] *By way of contrast, we will see that in dimension $d \geq 4$, in general, the* numbers *$e_k = D_{d,k}$ do not determine all $D_{i,j}$'s.*

k-Sets. We relax the condition of general position. A *k-set of S* is a set B of k points in S that can be separated from $S \backslash B$ by a hyperplane disjoint from S. We denote by $a_k = a_k(S)$ the number of k-sets of S and call $\overline{a} = \overline{a}(S) := (a_k)_{k \in \mathbb{Z}}$ the *vector of k-sets* (of S); note $a_0 = a_n = 1$ and $a_k = 0$ for $k \notin \{0, \ldots, n\}$.

Clearly, B is a k-set iff $(\emptyset, B)$ is a $(0, k)$ partition. This yields

$$a_k = D_{0,k}\ .$$

k-Sets and j-facets have received considerable attention in combinatorial and computational geometry (starting with papers by Lovász [16] and Erdős et al. [11] in the early 1970's) with particular interest in upper and lower bounds on their numbers. Despite of some progress in $\mathbb{R}^2$ and $\mathbb{R}^3$ in recent years, large gaps still remain (see [1, Chapter 6] or [17, Chapter 11] for surveys, and [7, 22, 23, 26] for very recent developments). In computational geometry k-sets play a role for higher-order Voronoi diagrams, halfspace range searching problems, analysis of randomized algorithms and so on (note also the related dual notion of k-levels in arrangements of hyperplanes). Recently, k-sets of the infinite set $\mathbb{N}_0^d$ – so-called *corner cuts* – have been investigated because of a relation to computational commutative algebra [20, 5, 25].

Faces of Polytopes. Let $\mathcal{P}$ be a convex d-polytope. We assume familiarity with the notion of i-dimensional faces, *i-faces* for short, of $\mathcal{P}$, cf. [14, 27]. By $f_i = f_i(\mathcal{P})$ we denote the number of i-faces, with $f_{-1} := 1$ (accounting for the empty face) and $f_d := 1$ (counting $\mathcal{P}$ as a d-face of itself); $f_i := 0$ for $i \notin \{-1, \ldots, d\}$.

If S is in general position and $\mathcal{P}$ is the convex hull $\operatorname{conv} S$ of S, then $\mathcal{P}$ is a simplicial d'-polytope, $d' := \min\{d, n-1\}$. (Simplicial means that every face apart from $\mathcal{P}$ is a simplex.) The convex hull of a set $A \in \binom{S}{i}$, $i \in \{0, \ldots, d\}$, constitutes an $(i-1)$-face F of $\mathcal{P}$ iff there is an oriented hyperplane h with

[7] As a side remark: If one knows for each oriented $(d-1)$-simplex the *number* of points on its positive side, then the actual *sets* of points on the positive sides of oriented $(d-1)$-simplices can be retrieved, see [12].

[8] Note, however, that an (i, j)-partition (A', B') may be obtained from many (d, k)-partitions.

[9] And we have the inequality $D_{i,j} \leq \sum_{k=j-d+i}^{j} \binom{d}{i}\binom{d-i}{j-k} g_k$; an estimate that readily allows improvement though if $i < d$.

$S \cap h = A$ and no point from S on the positive side of h (a *hyperplane supporting* $\mathcal{P}$ *in* F) – in other words, iff $(A, \emptyset)$ is an $(i, 0)$-partition of S and therefore,

$$f_{i-1}(\operatorname{conv} S) = D_{i,0} \ , \quad \text{for } i \neq d+1.$$

The case $i = d+1$ is special, in that $f_d = 1$ if $n \geq d+1$, and $f_d = 0$, otherwise, while $D_{d+1,0} = 0$, always.

k-Set Polytopes. General position is not assumed. The *k-set polytope* $Q_k(S)$ is the convex hull of the set

$$\sigma\binom{S}{k} := \left\{ \sigma(T) \,\middle|\, T \in \binom{S}{k} \right\} \ , \qquad \text{where } \sigma(T) := \textstyle\sum_{p \in T} p \ ;$$

(Note $\sigma\binom{S}{k} = \emptyset$ for $k \notin \{0, \ldots, n\}$, hence $Q_k(S) = \emptyset$ for such k's; and $\binom{S}{0} = \{\emptyset\}$, hence $Q_0(S)$ degenerates to the origin $\mathbf{0}$ in $\mathbb{R}^d$.) Beware that, in general, $Q_k(S)$ is not simplicial, even if S is in general position. We will shortly characterize the conditions for simpliciality (see Corollary 2.8 below), and we will characterize the types of faces that can occur (so-called hypersimplices, which are k'-set polytopes of some point set for some k', see Theorem 2.7 (b.2)).

k-Set polytopes have been introduced in [10] for proving upper bounds on the number of k-sets of dense point sets. Another application of k-set polytopes is the enumeration of k-sets via reverse search [3]. We refer also to the related notion of corner cut polytopes [20, 25], which are simply k-set polytopes of $\mathbb{N}_0^d$. These applications exploit a natural bijection between the vertices of a k-set polytope and the k-sets of the underlying point set S, see Figure 2.

We extend this relation in Theorems 2.1 and 2.7 to a bijection between the $(i-1)$-faces of a k-set polytope (where $i \in \{2, \ldots, d\}$) and the (i, j)-partitions for $j \in \{k-(i-1), \ldots, k-1\}$. This will be used to establish one of the relations among the numbers of (i, j)-partitions (Theorem 2.2 (7)).

Covectors (Oriented Matroids). $\langle \mathbf{v}, \mathbf{w} \rangle$ denotes the scalar product of two vectors (or points) $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^d$. Given $\mathbf{c} \in \mathbb{R}^d \backslash \{\mathbf{0}\}$ and $\alpha \in \mathbb{R}$, the *oriented hyperplane h with parameters* $(\mathbf{c}, \alpha)$ is the set $h := \{x \in \mathbb{R}^d \mid \langle c, x \rangle = \alpha\}$, with its *positive open halfspace* $h^+ := \{x \in \mathbb{R}^d \mid \langle c, x \rangle > \alpha\}$ and its *negative open halfspace* $h^- := \{x \in \mathbb{R}^d \mid \langle c, x \rangle < \alpha\}$; hence, $\mathbb{R}^d = h^+ \,\dot\cup\, h \,\dot\cup\, h^-$.

We assume some numbering $\{p_1, \ldots, p_n\}$ of the points in S. Every oriented hyperplane h in $\mathbb{R}^d$ defines a vector $\mathrm{U} = \mathrm{U}(h) \in \{+, -, 0\}^n$ by

$$(\mathrm{U})_i := \begin{cases} + & \text{if } p_i \in h^+, \\ 0 & \text{if } p_i \in h, \\ - & \text{if } p_i \in h^-, \end{cases}$$

where $(\mathrm{U})_i$ is the i-th entry of U. U is called a *covector of* S, and $\mathcal{L}(S)$ denotes the set of all covectors of S induced by all possible oriented hyperplanes. The set

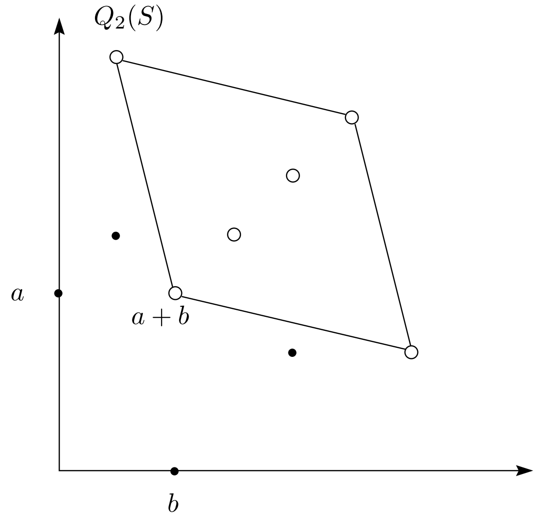


Figure 2: A set S_2 of 4 points in $\mathbb{R}^2$ (black) and the corresponding 2-set polytope.

of covectors of S induced by every possible oriented hyperplane in $\mathbb{R}^d$ determines the *oriented matroid* of S. In general, if a subset $\mathcal{L} \subseteq \{+, -, 0\}^n$ fulfills certain conditions, then it determines such an oriented matroid $\mathcal{M}(\mathcal{L})$ (see [4] for the full definition[10]). Oriented matroids which arise from sets of points are called *realizable*. The *support* of a covector U is the index set $\{i \mid (\mathrm{U})_i \neq 0\}$. Covectors of inclusion minimal support are called *cocircuits*. If all cocircuits have the same number of 0's, then the oriented matroid is called *uniform*, which is the case if it comes from a point set in general position. Cocircuits determine all covectors; in the uniform case, we can simply replace 0's in a cocircuit arbitrarily by any sign in $\{+, -, 0\}$ and we obtain a covector, and we obtain all of them in this way (this is basically a restatement of (3) in Remark 1 above).

There is an obvious correspondence between covectors and (i, j)-partitons: An oriented hyperplane h induces an (i, j)-partition iff it induces a covector where the number of 0's is i and the number of $+$'s is j; similarly, j-facets correspond to cocircuits with j the number of $+$'s. We do *not* claim our results to hold for oriented matroids (other than realizable ones), but we employ oriented matroids terminology for some of our proofs.

Notation and Conventions. Given sets $X, Y \subseteq \mathbb{R}^d$, we let $X + Y$ denote their sum $\{x+y \mid x \in X, y \in Y\}$, and we use $x+Y$ short for $\{x\}+Y$. For a point set $X \subseteq \mathbb{R}^d$, its affine hull is denoted by $\operatorname{aff} X$ and its convex hull by $\operatorname{conv} X$.

We assume the binomial coefficient $\binom{i}{j}$ to be defined for all i and j, where it is 0 unless $i \geq j \geq 0$. We use brackets for the indicator function for a predicate P: $[P] := 1$ if P is true and $[P] := 0$, otherwise. We use the sum convention that

[10]And, to be precise, an oriented matroid always contains the all-0 covector which we ignore here unless $n \leq d$.

the empty sum is the zero of the underlying monoid; e.g. for T an empty set of points in $\mathbb{R}^d$, $\sum_{p \in T} p = \mathbf{0}$, etc.

2 k-Set Polytopes and Linear Relations

Throughout this section, let S be a set of n points in $\mathbb{R}^d$, with explicit mention whenever general position is assumed.

