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Entering and Leaving j-Facets

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Abstract

Let S be a set of n points in d-space, no $i+1$ points on a common $(i-1)$-flat for $1 \leq i \leq d$. An oriented $(d-1)$-simplex spanned by d points in S is called *j-facet of S,* if there are exactly j points from S on the positive side of its affine hull. We show: (*) *For $j \leq n/2-2$, the total number of $(\leq j)$-facets (i.e. the number of i-facets with $0 \leq i \leq j$) in 3-space is maximized in convex position (where these numbers are known).* A large part of this presentation is a preparatory review of some basic properties of the collection of j-facets – some with their proofs – and of relations to well-established concepts and results from the theory of convex polytopes (h-vector, Dehn-Sommerville relations, Upper Bound Theorem, Generalized Lower Bound Theorem). The relations are established via a duality closely related to the Gale transform – similar to previous works by C. Lee, by K. Clarkson, and by K. Mulmuley.

A central definition is as follows. Given a directed line ℓ and a j-facet F of S, we say that ℓ *enters* F if ℓ intersects the relative interior of F in a single point, and if ℓ is directed from the positive to the negative side of F. One of the results reviewed is a tight upper bound of $\binom{j+d-1}{d-1}$ on the maximum number of j-facets entered by a directed line.

Based on these considerations, we also introduce a vector for a point relative to a point set, which – intuitively speaking – expresses 'how interior' the point is relative to the point set. This concept allows us to show that the statement (*) above is equivalent to the Generalized Lower Bound Theorem for d-polytopes with at most $d+4$ vertices.

Keywords: j-facets, k-sets, h-vector, Dehn-Sommerville relations, Upper Bound Theorem, Generalized Lower Bound Theorem, Gale transform.

1 Introduction

Let S be a set of n points in $\mathbb{R}^d$ *in general position,* i.e. no $i+1$ points on a common $(i-1)$-flat for $1 \leq i \leq d$. An oriented $(d-1)$-simplex spanned by d points in S is called a *j-facet of S*, if it has exactly j points from S on the positive side of its affine hull; hence, $j \in \mathbb{Z}$ and $0 \leq j \leq n-d$. There is an obvious correspondence between 0-facets and facets of the convex hull of S.

The maximum possible number of j-facets of an n-point set in $\mathbb{R}^d$ has raised some interest, starting with first bounds in the plane by Lovász [12] and Erdős, Lovász, Simmons, and Straus [8] in the early seventies. The currently best upper bound in the plane is of the order $n\sqrt[3]{j+1}$ due to Dey [7]. Planar point sets where the number of j-facets is of the order $n \cdot e^{\Omega(\sqrt{\log(j+1)})}$ for $2j \leq n-2$ are known due to a recent construction by Géza Tóth [19]. We refer the reader to [3, 2] for more references, also on the related problem of 'k-sets', and on geometric algorithms where the number of j-facets occurs in the analysis (but see also [18] for very recent developments on the upper bound in three dimensions).

The emphasis of the first part of this paper is on the structure of the collection of j-facets, and on relations to more established concepts in the theory of convex polytopes that go beyond the observation that 0-facets are facets of the convex hull. To this end, we define that a *directed line ℓ enters j-facet F*, if it intersects the relative interior of F in a single point, and if ℓ is directed from the positive to the negative side of F. If, instead, ℓ is directed from the negative to the positive side of F, then we say that *ℓ leaves F*.

Section 2 proves that no line can enter more than $\binom{j+d-1}{d-1}$ j-facets of a finite point set in $\mathbb{R}^d$. The proof mimics McMullen's proof of the bound on the entries of the h-vector of a simplicial convex polytope for the Upper Bound Theorem [13]. Section 3 will make this relation more explicit via a duality closely related to the Gale transform. (For example, this duality translates the Dehn-Sommerville relations to the fact that every directed line enters and leaves the same number of j-facets.) In slightly different settings – perhaps not as explicit, albeit essentially equivalent – such a relation has been worked out and exploited by C. Lee [10], K. Clarkson [5] and K. Mulmuley [15] (see also Remark 3 at the end of this paper).

An alternative proof of the bound on the number of j-facets entered by a line – by induction on the dimension – is given in Section 4. Based on the tools used in this proof, we also introduce a vector for a point relative to a point set, which expresses 'how interior' the point is relative to the point set. This vector relates to the g-vector for convex polytopes, and we can employ the rich theory developed there [17, 14]. In particular, the Generalized Lower Bound Theorem appears useful in our setting.

Finally, in Section 5 we close with a conclusion for the overall number of $(\leq j)$-facets (i.e. the total number of i-facets with $i \leq j$) of n-point sets. We show that for $j \leq n/2-2$, the number of $(\leq j)$-facets in $\mathbb{R}^3$ is maximized in convex position where these numbers are known to be $2(\binom{j+2}{2}n - 2\binom{j+3}{3})$ (this extends a corresponding result of N. Alon and E. Győri in the plane [1]). In fact, this statement can be shown to be equivalent to the Generalized Lower Bound Theorem for d-polytopes with at most $d+4$ vertices.

Conventions. We will use $(a_i)_i$ short for the sequence $(a_i)_{i=0}^{\infty} = (a_0, a_1, \ldots)$. Most of the sequences we introduce will be defined for all $i \in \mathbb{Z}$, mostly with $a_i = 0$ for $i < 0$. Similarly, $\sum_i a_i$ denotes $\sum_{i=0}^{\infty} a_i$. However, all the sequences $(a_i)_i$ we employ in such sums will vanish except for a finite number of terms.

The binomial coefficient $\binom{a}{b}$, $a, b \in \mathbb{Z}$, is defined to be 0 for $b < 0$ or $a < b$.

2 Lines entering j-facets

Let $S \subseteq \mathbb{R}^d$ be a set of n points in general position. Let ℓ be a directed line disjoint from all convex hulls of $d-1$ points in S. For $j \in \mathbb{Z}$, let $\overline{h}_j = \overline{h}_j(\ell, S)$ denote the number of

j-facets entered by line ℓ; hence, $\overline{h}_j = 0$ for $j < 0$ and for $j > n - d$.

Upper bounds on the $\overline{h}_j$'s. We derive a number of simple facts. First observe that a directed line penetrates the convex hull of S at most once. This translates to

Fact 2.1 $\overline{h}_0 \leq 1$. ◻

Next, let us consider the sum $s^0 := \sum_j \overline{h}_j$. This sum denotes the overall number of $(d-1)$-simplices spanned by d points in S that are intersected by line ℓ. It is not too difficult to see that the sum $s^1 := \sum_j j\overline{h}_j$ denotes the number of d-simplices spanned by $d+1$ points in S that are intersected by line ℓ: Given a j-facet F entered by ℓ, there are exactly j d-simplices with facet F which are intersected by ℓ and where the last point of intersection is in F. Similarly, for $k \in \mathbb{Z}$, $s^k = s^k(\ell, S) := \sum_j \binom{j}{k}\overline{h}_j$ gives the number of $(k+d)$-element subsets of S whose convex hull is met by line ℓ; we have $s^k = 0$ for $k < 0$ and for $k > n-d$. Now observe that none of the values s^k changes if we move a point in S parallel to ℓ again in general position – the vector $(s^k)_k$ is invariant under such motions. On the other hand, we have the following inversion formula for sequences $(a_i)_i$ and $(b_j)_j$ of real numbers (proof omitted).

$$\forall i \geq 0 : a_i = \sum_j \binom{j}{i} b_j \quad \Longleftrightarrow \quad \forall j \geq 0 : b_j = \sum_i (-1)^{i+j} \binom{i}{j} a_i \ . \tag{1}$$

It asserts that $(s^k)_k$ determines $(\overline{h}_j)_j$. That is, the sequence $(\overline{h}_j)_j$ is also invariant under motions of points parallel to ℓ.

Fact 2.2 *If $p \in S$ is replaced by some other point p' again in general position on the line through p parallel to ℓ, then the sequence $(\overline{h}_j)_j$ does not change.* ◻

In the next step we investigate the effect of removal of a point p in S, first the expected effect on the $\overline{h}$-sequence, if p is random. ($\mathbf{E}(X)$ denotes the expectation of random variable X.)

Fact 2.3 *For $j \in \mathbb{Z}$, $\mathbf{E}(\overline{h}_j(\ell, S \setminus \{p\})) = \frac{n-d-j}{n}\overline{h}_j + \frac{j+1}{n}\overline{h}_{j+1}$, where p is a random point chosen uniformly in S.*

Proof. For $0 \leq j \leq n-1-d$, a j-facet of $S \setminus \{p\}$ is either a j-facet of S with p one of the $n-d-j$ points on its negative side, or a $(j+1)$-facet of S with p one of the $j+1$ points on its positive side. For $j < 0$ and $j \geq n-d$ we get $\mathbf{E}(\overline{h}_j(\ell, S \setminus \{p\})) = 0$ as required. ◻

Fact 2.4 *For $j \in \mathbb{Z}$ and $p \in S$, $\overline{h}_j(\ell, S \setminus \{p\}) \leq \overline{h}_j$.*

Proof. For $0 \leq j \leq n-1-d$, Fact 2.2 allows us to move p so that it does not lie on the positive side of any $(j+1)$-facet of S entered by ℓ – without changing $\overline{h}_j$. Now the removal of p will not generate any new j-facets entered by ℓ. For $j < 0$ and $j \geq n-d$ the inequality is trivial. ◻

We have prepared all the ingredients for demonstrating the upper bounds for the $\overline{h}_j$'s. Facts 2.3 and 2.4 entail

$$\frac{n-d-j}{n}\overline{h}_j + \frac{j+1}{n}\overline{h}_{j+1} \leq \overline{h}_j \ ,$$

for all j, and so

$$\overline{h}_{j+1} \le \frac{j+d}{j+1}\overline{h}_j$$

for $j \ge 0$. Combined with Fact 2.1, this gives

$$\overline{h}_j \le \binom{j+d-1}{j} = \binom{j+d-1}{d-1}$$

for $j \ge 0$.