We define

$$f_i^{(k)} = f_i^{(k)}(S) := \begin{cases} f_i(Q_k(S)) \ , & \text{if } i \in \{-1, \ldots, d-1\}, \text{ and} \\ 0 \ , & \text{otherwise.} \end{cases}$$

Remark 2 *There is a small subtlety in the definition of $f_i^{(k)}$ that ought not to be swept under the rug. Let $d' \leq d$ be the dimension of $Q_k(S)$. If $d' = d$, then $f_{d'}^{(k)} = 0$; otherwise, $f_{d'}^{(k)} = 1$. The "logic" behind this proceeding is that we count the whole polytope $Q_k(S)$ as a face of itself only if it is contained in a hyperplane of the ambient space.*

Theorem 2.1 *For S in general position,*

$$f_{i-1}^{(k)} = \begin{cases} 1 \ , & \textit{if } i = 0, \\ D_{0,k} \ , & \textit{if } i = 1, \textit{ and} \\ \sum_{j=k-(i-1)}^{k-1} D_{i,j} \ , & \textit{otherwise.} \end{cases}$$

Remark 3 *It follows from Theorem 2.1 that*

$$f_{i-1}^{(k+1)} - f_{i-1}^{(k)} = D_{i,k} - D_{i,k-(i-1)} \ , \qquad \textit{provided } i \notin \{0, 1\}. \tag{4}$$

Therefore, by successive application of (4),

$$D_{i+1,k} = \sum_{m \geq 0} \left(f_i^{(k+1-im)} - f_i^{(k-im)} \right) \ , \qquad \textit{provided } i \notin \{-1, 0\};$$

in particular, $D_{2,k} = f_1^{(k+1)}$. That is, the $D_{i,j}$'s, $i \neq 1$, are determined by the $f_{i-1}^{(k)}$'s.

Via the Euler-Poincaré Formula, the theorem yields the linear relation (7) in Theorem 2.2 below. While (5) is obvious, (8) needs separate proof and can be seen as a generalization of the relation $d\, f_{d-1} = 2\, f_{d-2}$ for the f-vector of a simplicial d-polytopes (one of the Dehn-Sommerville Relations).

Theorem 2.2 *The following relations hold for S in general position.*

$$D_{0,0} = 1 \quad \textit{and} \quad D_{i,j} = D_{i,n-i-j} \ . \tag{5}$$

$$\sum_{j\in\mathbb{Z}} D_{i,j} = \begin{cases} 2\binom{n}{i}\varphi_{d-i}(n-i) \ , & \textit{for } n > i \geq 0 \textit{ and } i \leq d, \\ 1 \ , & \textit{for } n = i \leq d, \textit{ and} \\ 0 \ , & \textit{otherwise} \ , \end{cases} \tag{6}$$
$$\textit{where } \varphi_d(m) := \textstyle\sum_{\ell=0}^{d} \binom{m-1}{\ell} \textit{ for } m \geq 1.$$

$$D_{0,k} + \textstyle\sum_{i=2}^{d} \sum_{j=k-(i-1)}^{k-1} (-1)^{i-1} D_{i,j} = 1 - (-1)^d \ , \tag{7}$$
$$\textit{provided } k \in \{1, \ldots, n-1\} \textit{ and } n \geq d+1.$$

$$d\,(D_{d,j} + D_{d,j-1}) = 2\,D_{d-1,j} \ , \quad \textit{provided } n \geq d+1. \tag{8}$$

Remark 4 (Open Problem) *If $d \geq 3$, the statements of Theorem 2.1 saliently circumvent the numbers $D_{1,j}$. For $d = 2$, they are determined by the remaining entries of D (because of (8)), but for $d \geq 3$, we do not understand their relation to other entries.*

Remark 5 *For $d = 2$, (7) reads as*

$$D_{0,k} = D_{2,k-1} \ , \qquad (\textit{i.e. } a_k = e_{k-1}) \qquad \textit{for } n \geq 3 \textit{ and } k \in \{1, \ldots, n-1\},$$

which is the known simple relation between k-sets and $(k-1)$-facets in the plane.

If $d = 3$, then (7) amounts to

$$D_{0,k} - D_{2,k-1} + D_{3,k-2} + D_{3,k-1} = 2 \qquad \textit{for } n \geq 4 \textit{ and } k \in \{1, \ldots, n-1\}.$$

If we substitute in this relation the term $\frac{3}{2}(D_{3,k-1} + D_{3,k-2})$ for the term $D_{2,k-1}$ (according to (8)), we obtain[11]

$$D_{0,k} = \tfrac{1}{2}\,(D_{3,k-2} + D_{3,k-1}) + 2 \ , \qquad (\textit{i.e. } a_k = \tfrac{1}{2}(e_{k-2} + e_{k-1}) + 2)$$
$$\textit{for } n \geq 4 \textit{ and } k \in \{1, \ldots, n-1\},$$

as we have shown before in [2]. That is, again the vector of k-sets and the vector of j-facets determine each other. In Section 4 we will see that this is not the case in dimensions exceeding 3.

[11] Note $3D_{0,k} = D_{2,k-1} + 6$ for yet another relation that is a linear combination of (7) and (8) in case of $d = 3$.

Remark 6 *The relations in Theorem 2.1 are by no means a complete list of linear relations, not even of those known at this point. In particular, we have the Dehn-Sommerville Relations on $(D_{i,0})_{i\in\mathbb{Z}}$, and we have Mulmuley's relations [19] (mentioned in the introduction) with extensions in [1].*

Moreover, Gullikson and Hole [15] showed

$$\sum_{k\in\mathbb{Z}}(-1)^k a_k = 0 \qquad \textit{for odd } d.$$

Note here the relation $\sum_{j\in\mathbb{Z}}(-1)^j D_{d-1,j} = 0$ that follows immediately from (8).

The goal, of course, would be to supply a complete characterization of all linear relations, similar to the situation for the f-vector of simplicial polytopes, where we know that the Dehn-Sommerville Relations and linear relations thereof exhaust all possibilities, cf. [14, Section 9.2].

Proofs of Theorems 2.1 and 2.2 are postponed to the end of this section. We need some better understanding of k-set polytopes first.

Basic Properties of k-Set Polytopes. Recall that we have $Q_k(S) = \emptyset$ for $k \notin \{0,\dots,n\}$, $Q_0(S) = \{\mathbf{0}\}$, and $|Q_n(S)| = 1$. For the remaining values of k we get:

Lemma 2.3 *For $k \in \{1,\dots,n-1\}$, the dimensions of* $\operatorname{aff} S$ *and* $\operatorname{aff}\sigma\binom{S}{k}$ *are equal.*

Proof. We prove the stronger claim $\operatorname{aff} S = \operatorname{aff}(\frac{1}{k}\cdot\sigma\binom{S}{k})$ (recall $k > 0$).

The inclusion $\frac{1}{k}\cdot\sigma\binom{S}{k} \subseteq \operatorname{aff} S$ is immediate from the definitions of $\sigma\binom{S}{k}$ and affine combination.

For demonstrating $S \subseteq \operatorname{aff}(\frac{1}{k}\cdot\sigma\binom{S}{k})$, consider an arbitrary $p \in S$. Choose some $T \in \binom{S\setminus\{p\}}{k}$ (recall $k < n$). Now the equality

$$p = \left(\sum_{q\in T}\frac{1}{k}\cdot\sigma((T\cup\{p\})\setminus\{q\})\right) - (k-1)\frac{1}{k}\cdot\sigma(T)$$

shows that p is the affine combination of points in $\frac{1}{k}\cdot\sigma\binom{S}{k}$. ◻

If $\tau : \mathbb{R}^d \to \mathbb{R}^{d'}$, $x \mapsto v + Ax$, is an affine map that is injective on $\operatorname{aff} S$, then $\tau_k : x \mapsto kv + Ax$ is an affine map that is injective on $\operatorname{aff}\sigma\binom{S}{k} = \operatorname{aff} Q_k(S)$ with $Q_k(\tau(S)) = \tau_k(Q_k(S))$. Hence, $Q_k(\tau(S))$ and $Q_k(S)$ are affinely isomorphic. We will see that $Q_k(S)$ is determined up to affine isomorphism, if $n \le d+1$ and S in general position. (As a marginal note, observe that $Q_{n-k}(S) = Q_n(S) - Q_k(S)$.)

The *hypersimplex* $\Delta_{d-1}(k)$ (in $\mathbb{R}^d$) is the convex hull of those vertices of the d-cube $[0,1]^d$ whose coordinates sum up to k; $\Delta_{d-1}(1)$ is the standard $(d-1)$-simplex in $\mathbb{R}^d$ ([27, page 19]). Employing our terminology,

$$\Delta_{d-1}(k) = \operatorname{conv}\sigma\binom{U_d}{k} = Q_k(U_d)\ ,$$

where U_d denotes the set of $\{0,1\}$-points in $\mathbb{R}^d$ with exactly one 1-coordinate. Clearly, all points in $\sigma\binom{U_d}{k}$ are vertices of $\Delta_{d-1}(k)$, since they are among the vertices of the cube $[0,1]^d$.

Lemma 2.4 *If S is in general position and $n \leq d+1$, then $Q_k(S)$ is affinely isomorphic to the hypersimplex $\Delta_{n-1}(k)$.*

Proof. $\operatorname{conv} S$ is an $(n-1)$-dimensional simplex (due to general position and $n \leq d+1$), and thus affinely isomorphic to $\Delta_{n-1}(1)$ via an affine map $\tau : \mathbb{R}^d \to \mathbb{R}^n$, injective on $\operatorname{aff} S$ and with $\tau(S) = U_n$. By the preceding discussion $Q_k(S)$ is affinely isomorphic to $Q_k(U_n) = \Delta_{n-1}(k)$. $\square$

Remark 7 *Without going into further details, it is perhaps worthwhile to mention that if we embed S ($\subseteq \mathbb{R}^d$) in the hyperplane $\langle \mathbf{1}, x\rangle = 1$ in $\mathbb{R}^{d+1}$ ($\mathbf{1}$ the all-ones vector), then the k-set polytope of S is the cross-section of the zonotope $\operatorname{conv}\{\sigma(T) \,|\, T \in 2^S\}$ with the hyperplane $\langle \mathbf{1}, x\rangle = k$.*

Maximizing Sets and Vertices of k-Set Polytopes. Given a vector $\mathbf{c}$ in $\mathbb{R}^d \backslash \{\mathbf{0}\}$, we say that $T \in \binom{S}{k}$ *maximizes* $\mathbf{c}$ if

$$\langle \mathbf{c}, \sigma(T)\rangle \geq \langle \mathbf{c}, \sigma(T')\rangle, \quad \text{for all } T' \in \tbinom{S}{k};$$

in other words, $\sigma(T)$ lies in a supporting hyperplane of $Q_k(S)$ with normal vector $\mathbf{c}$.