Symmetry of $(\overline{h}_j)_{j\in\mathbb{Z}}$. We conclude this section by demonstrating the identity $\overline{h}_j = \overline{h}_{n-d-j}$. An $(n-d-j)$-facet entered by line ℓ corresponds to a j-facet left by ℓ by changing the orientation of the $(d-1)$-simplex. Hence, the identity claims that a directed line enters and leaves the same number of j-facets. The reader is encouraged to verify the relation via Fact 2.2, but we take a different path. First observe that

Fact 2.5 $\overline{h}_0 = \overline{h}_{n-d}$. □

For $j, k \in \mathbb{Z}$, define

$$\overline{h}_j^k := \sum_{i=0}^{n-d} \binom{i}{j} \cdot \overline{h}_i \cdot \binom{n-d-i}{k-j} .$$

$\overline{h}_j^k$ is the overall number of j-facets in $(k+d)$-element subsets of S entered by line ℓ, i.e. $\overline{h}_j^k = \sum_{Q\in\binom{S}{k+d}} \overline{h}_j(\ell, Q)$: For an i-facet of S to become a j-facet in a $(k+d)$-element subset of S, we have to select j from the i points on its positive side, $k-j$ from the $n-d-i$ points on its negative side, and all d points that span the i-facet. Because of Fact 2.5, we have $\overline{h}_0^k = \overline{h}_k^k$, and so

$$0 = \underbrace{\sum_{i=0}^{n-d} \overline{h}_i \cdot \binom{n-d-i}{k}}_{\overline{h}_0^k} - \underbrace{\sum_{i=0}^{n-d} \binom{i}{k} \cdot \overline{h}_i}_{\overline{h}_k^k} = \sum_{i=0}^{n-d} \binom{i}{k} \cdot (\overline{h}_{n-d-i} - \overline{h}_i) .$$

The inversion formula (1) tells us that these identities determine the terms $(\overline{h}_{n-d-i} - \overline{h}_i)$. Thus $\overline{h}_{n-d-i} - \overline{h}_i = 0$ for all $0 \le i \le n-d$ is the unique solution.

This counting argument makes explicit that the symmetry of the sequence $(\overline{h}_j)_j$ is an immediate consequence of the fact that the number of 0-facets entered equals the number of 0-facets left (Fact 2.5); this number happens to be 0 or 1, which is not essential in our proof, though.

We summarize the findings of this section.

Theorem 1 *Let S be a set of n points in $\mathbb{R}^d$ in general position, and let ℓ be a directed line disjoint from all convex hulls of $d-1$ points in S. The numbers $\overline{h}_j$ of j-facets of S entered by ℓ satisfy*
(i) $\overline{h}_j = \overline{h}_{n-d-j}$ *for all $j \in \mathbb{Z}$, and*
(ii)

$$\overline{h}_j \le \min\left\{\binom{j+d-1}{d-1}, \binom{n-j-1}{d-1}\right\}$$

for $0 \le j \le n-d$, and $\overline{h}_j = 0$, otherwise. □

The bound in (ii) is a consequence of (i) and $\overline{h}_j \leq \binom{j+d-1}{d-1}$. We will see later on that there are point sets and lines where this bound is attained for all j.

3 Convex polytopes and h-vectors

Let $\mathcal{S}$ be a finite multiset of points in $\mathbb{R}^d$. For $i \in \mathbb{Z}$, let $\tilde{f}_i = \tilde{f}_i(\mathcal{S})$ be the number of $(i+1)$-element subsets of $\mathcal{S}$ that are contained in a supporting hyperplane. For P a convex polytope and $i \in \mathbb{Z}$, let $f_i = f_i(P)$ be the number of i-faces of P, where we agree on $f_{-1} = 1$ and $f_d = 0$. If S is a set in general position (in particular, there are no multiple copies of the same point), then conv$S$ is a simplicial polytope and $\tilde{f}_i(S) = f_i(\text{conv}S)$ for all $i \in \mathbb{Z}$.

The *h-vector* $(h_j)_{j=0}^d = (h_j(P))_{j=0}^d$ of a simplicial convex polytope P can be defined as the unique sequence of numbers satisfying (recall (1))

$$\forall i,\ 0 \leq i \leq d: \quad f_{i-1} = \sum_{j=0}^{d} \binom{j}{d-i} \cdot h_j \ ,$$

cf [20]. We skip here the more geometric equivalent description of the h-vector via shellings. Important properties of the h-vector of a simplicial n-vertex d-polytope are:

- The *Dehn-Sommerville Relations*

$$\forall j,\ 0 \leq j \leq d: \quad h_j = h_{d-j} \ .$$

- The *Upper Bound Theorem* [13]

$$\forall j,\ 0 \leq j \leq d: \quad h_j \leq \min\left\{\binom{j+n-d-1}{n-d-1}, \binom{n-j-1}{n-d-1}\right\} \ ,$$

  and this bound is attained for all j for the convex hull of n points on the moment curve $\{(t^i)_{i=1}^d \,|\, t \in \mathbb{R}\}$.

- The *Generalized Lower Bound Theorem (GLBT)*[1]

$$\forall j,\ 1 \leq j \leq (d+1)/2: \quad h_{j-1} \leq h_j \ .$$

The only proof known for the GLBT goes via the g-theorem, which characterizes all possible h-vectors of simplicial d-polytopes [4, 17, 14].

Orthogonal dual. We describe a duality between sequences of n points in $\mathbb{R}^d$ and $\mathbb{R}^{n-d-1}$ that is closely related to the Gale transform, cf [9, 20] (see remark preceding Lemma 2). This will allow us to relate the h-vector of simplicial convex polytopes to the $\overline{h}$-sequences we have considered in Section 2.

[1]Sometimes, the statement is presented for $j \leq d/2$. But for d odd and $j = (d+1)/2$, we have $h_{j-1} = h_j$ because of the Dehn-Sommerville Relations.