Lemma 2.5 *Let $k \in \{1, \ldots, n\}$, $T \in \binom{S}{k}$, $\mathbf{c}$ a vector in $\mathbb{R}^d \backslash \{\mathbf{0}\}$ and let h be the oriented hyperplane with parameters $(\mathbf{c}, \alpha)$, where $\alpha := \min_{p \in T} \langle \mathbf{c}, p\rangle$. Then the following statements are equivalent.*

(a) $(S \backslash T) \cap h^+ = \emptyset$.

(b) *T maximizes $\mathbf{c}$.*

(c) *The sets in $\binom{S}{k}$ which maximize $\mathbf{c}$ are exactly those of the form*

$$(T \cap h^+) \cup R\ , \qquad R \in \binom{S \cap h}{|T \cap h|}\ .$$

Proof. By choice of α, we have $T \cap h^- = \emptyset$.
(c) $\Rightarrow$ **(b)** holds, since we can choose $R = T \cap h$, and $(T \cap h^+) \cup (T \cap h)$ equals T by our initial observation.
For **(b)** $\Rightarrow$ **(a)**, let p^* be some point in T with $\langle \mathbf{c}, p^*\rangle = \alpha$. If there exists a $q \in S \backslash T$ with $\langle \mathbf{c}, q\rangle > \alpha$, then

$$\langle \mathbf{c}, \sigma((T \cup \{q\}) \backslash \{p^*\})\rangle > \langle \mathbf{c}, \sigma(T)\rangle\ ,$$

– a contradiction to T maximizing $\mathbf{c}$. Therefore, if (b) holds, then $\langle \mathbf{c}, q\rangle \le \alpha$ for all $q \in S\backslash T$ and (a) holds.

Next we show **(a) $\Rightarrow$ (c)**. Put $A := S \cap h$ and $B := T \cap h^+$. Since $\langle \mathbf{c}, p\rangle = \alpha$ for all $p \in A$, $\langle \mathbf{c}, \sigma(B \cup R)\rangle$ attains the same value for all $R \in \binom{A}{|T\cap h|}$. This is also the value of $\langle \mathbf{c}, v(T)\rangle$, since $T = B \cup (T \cap h)$ (we use here $T \cap h^- = \emptyset$ again).

We are left to show that for all $T' \in \binom{S}{k}$ not of the form $B \cup R$, $R \in \binom{A}{|T\cap h|}$ there exists some $T'' \in \binom{S}{k}$ with $\langle \mathbf{c}, v(T'')\rangle > \langle \mathbf{c}, v(T')\rangle$. Suppose first that there is a point p in $T' \cap h^-$. Choose some $q \in T \setminus T'$ (this must exist, since $T' \neq T$ and $|T'| = |T|$ finite). We have $\langle \mathbf{c}, p\rangle < \alpha$ and $\langle \mathbf{c}, q\rangle \ge \alpha$; therefore $T'' := (T' \cup \{q\})\backslash\{p\}$ serves the purpose. Secondly, assume that $T' \cap h^- = \emptyset$, but $|T' \cap h| > |T \cap h|$; hence, $B\backslash T' \neq \emptyset$. Now choose some $p \in T' \cap h$ and some $q \in B\backslash T'$. Again, $T'' := (T' \cup \{q\})\backslash\{p\}$ is answering the purpose. Finally, if $T' \cap h^- = \emptyset$ and $|T' \cap h| = |T \cap h|$, then $T' \cap h^+ = B$, that is, T' is of the form excluded. For concluding $T' \cap h^+ = B$, we have eventually employed the precondition (a): $(S\backslash T) \cap h^+ = \emptyset$. ◻

Lemma 2.6 *Let S be in general position. For $k \in \{1, \ldots, n\}$ and $T \in \binom{S}{k}$, the following conditions are equivalent.*

(a) *T is a k-set.*

(b) *$\sigma(T)$ is a vertex of $Q_k(S)$.*

(c) *T maximizes some vector $\mathbf{c}$.*

(Moreover, if the conditions hold, then the cone of normal vectors of (oriented) supporting hyperplanes of $Q_k(S)$ at $\sigma(T)$ is precisely the set of normal vectors that are maximized by T.)

Proof. **(a) $\Rightarrow$ (b)**. For a k-set T there exists an oriented hyperplane h (with normal vector $\mathbf{c}$) such that $S \cap h = \emptyset$ and $S \cap h^+ = T$. While preserving these properties, we can perturb h so that all $\langle \mathbf{c}, p\rangle$, $p \in S$, are distinct; so let us assume this property. Set $\alpha := \min_{p\in T}\langle \mathbf{c}, p\rangle$. The oriented hyperplane $\hat{h}$ with parameters $(\mathbf{c}, \alpha)$ satisfies $|S\cap\hat{h}| = |T\cap\hat{h}| = 1$ and $(S\backslash T)\cap\hat{h}^+ = \emptyset$. It follows, by Lemma 2.5, that T is the unique set that maximizes $\mathbf{c}$, and thus $\sigma(T)$ is a vertex of $Q_k(S)$.

(b) $\Rightarrow$ (c). If $\sigma(T)$ is a vertex of $Q_k(S)$, then there is an oriented hyperplane h such that $\sigma(T) \in h$ and $h^+ \cap Q_k(S) = \emptyset$. That is, for $\mathbf{c}$ the normal vector of h, we have that that $\langle \mathbf{c}, \sigma(T')\rangle > \langle \mathbf{c}, \sigma(T)\rangle$ for no $T' \in \binom{S}{k}$. This constitutes that T maximizes $\mathbf{c}$.

(c) $\Rightarrow$ (a). If T maximizes some vector $\mathbf{c}$, then by Lemma 2.5 the hyperplane h with parameters $(\mathbf{c}, \min_{p\in T}\langle \mathbf{c}, p\rangle)$ has the property that $T \cap h^- = \emptyset$ and $(S\backslash T) \cap h^+ = \emptyset$. Since S is in general position, we can perturb h and obtain a hyperplane $\tilde{h}$ such that $S \cap \tilde{h}^+ = T$ and $S \cap \tilde{h} = \emptyset$. ◻

Remark 8 *The equivalence "T is a k-set $\Leftrightarrow$ $\sigma(T)$ is a vertex of $Q_k(S)$" is valid in general, i.e. without the general position assumption made in Lemma 2.6.*

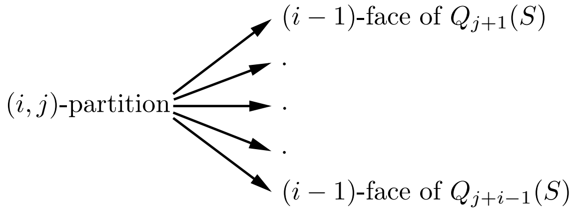


Figure 3: Visualization of Theorem 2.7, case $i \geq 2$.

Faces of k-Set Polytopes. We have prepared the grounds for the crucial result of this section, which will easily entail Theorem 2.1 and thereby Theorem 2.2 (7).

Theorem 2.7 *Let S be in general position and let (A, B) be an (i, j)-partition of S.*

(a.1) *If $i = 0$ (i.e. B is a j-set of S), then $\sigma(B)$ is a vertex of $Q_j(S)$.*

(a.2) *If $i \geq 2$, then for every $\ell \in \{1, \ldots, i-1\}$ the set*

$$\left\{\sigma(B \cup R) \,\middle|\, R \in \binom{A}{\ell}\right\} = \sigma(B) + \sigma\binom{A}{\ell}$$

is the vertex set of an $(i-1)$-face F' of $Q_{j+\ell}(S)$; we have that $F' = \sigma(B) + Q_\ell(A)$, an $(i-1)$-polytope affinely isomorphic to $\Delta_{i-1}(\ell)$.

Let F be an $(i-1)$-face of $Q_k(S)$.

(b.1) *If $i = 1$ (i.e. F is a vertex of $Q_k(S)$), then there is a unique k-set T of S with $F = \sigma(T)$.*

(b.2) *If $2 \leq i \leq d$, then there is exactly one (i, j)-partition (A', B') (for some $j \in \{k-(i-1), \ldots, k-1\}$) that induces F in the fashion described in (a.2); that is, $F = \sigma(B') + Q_{k-j}(A')$ and F is affinely isomorphic to the hypersimplex $\Delta_{i-1}(k-j)$.*

Proof. **(a.1)** is the implication (a) $\Rightarrow$ (b) from Lemma 2.6.

(a.2). Let h be an oriented hyperplane with $S \cap h = A$ and $S \cap h^+ = B$; let $(\mathbf{c}, \alpha)$ be the parameters of h. Consider $T^* = B \cup A^*$ for some $A^* \in \binom{A}{\ell}$. Note that A^* is nonempty, and so $\alpha = \min_{p \in T^*} \langle \mathbf{c}, p \rangle$. We have $(S \setminus T^*) \cap h^+ = \emptyset$ and so Lemma 2.5 tells us that the sets $B \cup R$, $R \in \binom{A}{\ell}$, are exactly those sets in $\binom{S}{j+\ell}$ that maximize $\mathbf{c}$. That is, there is a supporting hyperplane $\hat{h}$ of $Q_{j+\ell}(S)$ with normal vector $\mathbf{c}$ and with $\sigma\binom{S}{k} \cap \hat{h} = \sigma(B) + \sigma\binom{A}{\ell}$. That is, indeed $F' = \mathrm{conv}(\sigma(B) + \sigma\binom{A}{\ell})$ is a face of $Q_{j+\ell}(S)$. The remaining facts – in particular, that F' is an $(i-1)$-face and that all points in $\sigma(B) + \sigma\binom{A}{\ell}$ are vertices, follow from Lemmas 2.3 and 2.4, respectively.

(b.1). Note, as a word of warning, that $\sigma(T') = \sigma(T'')$ is possible for sets $T' \neq T''$ in $\binom{S}{k}$ – even with the general position assumption as we formulated it.

Clearly, if F is a vertex, then $F \in \sigma\binom{S}{k}$ and there has to be a set $T \in \binom{S}{k}$ with $F = \sigma(T)$. T has to be a k-set (Lemma 2.6), and there is an oriented hyperplane h with $|T \cap h| = 1$ and $S \cap (h \cup h^+) = T$. By Lemma 2.5 (c) it follows that T is the unique set that maximizes the normal vector of h and we are done.