For integers $0 \le d < n$, we call a matrix $A \in I\!R^{n\times d}$ *legal* if $A^\top\cdot\vec{\mathbf{1}} = \vec{\mathbf{0}}$ and if A has full rank d. We use $\vec{\mathbf{1}}$ and $\vec{\mathbf{0}}$ for vectors of all 1's and 0's, respectively, of appropriate dimension; here $\vec{\mathbf{1}} = 1^n$ and $\vec{\mathbf{0}} = 0^d$. We interpret matrix A as a sequence $S_A = (p_i)_{i=1}^n$ of n points in $I\!R^d$ in the obvious way: i-th row gives coordinates of p_i. The conditions for 'legal' translate to the facts that the origin is the center of gravity of the points in S_A, and that there is no hyperplane containing all points in S_A – an assumption much weaker than general position!

Given legal matrices $A \in I\!R^{n\times d}$ and $B \in I\!R^{n\times(n-d-1)}$, we call B *an orthogonal dual of* A, in symbols $A\perp B$, if $A^\top \cdot B = 0^{d\times(n-d-1)}$. In other words, the columns of A are orthogonal to the columns of B. That is, the columns of A span a linear vector space of dimension d orthogonal to the linear space of dimension $n-d-1$ spanned by the columns of B, and both spaces are orthogonal to $\vec{\mathbf{1}}$. Hence, given a legal matrix A, there is always an orthogonal dual B which is unique up to linear transformations. Clearly, $A\perp B \iff B\perp A$. (This convenient symmetry, enforced by the condition $A^\top\cdot\vec{\mathbf{1}} = \vec{\mathbf{0}}$, is the only difference to the standard Gale transform – apart from expository details.)

Lemma 2 *For $0 \le d < n$, let $A \in I\!R^{n\times d}$ and $B \in I\!R^{n\times(n-d-1)}$ be legal matrices with $A\perp B$, and let $S_A = (p_i)_{i=1}^n$ and $S_B = (p_i^*)_{i=1}^n$. For some $I \subseteq \{1,2,\ldots,n\}$, let $F := \{p_i \,|\, i \in I\}$ and $\overline{F^*} := \{p_i^* \,|\, i \notin I\}$.*
(i) *If F is contained in a supporting hyperplane of the points in S_A then $\mathbf{0} \in \mathrm{conv}\overline{F^*}$.*
(ii) *If $\mathbf{0} \in \mathrm{conv}F$, then $\overline{F^*}$ is contained in a supporting hyperplane of the points in S_B.*

Proof. Let F lie in a supporting hyperplane. That is, there is a vector $v \in I\!R^{d+1}$, such that for $\lambda = (\lambda_i)_{i=1}^n := (A\vec{\mathbf{1}})\cdot v$, we have $\lambda \ne \vec{\mathbf{0}}$, $\lambda_i \ge 0$ for all $1 \le i \le n$ and $\lambda_i = 0$ for $i \in I$. ($(A\vec{\mathbf{1}})$ denotes the matrix A with an extra column of 1's.) Moreover,

$$B^\top \cdot \lambda = \underbrace{B^\top \cdot (A\vec{\mathbf{1}})}_{0^{(n-d-1)\times(d+1)}} \cdot v = \vec{\mathbf{0}}$$

which means that the origin is a positive linear (and thus convex) combination of points p_i^* with $i \notin I$.

For the reverse direction (ii), let $\lambda \in I\!R^n$ be a vector that witnesses the fact that $\mathbf{0} \in \mathrm{conv}F$. That is, $0 \le \lambda \ne \vec{\mathbf{0}}$, $A^\top\lambda = \vec{\mathbf{0}}$, and $\lambda_i = 0$ for $p_i \notin F$; if $p_i \notin F$, then $i \notin I$. λ is orthogonal to the linear space spanned by the columns in A; consequently, it is in the linear space spanned by the columns of $(B\vec{\mathbf{1}})$, and there is a vector v with $(B\vec{\mathbf{1}})\cdot v = \lambda$. Hence, v corresponds to a supporting hyperplane that contains all p_i^* with $\lambda_i = 0$. Since $\lambda_i = 0$ for $i \notin I$, the hyperplane contains all points in $\overline{F^*}$. ◻

f- and h-vector under orthogonal duals. For $\mathcal{S}$ a finite multiset of points in $I\!R^d$, φ an i-flat, and $k \in Z\!\!Z$, let $s^k = s^k(f,\mathcal{S})$ denote the number of $(k+d+1-i)$-element subsets of $\mathcal{S}$ whose convex hull is intersected by φ. This generalizes our definition for lines from the previous section. We will employ it here also for points (i.e. 0-flats).

Lemma 3 *For $0 \le d < n$, let $A \in I\!R^{n\times d}$ and $B \in I\!R^{n\times(n-d-1)}$ be legal matrices with $A\perp B$, and let $\mathcal{S} \subseteq I\!R^d$ and $\mathcal{S}^* \subseteq I\!R^{n-d-1}$ be the multisets of points in S_A and S_B, respectively. Then*

$$\tilde{f}_i(\mathcal{S}) = s^{d-i-1}(\mathbf{0},\mathcal{S}^*) \quad\text{and}\quad \tilde{f}_i(\mathcal{S}^*) = s^{n-d-i-2}(\mathbf{0},\mathcal{S}) \ .$$

Proof. There is a bijection of $(i+1)$-element subsets of $\mathcal{S}$ contained in supporting hyperplanes and $(n-(i+1))$-element subsets of $\mathcal{S}^*$ that contain $\mathbf{0}$ in their convex hull. And $(d-i-1)+(n-d-1)+1 = n-(i+1)$. Therefore the left equality. The right equality follows from the symmetry of orthogonal duality. ◻

Theorem 4 (i) *If $(h_j)_{j=0}^d$ is the h-vector of a simplicial n-vertex d-polytope, then there is a set S of n points in general position in $\mathbb{R}^{n-d}$, and a line ℓ disjoint from all convex hulls of $(n-d)-1$ points in S, such that $\overline{h}_j(\ell, S) = h_j$ for $0 \le j \le d$.*
(ii) *Let S be a set of n points in general position in $\mathbb{R}^d$, and let ℓ be a line disjoint from all convex hulls of $d-1$ points in S. If ℓ intersects the convex hull of S, then there is a simplicial m-vertex $(n-d)$-polytope P with $m \le n$ and $h_j(P) = \overline{h}_j(\ell, S)$ for $0 \le j \le n-d$.*

Proof. Let P be a simplicial n-vertex d-polytope, and let V be the set of vertices of P. Since P is simplicial, a small perturbation of the vertex set of P that does not change its f-vector allows us to assume that $V \cup \{c\}$, c the centroid of V, is a set of $n+1$ points in general position in $\mathbb{R}^d$. Moreover, a translation of P allows us to assume that the origin is the centroid of V. Let $A \in \mathbb{R}^{n\times d}$ be a matrix which has the coordinates of the points in V in its rows. Now consider an orthogonal dual $B \in \mathbb{R}^{n\times(n-d-1)}$ of A, and let T be the multiset of points in S_B. General position of $V \cup \{\mathbf{0}\}$ implies that $T \cup \{\mathbf{0}\}$ is a *set* of $n+1$ points in general position (argument omitted). We have $f_i(P) = \tilde{f}_i(V) = s^{d-i-1}(\mathbf{0}, T)$. Now we lift $T \subseteq \mathbb{R}^{n-d-1}$ to a set $S \subseteq \mathbb{R}^{n-d}$ by adding to each point in T a $(n-d)$-th coordinate, such that S is in general position in $\mathbb{R}^{n-d}$ (random coordinates uniform from $[0, 1)$ will do with probability 1). Let ℓ denote the x_{n-d}-axis directed towards $x_{n-d} = +\infty$. Obviously, $s^{d-i-1}(\mathbf{0}, T) = s^{d-i-1}(\ell, S)$, and so – according to the relation between $s^k(\ell, S)$ and $\overline{h}_j = \overline{h}_j(\ell, S)$ we had derived in Section 2 –

$$\sum_{j=0}^{d} \binom{j}{d-i-1} \cdot h_j = \tilde{f}_i(V) = s^{d-i-1}(\ell, S) = \sum_{j=0}^{d} \binom{j}{d-i-1} \cdot \overline{h}_j \qquad (2)$$

for $-1 \le i \le d-1$. (2) implies $h_j = \overline{h}_j$ for $0 \le j \le d$ via (1), and we have completed the proof of statement (i).

For the proof of (ii), let $S \subseteq \mathbb{R}^d$ and ℓ as in the claimed statement, with $\ell \cap \mathrm{conv} S \ne \emptyset$. A suitable projection and perturbation gives a set $T \subseteq \mathbb{R}^{d-1}$ and $x \in \mathbb{R}^{d-1}$ such that $T \cup \{x\}$ is in general position, $x \in \mathrm{conv} T$ and $s^k(x, T) = s^k(\ell, S)$ for all $k \in \mathbb{Z}$. Let c be the centroid of T. Let us first assume that $c = x$. Then we apply a translation which maps $c = x$ to the origin $\mathbf{0}$. Now we apply the orthogonal dual construction as in (i) which gives us a set V of points in $\mathbb{R}^{n-(d-1)-1} = \mathbb{R}^{n-d}$. $P = \mathrm{conv} V$ is the requested $(n-d)$-polytope with at most n vertices (employ an identity similar to (2)). If $c \ne x$ then there is a hyperplane H normal to $c - x$ and disjoint from $\mathrm{conv} T$, such that we can apply a projective transformation π which makes H the hyperplane at infinity with $\pi(x)$ the centroid of $\pi(T)$ and $s^k(x, T) = s^k(\pi(x), \pi(T))$ for all $k \in \mathbb{Z}$ (detailed argument omitted). Now we can proceed as before to show (ii). ◻

The theorem shows that not only the proof of Theorem 1 mimics McMullen's proof of the Upper Bound Theorem – the statements are actually equivalent to the Dehn-Sommerville Relations and the Upper Bound Theorem. The fact that the Upper Bound Theorem is tight for points on the moment curve implies that the bounds in Theorem 1 are tight. We will not give a proof of the Generalized Lower Bound Theorem in the 'j-facet setting', but we will shortly interpret and use it in this setting.

4 Lines entering j-facets up to a point

Alternative proof for the bounds on the $\overline{h}_j$'s. Let S' be a set of n points in $\mathbb{R}^{d+1}$ in general position (it's $(d+1)$-space now!). Let ℓ be a directed line parallel to the x_{d+1}-axis and disjoint from all convex hulls of d points in S'. For $j \in \mathbb{Z}$, let $\overline{h}_j = \overline{h}_j(\ell, S')$.

Let S be the orthogonal projection of S' to the hyperplane $x_{d+1} = 0$ and let x be the projection of ℓ. That is, by removing the last coordinate, we can consider $S \cup \{x\}$ as a set of points in $\mathbb{R}^d$. A small perturbation of S' that does not change the $\overline{h}_j$'s allows us to assume that $S \cup \{x\}$ is in general position.