(b.2). Note that since F is an $(i-1)$-face with $i \geq 2$, $Q_k(S)$ has to be of dimension at least 1 and hence $k \in \{1, \ldots, n-1\}$.

Consider some supporting hyperplane h of $Q_k(S)$ with $Q_k(S) \cap h = F$. Let $\mathbf{c}$ be the normal vector of h and let $T \in \binom{S}{k}$ so that $\sigma(T)$ is a vertex of F. Now consider the hyperplane $\hat{h}$ with parameters $\mathbf{c}$ and $\alpha := \min_{p \in T} \langle \mathbf{c}, p \rangle$. Lemma 2.5 (c) tells us exactly which sets in $\binom{S}{k}$ maximize $\mathbf{c}$, namely those of the form $B' \cup R$, $R \in \binom{A'}{\ell}$, where $B' := T \cap \hat{h}^+$, $A' := S \cap \hat{h}$, and $\ell := |T \cap \hat{h}|$; $\ell > 0$ by choice of $\hat{h}$, and $\ell < |A'|$, since otherwise F is of dimension 0. That is, Lemma 2.3 is applicable and $F = \sigma(B') + Q_\ell(A')$ is of dimension $|A'| - 1$; therefore, $|A'| = i$. We have obtained the claimed (i, j) partition (A', B'), where $j := |B'| = |T| - \ell$ with $1 \leq \ell \leq i - 1$.

For the proof of uniqueness, let $\mathcal{T} := \{T \in \binom{S}{k} \mid \sigma(T) \text{ is vertex of } F\}$. Recall from (b.1) that every vertex has a unique set T that generates it. It follows that $\mathcal{T} = \{B' \cup R \mid R \in \binom{A'}{\ell}\}$ for A',B' and ℓ as above. Clearly this determines (A', B'), since $B' = \bigcap_{T \in \mathcal{T}} T$ and $A' = \bigcup_{T \in \mathcal{T}} T \backslash B'$. $\square$

Corollary 2.8 *For S in general position, $Q_k(S)$ is a simplicial polytope iff $d \leq 3$ or $k \notin \{2, \ldots, n-2\}$.*

Proof. Cases $d \in \{0, 1, 2\}$ are trivial. For $d = 3$, note that $\Delta_2(1)$ and $\Delta_2(2)$ are 2-simplices, while $\Delta_2(0)$ and $\Delta_2(3)$ degenerate to a point (so, in fact, they are 0-simplices). Hence, for $d = 3$ and for all k, all facets of $Q_k(S)$ are simplices and $Q_k(S)$ is simplicial.

However, for $d \geq 4$ and $2 \leq \ell \leq d-2$, the hypersimplex $\Delta_{d-1}(\ell)$ is a $(d-1)$-polytope with $\binom{d}{\ell} > d$ vertices – thus not a simplex. Hence, for $d \geq 4$ and $k \in \{2, \ldots, n-2\}$, $Q_k(S)$ is not simplicial. $Q_1(S) = \operatorname{conv} S$ and $Q_{n-1}(S) = \sigma(S) - Q_1(S)$, so this settles the cases $k \in \{1, n-1\}$ because of general position of S. The remaining cases are trivial, since for $k \notin \{1, \ldots, n-1\}$, the k-set polytope degenerates to a single point or the empty set. $\square$

Proof of Theorem 2.1. Case '$i = 0$,' i.e. the claim $f_{-1}^{(k)} = 1$, holds by definition (recall that every polytope enjoys the presence of an empty face: $f_{-1} = 1$).

Case '$i = 1$' claims that the number of k-sets of S is exactly the number of vertices of the k-set polytope. That fact is established by the bijection described in Theorem 2.7 (a.1) and (b.1).

Case '$i \in \{2, \ldots, d\}$' follows from the bijection between the $(i-1)$-faces of $Q_k(S)$ on the one side and the set

$$\{(A, B) \mid (A, B) \text{ is an } (i, j)\text{-partition of } S \text{ with } k - i + 1 \leq j \leq k - 1\}$$

on the other side, as it is described in Theorem 2.7 (a.2) and (b.2).

Finally, if $i \notin \{0, \ldots, d\}$, then $f_{i-1}^{(k)}$ is 0 (recall $f_d^{(k)} = 0$, in particular), and so is the sum $\sum_{j=k-i+1}^{k-1} D_{i,j}$, since $D_{i,j} = 0$ for $i \notin \{0, \ldots, d\}$. $\square$

Proof of Theorem 2.2 **(5)** is self-evident.

(6) We concentrate on the case $n > i \geq 0$ and $i \leq d$. First, for $i = 0$, we have to establish that there are $\varphi_d(n) = \binom{n-1}{d} + \binom{n-1}{d-1} + \cdots \binom{n-1}{0}$ ways of dissecting S by a hyperplane disjoint from S. This is actually a well-known fact (folklore), sometimes referred to as Cover's formula [6]; see also [9, Theorem 3.1], where this is stated in a different form, though. We present a proof for the sake of completeness. Moreover we want to provide an explicit bijection between *unordered hyperplane partitions* $\{\{S \cap h^-, S \cap h^+\} \mid S \cap h = \emptyset\}$ and at most d element subsets of $S \setminus \{a\}$, where $a \in S$ is some arbitrarily chosen *anchor point*.

Counting is obvious in $\mathbb{R}^1$. For the announced bijection, we can associate the trivial dissection $\{S, \emptyset\}$ with the empty set; a non-trivial dissection $\{B_0, B_1\}$, $\max B_0 < \min B_1$, is associated with $\min B_1$, if $a \in B_0$, and with $\max B_0$, if $a \in B_1$.

Now assume $d > 1$. Choose some generic line λ through a, so that the orthogonal projection of S on a hyperplane orthogonal to λ is in general position within this hyperplane. Let S' denote the projection of S. By induction hypothesis, there are $\varphi_{d-1}(n)$ unordered hyperplane partitions of S' in its affine hull; these are in correspondence to the dissections of S that can be realized by a hyperplane parallel to λ.

Given any other unordered hyperplane partition $\{B_0, B_1\}$, $a \in B_0$, consider the hyperplane h with $B_0 \subseteq S \cap (h^- \cup h)$ and $B_1 \subseteq S \cap (h^+ \cup h)$ that maximizes the distance between a and the point of intersection between λ and h. Note that the parameters of this hyperplane can be obtained from a linear program that is bounded, since no hyperplane parallel to λ realizes the partition $\{B_0, B_1\}$. Moreover, because of general position, there is a unique $A \in \binom{S \setminus \{a\}}{d}$ that determines h in the sense that the conditions $A \cap B_0 \subseteq S \cap (h^- \cup h)$ and $A \cap B_1 \subseteq S \cap (h^+ \cup h)$ lead to the same hyperplane h. The constraints in A are tight, that is, $A \subseteq h$. This set A will be associated with the partition $\{B_0, B_1\}$.

Why is every $A \in \binom{S \setminus \{a\}}{d}$ chosen exactly once? We describe the inverse map. Given such an A, let h_A be the oriented hyperplane with $A \subseteq h_A$ and $a \in h_A^-$. The projection $A' \cup \{a'\} \subseteq S'$ of $A \cup \{a\}$ to the hyperplane orthogonal to λ is a set of $d + 1$ points in general position in a $(d-1)$-flat. There is a unique Randon partition $\{A'_0, A'_1\}$ with $a' \in A'_0$; that is $A' \cup \{a'\} = A'_0 \,\dot\cup\, A'_1$ and conv $A'_0 \cap$conv $A'_1 \neq \emptyset$ (consult Radon's Theorem, cf. [8, Theorem 2.1 and remark after Theorem 9.1]). For A_0 and A_1 the preimages of A'_0 and A'_1, respectively, set $B_0 := A_0 \cup (S \cap h_A^-)$ and $B_1 := A_1 \cup (S \cap h_A^+)$. The partition $\{B_0, B_1\}$ is indeed realizable by a hyperplane (an appropriate perturbation of h_A), and it is not difficult to see that $\{B_0, B_1\}$ is the unique partition that is mapped to A.

This completes the proof that there are $\varphi_d(n)$ unordered hyperplane partitions, and this shows that, for $n > 0$, $\sum_{j\in\mathbb{Z}} D_{0,j} = 2\varphi_d(n)$. In fact, after fixing some point $a \in S$ and an ordered sufficiently generic orthogonal basis, the proof establishes the claimed bijection between hyperplane partitions and at most d element subsets of $S\backslash\{a\}$.

The identity for $n > i > 0$ and $i \leq d$ is now easy to obtain. We simply consider each i-tuple A of points in S. We choose a generic $(d-i)$-flat κ disjoint from aff A. Every point p in $S_0 := S\backslash A$ is mapped to the intersection of aff$(A \cup \{p\})$ with κ, which results in a set S'_0 of $n-i$ points. In κ, there are $\varphi_{d-i}(n-i)$ unordered partitions (by $(d-i-1)$-flats in κ), which correspond to the ways a hyperplane h with $S \cap h = A$ can partition $S\backslash A$ (details omitted).

For a proof of **(7)** recall the Euler-Poincaré Formula for the f-vector of a d-dimensional polytope, [27, Corollary 8.17].

$$f_0 - f_1 + \cdots + (-1)^{d-1} f_{d-1} = 1 - (-1)^d \ . \tag{9}$$

If the presumptions $k \in \{1, \ldots, n-1\}$ and $n \geq d+1$ are satisfied, $Q_k(S)$ is a d-dimensional polytope (see Lemma 2.3), we can substitute the findings from Theorem 2.1 in (9) with $f_i^{(k)}$ for f_i, and we readily obtain (7).

The final relation **(8)** claims $d\,(D_{d,j} + D_{d,j-1}) = 2\,D_{d-1,j}$, provided $n \geq d+1$. We employ double counting for a proof.

Fix some j. We assign to every (d,j)-partition (A,B) the set

$$\Gamma(A,B) := \{(A\backslash\{p\}, B) \,|\, p \in A\}$$

By appropriate small rotation of the hyperplane inducing (A,B) it is easily seen that each of these pairs is a $(d-1,j)$-partition of S; and obviously there are $d = |A|$ of them. In a similar fashion, every $(d,j-1)$-partition (A,B) maps to a set

$$\Gamma(A,B) := \{(A\backslash\{p\}, B \cup \{p\}) \,|\, p \in A\}$$

of d distinct $(d-1,j)$-partitions of S.