We choose a directed line λ in $\mathbb{R}^d$ through x that is disjoint from all convex hulls of $d-1$ points in S. For $i \in \mathbb{Z}$, we let $\hat{h}_i = \hat{h}_i(x, \lambda, S)$ be the number of i-facets of S entered by λ before x (i.e. with x on the negative side).

We want to argue that

$$\overline{h}_j - \overline{h}_{j-1} = \hat{h}_j - \hat{h}_{n-d-j} \ , \tag{3}$$

for all $j \in \mathbb{Z}$. Before we proceed with this argument, note that $\hat{h}_i \le \overline{h}_i(\lambda, S)$. If we know that $\overline{h}_i(\lambda, S) \le \binom{i+d-1}{d-1}$, then from (3) it follows that

$$\overline{h}_j = \sum_{i=0}^{j} (\hat{h}_i - \hat{h}_{n-d-i}) \le \sum_{i=0}^{j} \hat{h}_i \le \sum_{i=0}^{j} \binom{i+d-1}{d-1} = \binom{j+(d+1)-1}{(d+1)-1}$$

and we have an inductive proof of the upper bound $\binom{j+d-1}{d-1}$ starting in dimension $d = 1$.

So why does (3) hold? We count the number $s^k = s^k(x, S)$ of $(k+d+1)$-element subsets of S whose convex hulls contain x, or, equivalently, the number $s^k(\ell, S')$ of $(k+(d+1))$-element subsets of S' whose convex hull is intersected by ℓ. For $k \in \mathbb{Z}$,

$$s^k(\ell, S') = \sum_i \underbrace{\binom{i}{k}}_{\binom{i+1}{k+1} - \binom{i}{k+1}} \cdot \overline{h}_i = \sum_i \binom{i}{k+1} \cdot (\overline{h}_{i-1} - \overline{h}_i) \ , \tag{4}$$

where the first equality was derived in Section 2.

We develop the numbers s^k 'directly' in $\mathbb{R}^d$ in the set S. To this end we observe a point ξ moving on λ towards x. As ξ enters an i-facet of S, it exits $\binom{i}{k+1}$ convex hulls of $k+d+1$ points, and it penetrates $\binom{n-d-i}{k+1}$ convex hulls of $k+d+1$ points in S'. This shows that

$$s^k(x, S) = \sum_{i=0}^{n-d} \hat{h}_i \cdot \left(\binom{n-d-i}{k+1} - \binom{i}{k+1} \right) = \sum_i \binom{i}{k+1} \cdot (\hat{h}_{n-d-i} - \hat{h}_i) \tag{5}$$

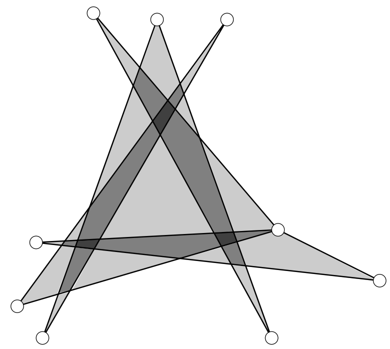
for $k \in \mathbb{Z}$. $s^k(\ell, S') = s^k(x, S)$, (4), and (5) imply (3) via (1).

Apart from the alternative proof of the upper bounds for the $\overline{h}_j$'s, we want to point out two implications of (3). First, the difference $\hat{h}_j - \hat{h}_{n-d-j}$ does not depend on the choice of line λ through x. Second, since we know from the GLBT that $\overline{h}_j \ge \overline{h}_{j-1}$ for $2j \le (n-(d+1)) + 1 = n - d$, we can conclude that $\hat{h}_j - \hat{h}_{n-d-j} \ge 0$ for $2j \le n - d$. In other words, the GLBT says that for $2j \le n - d$, we can never leave more j-facets than we enter j-facets as we move along a line starting at a point outside the convex hull of S.

The g-values of a point relative to S. Let S be a set of n points in $\mathbb{R}^d$ in general position, let x be a point not in S such that $S \cup \{x\}$ is in general position, and, let λ be a directed line through x which is disjoint from all convex hulls of $d-1$ points in S. We define

$$g_j = g_j(x,S) := \hat{h}_j(x,\lambda,S) - \hat{h}_{n-d-j}(x,\lambda,S) \ .$$

for $0 \le j \le n-d$. Recall that g_j does not depend on the choice of λ.



Illustrating the function $g_3(x,S)$ for a set S of nine points in the plane.
Darker shading indicates larger $g_3(x,S)$ for points x in that area.

Lemma 5 (i) *$g_j = -g_{n-d-j}$ for $0 \le j \le n-d$.*
(ii) *For $n-d$ even, $g_{(n-d)/2} = 0$.*
(iii) *$g_j \ge 0$ for $0 \le 2j \le n-d$.*
(iv) *$s^k(x,S) = -\sum_{i=0}^{n-d} \binom{i}{k+1} g_i(x,S)$ for all $k \in \mathbb{Z}$.*
(v) *$x \notin \mathrm{conv}S$ iff $g_j(x,S) = 0$ for all j, $0 \le j \le n-d$.* □

Recall that (iii) is equivalent to the GLBT for simplicial $(n-d-1)$-polytopes with at most n vertices. While this statement seems to be difficult to prove, the reader is encouraged to verify it for $j < (n-d)/d$ via centerpoints (see [10]): Given $S \subseteq \mathbb{R}^d$, a point $c \in \mathbb{R}^d$ is called *centerpoint* if every hyperplane containing c has at most $d|S|/(d+1)$ points from S on either side. Such a centerpoint exists for every finite point set.

In the next section we will use

$$\Gamma_j = \Gamma_j(S) := \sum_{p \in S} g_j(p, S \setminus \{p\})$$

for $0 \le j \le (n-1)-d$, and

$$\Sigma^k = \Sigma^k(S) := -\sum_{i=0}^{(n-1)-d} \binom{i}{k+1} \Gamma_i \ . \tag{6}$$

for $k \in \mathbb{Z}$. We record the immediate implications of Lemma 5 to the introduced values.

Lemma 6 (i) *For $0 \le j \le (n-1)-d$, $\Gamma_j = -\Gamma_{(n-1)-d-j}$.*
(ii) *For $(n-1)-d$ even, $\Gamma_{(n-d-1)/2} = 0$.*
(iii) *$\Gamma_j \ge 0$ for $2j \le (n-1)-d$.*
(iv) *For all $k \in \mathbb{Z}$, Σ^k is the number of pairs (p,Q), $Q \in \binom{S}{k+d+1}$, $p \in S \setminus Q$, with $p \in \mathrm{conv}Q$.*
(v) *S is in convex position iff $\Gamma_j = 0$ for all j, $0 \le j \le (n-1)-d$.* □

5 A conclusion

Given a set S of n points in $\mathbb{R}^d$ in general position, we denote by $e_j = e_j(S)$ the number of j-facets of S and we set $E_j = E_j(S) := \sum_{i \le j} e_i(S)$. We show a tight upper bound on E_j in 3-space for $2j \le n-4$. Two simple facts we will need below: $e_j = e_{n-d-j}$ and $E_{n-d} = 2\binom{n}{d}$.

First, we count the number of 0-facets of $(k+d)$-element subsets of S, i.e. $e_0^k := \sum_{Q \in \binom{S}{k+d}} e_0(Q)$, in terms of the E_j's.

$$\begin{aligned} e_0^k &= \sum_j \binom{n-d-j}{k} \underbrace{e_j}_{e_{n-d-j}} = \sum_{j=0}^{n-d} \binom{j}{k} \underbrace{e_j}_{E_j - E_{j-1}} \\ &= -\binom{0}{k} \underbrace{E_{-1}}_{0} + \sum_{j=0}^{n-d-1} \underbrace{\left(\binom{j}{k} - \binom{j+1}{k} \right)}_{-\binom{j}{k-1}} E_j + \binom{n-d}{k} \underbrace{E_{n-d}}_{2\binom{n}{d}} \\ &= 2 \overbrace{\binom{n}{k+d}\binom{k+d}{d}}^{\binom{n}{d}\binom{n-d}{k}} - \sum_{j=0}^{n-d-1} \binom{j}{k-1} E_j \end{aligned}$$

Second we count the number of vertices of the convex hulls of $(k+d)$-element subsets of S, i.e. $f_0^k := \sum_{Q \in \binom{S}{k+d}} f_0(\text{conv}Q)$.

$$f_0^k + \Sigma^{k-2} = (k+d)\binom{n}{k+d}$$

since every pair (p, Q), $Q \in \binom{S}{k+d-1}$, $p \in S \setminus Q$, contributes either one to f_0^k (if $p \notin \text{conv}Q$) or one to Σ^{k-2} (if $p \in \text{conv}Q$). We substitute Σ^{k-2} according to (6):

$$f_0^k = (k+d)\binom{n}{k+d} + \sum_{j=0}^{n-d-1} \binom{j}{k-1} \Gamma_j \ .$$

In the plane, $e_0^k = f_0^k$ yields

$$\sum_{j=0}^{n-3} \binom{j}{k-1} (E_j + \Gamma_j) = \binom{n}{k+2} \underbrace{\left(2\binom{k+2}{2} - (k+2)\right)}_{k(k+2)} \ .$$

This equality is satisfied for and only for $E_j + \Gamma_j = (j+1)n$. In 3-space, Euler's Relation gives $e_0^k = 2f_0^k - 4\binom{n}{k+3}$ and

$$\sum_{j=0}^{n-4} \binom{j}{k-1} (E_j + 2\Gamma_j) = \binom{n}{k+3} \underbrace{\left(2\binom{k+3}{3} - 2(k+3) + 4\right)}_{k(k+1)(k+5)/3} \ .$$

Here, $E_j + 2\Gamma_j = 2(\binom{j+2}{2}n - 2\binom{j+3}{3})$ constitutes the unique solution.

Lemma 7 (i) *In the plane,* $E_{n-2} = 2\binom{n}{2}$ *and* $E_j = (j+1)n - \Gamma_j$ *for* $0 \le j \le n-3$.
(ii) *In 3-space,* $E_{n-3} = 2\binom{n}{3}$ *and* $E_j = 2(\binom{j+2}{2}n - 2\binom{j+3}{3} - \Gamma_j)$ *for* $0 \le j \le n-4$. ◻

Lemma 6 (iii) and (v) provide

Corollary 8 (i) *In the plane,* $E_j \le (j+1)n$ *for* $0 \le 2j \le n-3$ *with equality for* S *in convex position.*
(ii) *In 3-space,* $E_j \le 2(\binom{j+2}{2}n - 2\binom{j+3}{3})$ *for* $0 \le 2j \le n-4$ *with equality for* S *in convex position.* ◻

Bound (i) has been previously established in [1] and [16]. Bound (ii) was known for $j \le n/4 - 2$, [2]. The restriction of '$2j \le n-d-1$' is a crucial threshold for exact E_j-bounds, since, for $n-d$ even, $E_{(n-d)/2} = \binom{n}{d} + e_{(n-d)/2}/2$.