If we can show that every $(d-1,j)$-partition (A',B') appears in exactly two such sets, the asserted identity is verified. Let h be an oriented hyperplane inducing (A',B'). We can rotate h about the $(d-2)$-flat aff A' in two directions until we hit points p and q, respectively (both not in A'); we have $p \neq q$, since $n \geq d+1$. If $p \in B'$, then we have reached a $(d,j-1)$-partition $(A' \cup \{p\}, B'\backslash\{p\})$; obviously, $(A',B') \in \Gamma(A' \cup \{p\}, B'\backslash\{p\})$. If $p \notin B'$, we have $(A',B') \in \Gamma(A' \cup \{p\}, B')$, with $(A' \cup \{p\}, B')$ a (d,j)-partition. The same applies to q instead of

p. Every hyperplane inducing (A', B') leads to the same points p and q, and we are done. $\square$

3 (i, j)-Partitions under Continuous Motion

In this section we let S and S' denote two sets of $n \geq d+2$ points[12] each in general position in $\mathbb{R}^d$, $d \geq 2$, and we will use a tacitly assumed bijection $p \mapsto p'$ between S and S'. We want to investigate the numbers of (i, j)-partitions under continuous motion of the underlying point set. S and S' can be thought of as the configuration of the moving point set right before and right after an event that changes some of the $D_{i,j}$'s. Such considerations have been exploited frequently, take the original proof of Tverberg's Theorem [24] as a prominent example in discrete geometry, and see [15, 2] for examples in the context of k-sets and j-facets.

What might change the $D_{i,j}$'s? We have seen that the (d, k)-partitions determine all (i, j)-partitions (Remark 1). As long as no point moves over a hyperplane determined by some other d points, we are save. Otherwise, the $d+1$ points $p_1, \ldots, p_{d+1}$ involved[13] change their sign[14] $\chi(p_1, \ldots, p_{d+1})$, and in a generic motion, this will be the only $(d+1)$-point subset that does so.

Mutations and Mutation Kernel. The pair (S, S') is called a *mutation* if there is a set $X \in \binom{S}{d+1}$ so that for a sequence $(q_1, \ldots, q_{d+1})$ of $d+1$ distinct points in S

$$\chi(q_1, \ldots, q_{d+1}) \neq \chi(q'_1, \ldots, q'_{d+1}) \quad \text{iff } \{q_1, \ldots, q_{d+1}\} = X.$$

The set X is called *mutation kernel* of the mutation (S, S').

We are interested in the increments $D_{i,j}(S') - D_{i,j}(S)$, and we will see that this change depends on two integer parameters of the mutation only (apart from d and n). (i) In order to introduce the first parameter observe that all hyperplanes spanned by d points in X partition $S \backslash X$ in the same way into two sets B_0 and B_1. This fact is obvious if one keeps in mind that the simplex spanned by X is 'almost flat' before and after it changes its sign. We will have to verify that this is actually guaranteed by our definition of a mutation. The size, ℓ, of B_0 determines one of the two parameters. (ii) For the second parameter note that if we choose d points in X, then the hyperplane spanned by these points may have the unique remaining point in X either on the same or on the opposite side of B_0.

[12] If $n \leq d+1$, the $D_{i,j}$'s do not depend on the configuration but on n and d only; the same is true, if $d = 1$. These cases are of no interest to us here.

[13] None of these points is distinguished, all of them move over the hyperplane determined by the remaining d points.

[14] Note that if these points change their sign in some order, then they change their sign in each orders.

The number, m, of d-point subsets of X where the remaining point in X lies on the same side as[15] B_0 is the second parameter we need to consider. (The choice of B_0 among B_0 and B_1 was arbitrary, so depending on this choice, the parameters may be (m, ℓ) or $(d+1-m, n-(d+1)-\ell)$). A more formal introduction of these parameters will be given shortly.

The Simplex Spanned by the Mutation Kernel is 'Almost Flat.' Let us assume for the rest of this section that (S, S') is a mutation with mutation kernel $X = \{p_1, \ldots, p_{d+1}\}$. The following lemma states that the hyperplane spanned by $\{p_1, \ldots, p_{d-1}, p_d\}$ separates points $\hat{p}, \hat{\hat{p}} \in S \backslash X$ iff the hyperplane spanned by $\{p_1, \ldots, p_{d-1}, p_{d+1}\}$ does so; in fact, iff the hyperplane spanned by $\{p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{d+1}\}$ does so for $i = 1, \ldots, d+1$, since we can apply the lemma to any permutation of $(p_1, \ldots, p_{d+1})$. In other words, all hyperplanes spanned by d points in X separate $S \backslash X$ in the same manner.

Lemma 3.1 *For a mutation (S, S') with mutation kernel $X := \{p_1, \ldots, p_{d+1}\}$ and for $\hat{p}, \hat{\hat{p}} \in S \backslash X$, we have*

$$\chi(\lambda, p_d, \hat{p}) \cdot \chi(\lambda, p_d, \hat{\hat{p}}) = \chi(\lambda, p_{d+1}, \hat{p}) \cdot \chi(\lambda, p_{d+1}, \hat{\hat{p}})$$

where λ is short for the sequence $p_1, \ldots, p_{d-1}$.

Proof. The three-term Grassmann-Plücker identity (cf. [4, 21]) implies that

$$\{\chi(\lambda, p_d, \hat{p}) \cdot \chi(\lambda, p_d, \hat{\hat{p}}),\ -\chi(\lambda, p_{d+1}, \hat{p}) \cdot \chi(\lambda, p_{d+1}, \hat{\hat{p}}),\ \chi(\lambda, p_d, p_{d+1}) \cdot \chi(\lambda, \hat{p}, \hat{\hat{p}})\}$$

contains $\{-1, +1\}$ or equals $\{0\}$. The value 0 contradicts general position. Now, if the asserted identity in the lemma fails to hold, this implies the value of $\chi(\lambda, p_d, p_{d+1}) \cdot \chi(\lambda, \hat{p}, \hat{\hat{p}})$ (it has to be $-\chi(\lambda, p_d, \hat{p}) \cdot \chi(\lambda, p_d, \hat{\hat{p}})$).

If the value of $\chi(\lambda, p_d, p_{d+1}) \cdot \chi(\lambda, \hat{p}, \hat{\hat{p}})$ is fixed by $-\chi(\lambda, p_d, \hat{p}) \cdot \chi(\lambda, p_d, \hat{\hat{p}})$, then – by definition of a mutation – it remains unchanged under the mapping $p \mapsto p'$; a contradiction to the mutation condition which lets $\chi(\lambda, p_d, p_{d+1})$ change its sign under the mapping, but not so the sign $\chi(\lambda, \hat{p}, \hat{\hat{p}})$. ◻

At this point we prefer to pass over to covector terminology. To this end assume an underlying ordering $\{p_1, \ldots, p_n\}$ of S with $\{p_1, \ldots, p_{d+1}\} = X$. We set $\overline{+} := -$, $\overline{-} := +$, $\overline{0} := 0$, and for sequences $\mathrm{v} = v_1 \ldots v_s$ and $\mathrm{w} = w_1 \ldots w_t$ over $\{+, -, 0\}$, we let $\overline{\mathrm{v}} := \overline{v_1} \ldots \overline{v_s}$ and $\mathrm{vw} := v_1 \ldots v_s w_1 \ldots w_t$. Moreover, we let $\operatorname{pert} \mathrm{v}$ be the set of all vectors which agree with v on all nonzero positions of v – in accordance with the analogous definition for hyperplane partitions in (2). Recall that a cocircuit is a covector where the number of 0's is d, and that the set $\mathcal{L}(S)$ of all covectors of S can be written as the union of all sets $\operatorname{pert} \mathrm{c}$ over all cocircuits c of S.

[15] The reader may justifiably worry, what that means if B_0 is empty – a forthcoming more formal definition will clarify.

Formal Introduction of Mutation Parameters. Now the only cocircuits of S which are not cocircuits of S' are those induced by oriented hyperplanes spanned by d points in X, i.e. the positions of 0's in these cocircuits (d of them) are among the first $d+1$ positions; obviously, there are $2(d+1)$ such cocircuits. Moreover, Lemma 3.1 shows that there is a vector[16] $\mathrm{z} \in \{+,-\}^{n-(d+1)}$ such that all cocircuits with all their 0's in the first $d+1$ positions are from $\{+,-,0\}^{d+1}\mathrm{z}$ or $\{+,-,0\}^{d+1}\overline{\mathrm{z}}$; note that z cannot be the empty sequence due to our assumption $n \geq d+2$. We denote those cocircuits by c_i and $\overline{\mathrm{c}_i}$, $i = 1, \ldots, d+1$, where

$$\mathrm{c}_i = \overbrace{0\ldots0}^{i-1} g_i \overbrace{0\ldots0}^{d+1-i} \mathrm{z}\ , \qquad g_i \in \{+,-\}.$$

For S', these cocircuits are substituted for by c'_i and $\overline{\mathrm{c}'_i}$, $i = 1, \ldots, d+1$, where

$$\mathrm{c}'_i = \overbrace{0\ldots0}^{i-1} \overline{g_i} \overbrace{0\ldots0}^{d+1-i} \mathrm{z}\ .$$

Now we call the pair (S, S') an *(m, ℓ)-mutation*, for m the number of $+$'s among the g_i's and ℓ the number of $+$'s in z.

Lemma 3.2 *If (S, S') is an (m, ℓ)-mutation, then (i) it is also a $(d+1-m, n-(d+1)-\ell)$-mutation, and (ii) (S', S) is a $(d+1-m, \ell)$-mutation. Moreover, (iii) $1 \leq m \leq d$ and (iv) $0 \leq \ell \leq n-(d+1)$.*

Proof. The only item that needs some consideration is (iii). Suppose $m = 0$, that is $g_i = -$ for all $i = 1, \ldots, d+1$. That is, all points with corresponding entry $+$ in z are separated from the interior of conv X by *all* hyperplanes spanned by facets of conv X. No point can satisfy this, so z has all $-$'s. But now switch to S' and apply the argument to X' and the cocircuits $\overline{\mathrm{c}'_i}$ to obtain a contradiction.

If $m = d+1$, apply the reasoning to X and the cocircuits $\overline{\mathrm{c}_i}$. ◻ .

Switching Covectors. We have by now complete control of the changes in the set of cocircuits from S to S': The c_i 's and $\overline{\mathrm{c}_i}$'s go, and the c'_i 's and $\overline{\mathrm{c}'_i}$'s come. We know that the c_i's stand for m of the $(\ell+1)$-facets of S and $(d+1)-m$ of the ℓ-facets of S, and similarly for the other cocircuits involved. So we could easily derive the increments $e_j(S') - e_j(S)$ in terms of m and ℓ (and d and n, of course) now. We head for the general setting, instead.

Recall that all covectors of S can be obtained as perturbations of cocircuits. It follows, that all covectors in the symmetric difference $\mathcal{L}(S) \oplus \mathcal{L}(S')$ must be perturbations of one of c_i, $\overline{\mathrm{c}_i}$, c'_i or $\overline{\mathrm{c}'_i}$, $i = 1, \ldots, d+1$, and thus have to equal z or $\overline{\mathrm{z}}$ in their last $n-(d+1)$ entries.

[16] A vector representing the unique partition of $S \backslash X$ by hyperplanes spanned by d points in the mutation kernel X. There are two of them, namely z and $\overline{\mathrm{z}}$.

The issue remaining is the following: Do the cocircuits C_i, $\overline{\mathrm{C}_i}$, C'_i or $\overline{\mathrm{C}'_i}$, $i = 1, \ldots, d+1$, and perturbations thereof tell us everything about covectors of the form $\{+,-,0\}^{d+1}\mathrm{Z}$ or $\{+,-,0\}^{d+1}\overline{\mathrm{Z}}$? After all, such covectors appear also as perturbations of other cocircuits. The following lemma clarifies the picture.

Lemma 3.3 *Let (S, S') be a mutation, with C_i's and Z as defined above. Then* all *covectors in $\mathcal{L}(S)$ of the form $\{+,-,0\}^{d+1}\mathrm{Z}$ are in $\bigcup_{i=1}^{d+1} \operatorname{pert} \mathrm{C}_i$.*

Proof. Note that for $\mathrm{V} \in \{+,-,0\}^{d+1}$, we have $\mathrm{VZ} \in \bigcup_{i=1}^{d+1} \operatorname{pert} \mathrm{C}_i$ iff $(\mathrm{V})_i = g_i$ for some $i = 1, \ldots, d+1$; that is, iff $\overline{g_1} \ldots \overline{g_{d+1}}\,\mathrm{Z} \notin \operatorname{pert} \mathrm{VZ}$. Hence, the assertion of the lemma is equivalent to $\overline{g_1} \ldots \overline{g_{d+1}}\,\mathrm{Z} \notin \mathcal{L}(S)$; for sufficiency recall that every covector forces all of its permutations to be covectors.

Now let us restrict ourselves to the subset $P = \{p_1, \ldots, p_{d+2}\}$ of S. If, indeed, $\overline{g_1} \ldots \overline{g_{d+1}}\,\mathrm{Z} \in \mathcal{L}(S)$, this shows that $\mathcal{L}(P) \supseteq \{+,-\}^{d+1}(\mathrm{Z})_1$, and since every covector forces its complementary vector to be a covector, we have $\mathcal{L}(P) \supseteq \{+,-\}^{d+2}$. So P realizes all of its 2^{d+2} ordered partitions as ordered hyperplane partitions – too much is too much, as Theorem 2.2(6) tells us. ◻

All in all, we have shown that $\mathcal{L}(S) \oplus \mathcal{L}(S')$ equals

$$\left(\bigcup_{i=1}^{d+1} \operatorname{pert} \mathrm{C}_i \;\oplus\; \bigcup_{i=1}^{d+1} \operatorname{pert} \mathrm{C}'_i \right) \dot\cup \left(\bigcup_{i=1}^{d+1} \operatorname{pert} \overline{\mathrm{C}_i} \;\oplus\; \bigcup_{i=1}^{d+1} \operatorname{pert} \overline{\mathrm{C}'_i} \right)$$

which leaves us with a counting exercise. Given i and j, the set of vectors of the form $\{+,-,0\}^{d+1}\,\mathrm{Z}$ contains $\binom{d+1}{i}\binom{d+1-i}{j-\ell}$ vectors with i the number of 0's and j the number of $+$'s. How many of these are *not* in $\bigcup_{i=1}^{d+1} \operatorname{pert} \mathrm{C}_i$, i.e. have $\overline{g_1} \ldots \overline{g_{d+1}}\,\mathrm{Z}$ as perturbation? For that we would have to switch $(d+1)-m-(j-\ell)$ of the $\overline{g_i} = +$ to 0 (which leaves $(j-\ell)$ $+$'s among the first $d+1$ positions) and switch $i - ((d+1) - m - (j-\ell))$ of the $\overline{g_i} = -$ to 0 in order to have i to be the number of 0's. This makes

$$\begin{aligned} \binom{d+1-m}{(d+1)-m-(j-\ell)} \binom{m}{i-((d+1)-m-(j-\ell))} &= \\ \binom{m}{d+1-i-(j-\ell)} \binom{d+1-m}{j-\ell} &=: \; T_{i,j}(m,\ell) \end{aligned}$$

sequences (with i 0's and j $+$'s) not appearing in $\bigcup_{i=1}^{d+1} \operatorname{pert} \mathrm{C}_i$. Summing up, the number of covectors of (i,j)-partitions in $\bigcup_{i=1}^{d+1} \operatorname{pert} \mathrm{C}_i$ is $\binom{d+1}{i}\binom{d+1-i}{j-\ell} - T_{i,j}(m,\ell)$, and the number of covectors of (i,j)-partitions in $\bigcup_{i=1}^{d+1} \operatorname{pert} \mathrm{C}'_i$ is $\binom{d+1}{i}\binom{d+1-i}{j-\ell} - T_{i,j}(d+1-m,\ell)$. That is, the increment of (i,j)-partitions of the form $\{+,-,0\}^{d+1}\,\mathrm{Z}$ is $T_{i,j}(m,\ell) - T_{i,j}(d+1-m,\ell)$. An analogous analysis (with $n-(d+1)-\ell$ for ℓ, and $d+1-m$ for m) for covectors of the form $\{+,-,0\}^{d+1}\,\overline{\mathrm{Z}}$ finally yields the following result.

Theorem 3.4 *If (S, S') is an (m, ℓ)-mutation, then*

$$D_{i,j}(S') - D_{i,j}(S) = \delta_{i,j}^{(m,\ell)} \ ,$$

where

$$\begin{aligned}\delta_{i,j}^{(m,\ell)} \quad := \quad & T_{i,j}(m,\ell) - T_{i,j}(d+1-m,\ell) \\ & + T_{i,j}(d+1-m, n-(d+1)-\ell) - T_{i,j}(m, n-(d+1)-\ell).\end{aligned}$$

$\boxminus$

Remark 9 $T_{i,j}(m,\ell) \neq 0$ *iff* $\ell + \max\{0, d+1-m-i\} \leq j \leq \ell + d + 1 - \max\{m, i\}$. *For example, relevant for the changes in the number of j-facets, $T_{d,j}(m,\ell)$ vanishes unless $\ell \leq j \leq \ell+1$, and thus $\delta_{d,j}^{(m,\ell)} = 0$ unless $j \in \{\ell, \ell+1, n-(d+1)-\ell, n-d-\ell\}$. And in view of k-sets, $T_{0,j}(m,\ell)$ vanishes unless $j = \ell + d + 1 - m$, and thus $\delta_{0,j}^{(m,\ell)} = 0$ unless $j \in \{\ell+m, \ell+d+1-m, n-\ell-m, n-\ell-d-1+m\}$.*

In the 'balanced' situation $m = d+1-m$, the increment $\delta_{i,j}^{(m,\ell)}$ vanishes for all i, j, and ℓ. That is, for d odd, a $((d+1)/2, \ell)$-mutation leaves the $D_{i,j}$'s untouched. This is of particular interest in $\mathbb{R}^3$, where a motion preserving convex position of a point set encounters such balanced $(2, \ell)$-mutations only.

With a little help of the just given remarks, the following implications for j-facets and k-sets are easy to obtain.

Corollary 3.5 *Let (S, S') be an (m, ℓ)-mutation.*

(a) $e_j(S') = e_j(S)$ *for* $j \notin \{\ell, \ell+1, n-d-(\ell+1), n-d-\ell\}$,

$$\begin{aligned} & e_\ell(S') - e_\ell(S) = e_{n-d-\ell}(S') - e_{n-d-\ell}(S) \\ & \qquad = (2m-d-1) \cdot (1 - [2\ell = n-d-1] + [2\ell = n-d]) \ , \quad \textit{and} \\ & e_{\ell+1}(S') - e_{\ell+1}(S) = e_{n-d-(\ell+1)}(S') - e_{n-d-(\ell+1)}(S) \\ & \qquad = (d+1-2m) \cdot (1 - [2\ell = n-d-1] + [2\ell = n-d-2]) \ . \end{aligned}$$

(b) $a_k(S') = a_k(S)$ *for* $k \notin \{\ell+m, \ell+d+1-m, n-\ell-m, n-\ell-d-1+m\}$,

$$\begin{aligned} & a_{\ell+m}(S') - a_{\ell+m}(S) = a_{n-\ell-m}(S') - a_{n-\ell-m}(S) \\ & \qquad = [2m \neq d+1] \cdot (-1 + [2\ell = n-d-1] - [2\ell = n-2m]) \ , \quad \textit{and} \\ & a_{\ell+d+1-m}(S') - a_{\ell+d+1-m}(S) = a_{n-\ell-d-1+m}(S') - a_{n-\ell-d-1+m}(S) \\ & \qquad = [2m \neq d+1] \cdot (1 - [2\ell = n-d-1] + [2\ell = n-2d-2+2m]) \ . \end{aligned}$$

$\boxminus$

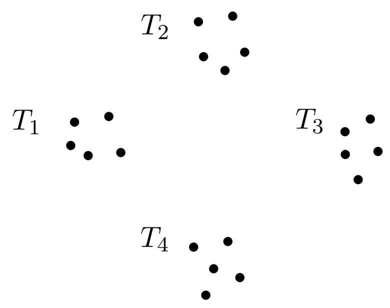


Figure 4: Projection of a set S in $\mathbb{R}^4$ with four $(d+1)$-sets.