For constant dimension d, an asymptotic bound of the order $n^{\lfloor d/2 \rfloor}(j+1)^{\lceil d/2 \rceil}$ – asymptotically tight for points on the moment curve – is known, [6].

Remark 1 Let us write GLBT(d,n) for the statement of the Generalized Lower Bound Theorem for simplicial d-polytopes with at most n vertices. We have seen that GLBT$(d,d+3)$ implies Corollary 8 (i), and GLBT$(d,d+4)$ implies part (ii) of that corollary. In fact, one can show now that GLBT$(d,d+3)$ is equivalent to (i) and GLBT$(d,d+4)$ is equivalent to (ii). That is, [1] and [16] have shown GLBT$(d,d+3)$.

The argument proceeds as follows. Suppose we have $d+4$ points in general position in $\mathbb{R}^d$, whose convex hull violates GLBT$(d,d+4)$. By the duality described in Section 3 this corresponds to a set of $n = d+4$ points in $\mathbb{R}^4$ and a directed line ℓ such that $\overline{h}_{j-1} > \overline{h}_j$ for some $2j \le d+1 = n-3$. Now we project this point set parallel to ℓ to obtain a 3-dimensional n-point set S with a point x with $g_j(x,S) < 0$. Note that we can project S to a sphere centered at x without changing $g_j(x)$: clearly, such a projection will not change $s^k(x)$, $k \in \mathbb{Z}$, and so, due to Lemma 5(iv), it will not change the $g_j(x)$'s. Let S' be this projected set together with x, i.e. $|S'| = n+1$. Since all points in S' apart from x are extreme, we have $\Gamma_j(S') = g_j(x, S' \setminus \{x\}) < 0$, where $2j \le n-3 = |S'|-4$. Now Lemma 7 infers the fact that S' has more $(\le j)$-facets than a set of $n+1$ points in convex position.

Remark 2 It is not clear how the bounds in Corollary 8 generalize to higher dimensions. All we can claim at this point (without providing the proof here) is that *if* the number of $(\le j)$-facets in 4-space is maximized in convex position for $2j \le n-5$, then it is maximized for points on the moment curve, or, more generally, by the vertex sets of neighborly polytopes (where these numbers are known).

Remark 3 We have mentioned relations to other papers in the introduction. In Lee's contribution [10] the duality is worked out, and a winding number is introduced, equivalent to the g_j-values of a point we defined here. Also a proof of GLBT$(d,d+3)$ in this dual setting is presented.

In [5] Clarkson presents a nice probabilistic proof for an upper bound of $\binom{j+d-1}{d-1}$ for the number of so-called local minima in j-levels of arrangements of hyperplanes in d-space. This translates to the bounds for the number of j-facets entered by a line (by polar duality). He uses LP-duality to show that this way he gave a new proof of the Upper Bound Theorem.

Finally, Mulmuley considers in [15] so-called h-matrices of bounded k-complexes of arrangements of hyperplanes. 'Our' h- and $\overline{h}$-vector appears in such an h-matrix as the first row and column. Again, properties are derived similar to the Upper Bound Theorem and Dehn-Sommerville Relations.

One difference between our setting and the ones (related by polar duality) in [5] and [15] is that they have to add extra objects in order to ensure boundedness – an issue that never occurs in our scenario.

REMARK 4 A *k-set* of a finite set S in $\mathbb{R}^d$ is a subset K of S that can be separated from $S \setminus K$ by a hyperplane. By the relation between k-sets and j-facets mentioned in [2, Theorem 3], Corollary 8 implies that for $k \leq n/2 - 1$ the number of $(\leq k)$-sets of n-point sets in $\mathbb{R}^3$ is maximized in convex position.

REMARK 5 We refer to a paper by J. Linhart [11], since he proves the same bound for a similar problem. Let us briefly translate his setting to a scenario comparable to ours. We are given a set S' of $n+2$ points in general position in $\mathbb{R}^d$. Let x and y be two distinct points in S', and $S := S' \setminus \{x, y\}$. For $0 \leq j \leq n+1-d$, we denote by $\hat{e}_j$ the number of j-facets of $S \cup \{x\}$ incident to x and with y on its positive side; $\hat{E}_j := \sum_{i=0}^{j} \hat{e}_j$. Then Linhart proves that *for all* $0 \leq j \leq n-(d-1)$ we have $\hat{E}_j \leq (j+1)n$, if $d = 3$, we have $\hat{E}_j \leq 2(\binom{j+2}{2}n - 2\binom{j+3}{3})$, if $d = 4$, and we have $\hat{E}_j \leq n(\binom{j+2}{2}(n-1) - 2\binom{j+3}{3})/2$, if $d = 5$.

So how does this relate to our problem of counting all j-facets? If x can be separated from S by a hyperplane H, then we can consider S'', the set of intersections of the segments $\overline{xp}$, $p \in S$, with the hyperplane H. Clearly, there is a bijection between the j-facets of S incident to x on one hand, and the j-facets of S'' in H on the other hand. That is, on one hand, the bound we obtained here for $(\leq j)$-facets in 3-space implies Linhart's bound in 4-space only when x is separable; on the other hand, we are not restricted to j-facets containing a specific point y. Hence the results are incomparable. It explains why Linhart's bounds are valid for all j, while this cannot be the case for our problem.

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