4 Vector $\overline{a}(S)$ versus Vector $\overline{e}(S)$

In this section we show that for $d \geq 4$ the information given by the vector $\overline{e}(S)$ is in general not sufficient to determine the vector $\overline{a}(S)$ – nor vice versa. We show that for every $d \geq 4$ and every $n \geq 4(d+1)$ there are sets S, S' in $\mathbb{R}^d$ of n points in general position with $\overline{e}(S) = \overline{e}(S')$ and $\overline{a}(S) \neq \overline{a}(S')$. An analogous statement is shown for the case when the roles of $\overline{a}$ and of $\overline{e}$ are swapped. This contrasts the situation in $\mathbb{R}^2$ and $\mathbb{R}^3$ (see Remark 5) where the vectors $\overline{e}(S)$ and $\overline{a}(S)$ determine each other.

Theorem 4.1 **(a)** *For any $d \geq 4$ and $n \geq 4(d+1)$ there are sets S, S' in $\mathbb{R}^d$ of n points in general position with $\overline{e}(S) = \overline{e}(S')$ and $\overline{a}(S) \neq \overline{a}(S')$. More precisely, for each $d \geq 4$, $k \geq 1$, $\ell \geq 0$ and $n \geq 4k(\ell+d+1)$ there are sets $S_{k,\ell}$, $S'_{k,\ell}$ in $\mathbb{R}^d$ of n points in general position such that $\overline{e}(S_{k,\ell}) = \overline{e}(S'_{k,\ell})$ and*

$$a_{l+m}(S_{k,\ell}) - a_{\ell+m}(S'_{k,\ell}) = k,$$

where

$$m = \begin{cases} d/2 & \text{if } d \text{ is even,} \\ (d+5)/2 & \text{otherwise.} \end{cases}$$

(b) *For any $d \geq 4$ and $n \geq \frac{3}{2}(d+1)^2$ there are sets S', S'' in $\mathbb{R}^d$ of n points in general position with $\overline{a}(S') = \overline{a}(S'')$ and $\overline{e}(S') \neq \overline{e}(S'')$. In more concrete terms, for each $d \geq 4$, $\ell \geq \lfloor d/2 \rfloor - 1$ and $n \geq (\ell+d+1)(d+1)$ there are sets S'_ℓ, S''_ℓ in $\mathbb{R}^d$ of n points in general position such that $\overline{a}(S'_\ell) = \overline{a}(S''_\ell)$ and*

$$e_{\ell-m}(S'_\ell) - e_{\ell-m}(S''_\ell) = 1,$$

where

$$m = \begin{cases} d/2 - 1 & \text{if } d \text{ is even,} \\ (d-3)/2 & \text{otherwise.} \end{cases}$$

Proof. **(a)** Assume that $d \geq 4$. In order to illustrate the idea of the proof we start with the special case of d even, $k = 1$, and $\ell = 0$. We let S be a set of $4(d+1)$ points in general position such that there are four pairwise disjoint $(d+1)$-sets of S denoted as $T_1, \ldots, T_4$, see Figure 4. The set T_1 itself is chosen in such a way that it allows a $(d/2+2, 0)$-mutation (of the whole set with mutation kernel in T_1), and T_2, T_3, and T_4 are chosen in such a way that each of these sets allows a $(d/2, 0)$-mutation. Let S' be a set obtained by "executing" a corresponding mutation (as required) for each of the sets $T_1, \ldots, T_4$. By Corollary 3.5 we have $\overline{e}(S) = \overline{e}(S')$, but on the other hand we have by the same corollary

$$\begin{aligned} a_{d/2+2}(S') &= a_{d/2+2}(S) - 1, \\ a_{d/2-1}(S') &= a_{d/2-1}(S) + 1, \\ a_{d/2}(S') &= a_{d/2}(S) - 3, \\ a_{d/2+1}(S') &= a_{d/2+1}(S) + 3, \end{aligned}$$

and so $\overline{a}(S) \neq \overline{a}(S')$.

The general case is based on the same scheme. We construct two sets S and S' of n points each in $\mathbb{R}^d$ such that S' is obtained by moving the points in S under a sequence of mutations. Let d, k, ℓ, and n be fixed and fulfill the conditions described in the assertion of the theorem.

Assume first that d is even. We put $t = 4k$, $m' = d/2+2$ and $m'' = d/2$. Assume that S is constructed in such a way that it has pairwise disjoint $(\ell+d+1)$-sets $T_1, \ldots, T_t$; this is possible for example if $T_1, \ldots, T_t$ are sufficiently close to the surface of a sphere. The remaining $n - t(\ell + d + 1)$ points are placed at the center of the sphere. Furthermore, for each $i \in \{1, \ldots, t\}$ we deform T_i in such a way that for a given integer $m_i \in \{1, \ldots, d\}$ a subset of $d+1$ points in T_i forms a mutation kernel of a (potential) (m_i, ℓ)-mutation. It is not hard to see that such a construction is possible. Now S' is obtained by "executing" the assigned mutations, where $m_1 = \ldots = m_k = m$ and $m_{k+1} = \ldots = m_t = m'$. By Corollary 3.5 exactly three (m', ℓ)-mutations compensate the changes of the vector of j-facets caused by one (m, ℓ)-mutation, and so we have immediately $\overline{e}(S) = \overline{e}(S')$. Since m, m', $d+1-m$ and $d+1-m'$ are pairwise distinct integers (m and $d+1-m$ have different parity) and since $2\ell < n - 2d - 2$ (and so the 'special terms 'of Corollary 3.5 like $[2l = n-d-1]$ etc. can be ignored), we have by the same corollary that

$$\begin{aligned} a_{\ell+m}(S') &= a_{\ell+m}(S) - k, \\ a_{\ell+d+1-m}(S') &= a_{\ell+d+1-m}(S) + k, \\ a_{\ell+m'}(S') &= a_{\ell+m'}(S) - 3\,k, \\ a_{\ell+d+1-m'}(S') &= a_{\ell+d+1-m'}(S) + 3\,k. \end{aligned}$$

If d is odd, then we put $t = 3k$, $m = (d+5)/2$ and $m' = (d-1)/2$. The remainder of the proof goes through as above.

(b) Assume that d, ℓ and n are given and fulfill the conditions of the theorem statement. First we treat the case that d is even. Let S, S' and S'' be sets of

n points in general position in $\mathbb{R}^d$ with following properties. We construct S in such a way that it has $(\ell+d+1)$-sets $T_0,\ldots,T_t$, where $t=d$. Furthermore, the $(\ell+d+1)$-set T_0 is deformed in such a way that it allows a $(1,\ell)$-mutation M_0, and for each $i\in\{1,\ldots,t\}$ we deform T_i in such a way that the set T_i allows an $(d/2,\ell-d/2+i)$-mutation M_i. Now S' is obtained from S by "executing" the mutation M_0, and S'' is obtained from S by executing the mutations $M_1,\ldots,M_t$. Next we show that $\overline{a}(S')=\overline{a}(S'')$ but $\overline{e}(S')\neq\overline{e}(S'')$. We have $2\ell<n-2d-2$ and so by Corollary 3.5 it follows that

$$\begin{aligned}
a_{l+1}(S') &= a_{l+1}(S)-1,\\
a_{n-l-1}(S') &= a_{n-l-1}(S)-1,\\
a_{l+d}(S') &= a_{l+d}(S)+1,\\
a_{n-l-d}(S') &= a_{n-l-d}(S)-1.
\end{aligned}$$

Furthermore, for each $i\in\{1,\ldots,t\}$, the mutation M_i changes the vector of k-sets in the following way:

$$\begin{array}{ll}
e_{l+i}(S) \text{ and } e_{n-l-i}(S) & \text{are changed by } -1,\\
e_{l+i+1}(S) \text{ and } e_{n-l-i-1}(S) & \text{are changed by } +1.
\end{array}$$

The cumulative effect of these changes is the same for S'' as given above for S' and so $\overline{a}(S')=\overline{a}(S'')$. On the other hand, by Corollary 3.5 (and the fact that $2\ell<n-2d-2$) we have

$$\begin{aligned}
e_{\ell}(S') &= e_{\ell}(S)+1-d,\\
e_{n-d-\ell}(S') &= e_{n-d-\ell}(S)+1-d,\\
e_{\ell+1}(S') &= e_{\ell+1}(S)+d-1,\\
e_{n-d-\ell-1}(S') &= e_{n-d-\ell-1}(S)+d-1,
\end{aligned}$$

which are the only differences between $\overline{e}(S)$ and $\overline{e}(S')$. Moreover, we have for example

$$e_{\ell-d/2+1}(S'')=e_{\ell-d/2+1}(S)-1,$$

and it follows that $\overline{e}(S')\neq\overline{e}(S'')$.

If d is odd, then S' is the same as above and S'' is obtained from S by exactly $t=(d-1)/2$ mutations $M_1,\ldots,M_t$, where for $i\in\{1,\ldots,t\}$ M_i is an $((d-1)/2,l-(d+1)/2+2i)$-mutation. The remainder of the proof goes through as above. ◻

5 (i,j)-Partitions on the Moment Curve

The *moment curve* in $\mathbb{R}^d$ is the set $M_d=\{(t,t^2,\ldots,t^d)\,|\,t\in\mathbb{R}\}$. We denote by $S_{n,d}=\{p_1,\ldots,p_n\}$ a set of n points on the moment curve, with the numbering consitent with the order of occurrence on the curve. In this section we derive a formula for the numbers of (i,j)-partitions of $S_{n,d}$ (Theorem 5.1). Such formulas have been known for cases $i = 0$ [15] and $i = d$ [1].

Let us define

$$\binom{a}{b}_{-1} = \begin{cases} 1 & \text{if } a = b = -1 \\ \binom{a}{b} & \text{otherwise.} \end{cases}$$

Theorem 5.1 *For $n \geq d+1$ we have*

$$D_{i,j}(S_{n,d}) = \sum_{s=0}^{d} B(n,j,i,s)$$

where, for $q \in \mathbb{N}_0$, $B(n,j,i,2q)$ equals

$$\sum_{t_1=0}^{i}\sum_{t_2=0}^{i-t_1}\left(\binom{q+1}{t_1}\binom{j-1}{q-t_1}_{-1}\binom{q}{t_2}\cdot \binom{n-i-j-1}{q-t_2-1}_{-1}\binom{2q-t_1-t_2}{2q-i}\right)$$
$$+\sum_{t_1=0}^{i}\sum_{t_2=0}^{i-t_1}\left(\binom{q}{t_1}\binom{j-1}{q-t_1-1}_{-1}\binom{q+1}{t_2}\cdot \binom{n-i-j-1}{q-t_2}_{-1}\binom{2q-t_1-t_2}{2q-i}\right)$$

and, for $q \in \mathbb{N}$, $B(n,j,i,2q-1)$ equals

$$2\sum_{t_1=0}^{i}\sum_{t_2=0}^{i-t_1}\left(\binom{q}{t_1}\binom{j-1}{q-t_1-1}_{-1}\binom{q}{t_2}\cdot \binom{n-i-j-1}{q-t_2-1}_{-1}\binom{2q-1-t_1-t_2}{2q-1-i}\right).$$

For $i = 0$ the formula in Theorem 5.1 reduces to (10) as below which can be found also in [15]. The formula for $e_j(S_{n,d})$ can be derived from Theorem 5.1 using Vandermonde's convolution [13].

Corollary 5.2 *For $n \geq d+1$ we have*

$$a_j(S_{n,d}) = D_{0,j}(S_{n,d}) = \sum_{s=0}^{d} B(n,j,s) \tag{10}$$

where, for $q \in \mathbb{N}$,

$$B(n,j,2q-1) := 2\binom{j-1}{q-1}\binom{n-j-1}{q-1}$$

and, for $q \in \mathbb{N}_0$,

$$B(n, j, 2q) := \binom{j-1}{q}\binom{n-j-1}{q-1} + \binom{j-1}{q-1}\binom{n-j-1}{q}.$$

Furthermore, for $d = 2q - 1$ we have

$$e_j(S_{n,d}) = D_{d,j}(S_{n,d}) = 2\binom{j+q-1}{q-1}\binom{n-j-q}{q-1}$$

and for $d = 2q$ we have

$$e_j(S_{n,d}) = D_{d,j}(S_{n,d}) = \binom{j+q-1}{q-1}\binom{n-j-q}{q} + \binom{j+q}{q}\binom{n-j-q-1}{q-1}.$$

The proof of Theorem 5.1 is postponed to the end of this section.

An *ordered partition of a set* S is a t-tuple $(S_1, \dots, S_t)$ of (possibly empty) sets such that $S = S_1 \dot\cup S_2 \dot\cup \dots \dot\cup S_t$ and for S_a, S_b with $a < b$ it holds that the indices of the points in S_a are smaller than the indices of the points in S_b. The sets $S_1, \dots, S_t$ are called *blocks*. Let h be the hyperplane $h_0 + h_1x_1 + \dots + h_dx_d = 0$ in $\mathbb{R}^d$. The points of intersection of h with the moment curve M_d correspond to the roots of the polynomial

$$f(t) = h_0 + h_1t + \dots + h_dt^d.$$

The graph of $f(t)$ is divided up by at most d intersections with the axis t into segments above, on and below the axis t (with zero or more points in $S_{n,d}$ in each segment). This gives rise to a following definition. A *PZN-partition of* $S_{n,d}$ *(induced by* h*)* is an ordered partition of $S_{n,d}$ such that the consecutive blocks are determined by the consecutive segments of the graph of $f(t)$: the ith block contains exactly the points in the ith segment. In addition, the blocks are *colored* by P, Z and N depending whether the segment of the block is above, on, or below the axis t, respectively. The order of blocks in the partition is determined by the order how the segments of the graph of $f(t)$ are traversed when t goes from $-\infty$ to $+\infty$.

It is not hard to see that the PZN-partitions of $S_{n,d}$ are exactly the ordered partitions of $S_{n,d}$ with blocks colored P, Z and N which fulfill the following conditions:

- the first block is a P-block or an N-block,
- if B is not the last block and it is a P-block (N-block), then B is directly followed by a Z-block and an N-block (a P-block) (i.e. the sequence is PZN or NZP),

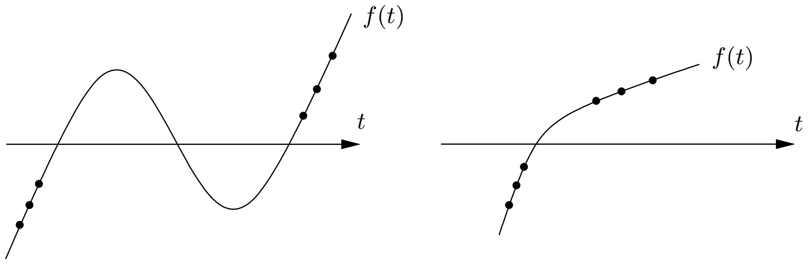


Figure 5: Both PZN-partitions correspond to the same $(0, 3)$-partition

- each Z-block has cardinality 0 or 1,
- the number of Z-blocks is at most d.

If the total cardinality of the P-blocks is j and the total cardinality of the Z-blocks is i, then such PZN-partition corresponds to an (i, j)-partition of $S_{n,d}$. Unfortunately, many different PZN-partitions can yield the same (i, j)-partition. This is illustrated in Figure 5. To overcome this problem, we introduce the following notion. A PZN-partition is called a *minimal PZN-partition*, if

1. all Z-blocks before the first non-empty P-block or N-block are non-empty,
2. each Z-block directly preceding an empty P-block or an empty N-block is non-empty.

Lemma 5.3 *Among all PZN-partitions of $S_{n,d}$ which correspond to the same (i, j)-partition of $S_{n,d}$ there is exactly one minimal PZN-partition.*

Proof. The proof of existence is easy and left to the reader. Assume that $\mathcal{P}$ and $\mathcal{P}'$ are two minimal PZN-partitions of $S_{n,d}$ which correspond to the same (i, j)-partition of $S_{n,d}$ for some $i \in \{0, \ldots, d\}$ and $j \in \{0, \ldots, n-i\}$. We show first that $\mathcal{P}$ and $\mathcal{P}'$ have the same non-empty P-blocks and the same non-empty N-blocks (each non-empty block is identified by its color and by the points it contains). Assume that there is a P-block (N-block) of $\mathcal{P}$ not present in $\mathcal{P}'$. Then $\mathcal{P}'$ must contain two P-blocks (N-blocks) B_1, B_2 with $B \subseteq B_1 \cup B_2$. Since B_1 and B_2 contain consecutive points on M_d, all blocks between B_1 and B_2 must be empty. Especially, there is an empty N-block (P-block) directly preceded by an empty Z-block between B_1 and B_2, which contradicts 2. Since the non-empty Z-blocks are determined by the (i, j)-partition, $\mathcal{P}$ and $\mathcal{P}'$ have the same non-empty blocks.

Next we show that $\mathcal{P}$ and $\mathcal{P}'$ have the same empty blocks (empty blocks are identified by their color and one non-empty block directly proceeding or directly following the empty block). Assume that B is an empty Z-block of $\mathcal{P}$. Then the P-block (N-block) B' directly following B must be non-empty by 2, furthermore there is a non-empty P-block or N-block B'' before B closest to B (by 1) in $\mathcal{P}$.

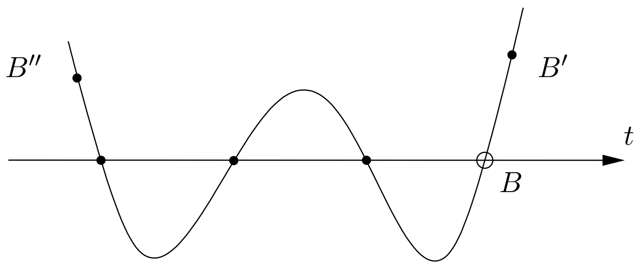


Figure 6: Illustration of the argument that B must exist in $\mathcal{P}'$

The number of Z-blocks between B'' and B' in $\mathcal{P}'$ is the same as in $\mathcal{P}$, since B'' has the same color in both partitions and both partitions have the same number of non-empty Z-blocks between B'' and B' (parity argument), see Figure 6. Since each (empty) P-block or N-block between B'' and B' is directly preceded by a non-empty Z-block in both $\mathcal{P}$ and $\mathcal{P}'$ we follow that B must exist in $\mathcal{P}'$. Now assume that B is an empty P-block (N-block) in $\mathcal{P}$. If B is the first block of $\mathcal{P}$, then clearly B also exists in $\mathcal{P}'$ by 1 and by the fact, that the first non-empty P-block or N-block has the same color in both $\mathcal{P}$ and $\mathcal{P}'$. Otherwise there is a non-empty P-block or N-block B' before B closest to B. Then B must occur in $\mathcal{P}'$ by a similar argument as before. ◻

Proof of Theorem 5.1. By the last lemma we have to count the minimal PZN-partitions such that the compounded cardinality of the P-blocks is j and the compounded cardinality of the Z-blocks is i. We classify the PZN-partitions by the number of their Z-blocks, and so let $B(n, j, i, s)$ be the number of such minimal PZN-partitions of $S_{n,d}$ with exactly s Z-blocks each. Among them we count the PZN-partitions which start with a P-block and have t_1 empty P-blocks and t_2 empty N-blocks. If $s = 2q - 1$, then the number of P-blocks is q and the number of N-blocks is also q. There are $\binom{q}{t_1}$ possibilities to choose the empty P-blocks among all P-blocks and $\binom{j-1}{q-t_1-1}_{-1}$ ways to partition a set of j points of $S_{n,d}$ into $q - t_1$ remaining non-empty P-blocks (if $j = 0$ and $q - t_1 = 0$, then all P-blocks are empty and we have exactly one choice). It is not hard to see that we can choose the empty N-blocks among all N-blocks independently of the choice of the empty P-blocks. There are $\binom{q}{t_2}$ choices for the empty N-blocks, and we can partition the $n - i - j$ points in $S_{n,d}$ into $q - t_2$ remaining non-empty N-blocks in $\binom{n-j-i-1}{q-t_2-1}_{-1}$ ways.

By 1 and 2 it is clear that for each empty P-block or N-block we must make a unique Z-block non-empty and that in total $t_1 + t_2$ Z-blocks become non-empty. The remaining $i - t_1 - t_2$ points in $S_{n,d}$ can be put into the remaining $s - t_1 - t_2$ Z-blocks, which is possible in $\binom{s-t_1-t_2}{s-i}$ ways. Since the same calculation also holds for the PZN-blocks which start with an N-block, the value of $B(n, j, i, s)$ is

$$2\sum_{t_1=0}^{i}\sum_{t_2=0}^{i-t_1}\left(\binom{q}{t_1}\binom{j-1}{q-t_1-1}_{-1}\binom{q}{t_2}\right.$$
$$\left.\binom{n-i-j-1}{q-t_2-1}_{-1}\binom{2q-1-t_1-t_2}{2q-1-i}\right).$$

The case $s = 2q$ is handled analogously. ◻

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