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Convex Quadrilaterals and k-Sets

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Abstract

We prove that the minimum number of convex quadrilaterals determined by n points in general position in the plane – or in other words, the rectilinear crossing number of the complete graph K_n – is at least $(\frac{3}{8} + 10^{-5})\binom{n}{4} + O(n^3)$. This is closely related to the rectilinear crossing number of complete graphs and to Sylvester's Four Point Problem from the theory of geometric probabilities.

As our main tool, we prove a lower bound on the number of $(\leq k)$-sets of the point set: for every $k \leq n/2$, there are at least $3\binom{k+1}{2}$ subsets of size at most k that can be separated from their complement by a straight line.

1 Introduction

Let S be a set of n points in general position in the plane, i.e. no three points are collinear. Four points in S may or may not form the vertices of a convex quadrilateral; if they do, we call this subset of 4 elements *convex*. We are interested in the number $\square(S)$ of convex 4-element subsets. This can of course be as large as $\binom{n}{4}$, if S is in convex position, but what is its minimum?

There are two other, equivalent ways of reformulating the problem: As the well-known *Four Point Problem*, which we review below, or as that of finding the *rectilinear crossing number* of the complete n-graph K_n, i.e., to determine the minimum number of crossings in a drawing of K_n in the plane with straight edges and the nodes in general position.

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We note here that the rectilinear crossing number of complete graphs also determines the rectilinear crossing number of random graphs (provided the probability for an edge to appear is at least $\frac{\ln n}{n}$), as was shown by Spencer and Tóth [16].

It is easy to see that for $n = 5$ we get at least one convex 4-element subset, from which it follows by straightforward averaging that at least 1/5 of the 4-element subsets are convex for every $n \geq 5$. Similarly, if we have a lower bound for some fixed number n_0 of points, then this gives a lower bound for every $n \geq n_0$. The best lower bound proved by this method is due to Aichholzer, Aurenhammer, and Krasser [2]: they obtained the lower bound of $0.3115\binom{n}{4}$ by inspecting all configurations of 11 points.

The first lower bound based on a different method was $\frac{53-5\sqrt{13}}{216}\binom{n}{4}+O(n^3) \approx 0.3288\binom{n}{4}$, which was established in [19].

Upper bounds on the minimum number of convex quadrilaterals were given by constructions of Jensen [12], Singer [15] and others; the best upper bound is $0.3807\binom{n}{4}$, also due to Aichholzer, Aurenhammer and Krasser [2]. The best construction "by hand", i.e., without a computer-generated base case, yields an upper bound of $0.3838\binom{n}{4}$ and is due to Brodsky, Durocher, and Gethner [6].

As mentioned above, there is yet another equivalent way of looking at the question: For a (Borel) probability ditribution μ in the plane, let $\square(\mu)$ denote the probability that four independent μ-random points form a convex quadrilateral. We assume that every line has μ-measure zero, so that degenerate configurations occur only with probability zero. Then,

$$\inf_{\mu} \square(\mu) = \lim_{n\to\infty} \max_{|S|=n} \frac{\square(S)}{\binom{n}{4}},$$

as was pointed out by Scheinerman and Wilf [14]. (It follows from the averaging argument outlined above that the sequence on the right-hand side is monotonically increasing, so that the limit exists.) This remains true even if we restrict our attention to uniform distributions on bounded open sets, or on regions bounded by a simple closed Jordan curve.

In this "continuous" set-up, the problem is known as Sylvester's Four Point Problem, because it was first posed by Sylvester [17] in 1864 (without, however, adressing the issue of the dependence on the underlying distribution). At first, investigations focussed on the case of a uniform distribution μ_K on a *convex* body K in the plane. For this special case, the problem was solved by Blaschke [5], who showed that

$$\frac{2}{3} \leq \square(\mu_K) \leq 1 - \frac{35}{12\pi^2} \approx 0.704,$$

and that both inequalities are sharp: The lower bound is attained iff K is a triangle, and the upper bound iff K is an ellipse.

We work in the set-up of finite point sets. Our main theorem is the following lower bound:

Theorem 1. *Let S be a set of n points in the plane in general position. Then the number $\square(S)$ of convex quadrilaterals determined by S is at least*

$$(3/8+\varepsilon)\binom{n}{4}+O(n^3) > 0.37501\binom{n}{4},$$

where $\varepsilon \approx 1.0887 \cdot 10^{-5}$.

We note that a lower bound of $3/8\binom{n}{4}$ has been proved independently by Ábrego and Fernández-Merchant [1], using methods similar to ours.

The small ε is significant for the following reason. If one drops the requirement that the edges be represented by straight line segments and instead allows general Jordan arcs, then K_n *can* be drawn with $3/8\binom{n}{4}+O(n^3)$ crossings [11, 13]. So, while it is well-known that the ordinary crossing number and the rectilinear crossing number of complete graphs differ (the smallest n for which they differ is 8, see [15]), our lower bound shows that the difference lies in the asymptotically relevant term.

The first ingredient of our proof is an expression of the number of convex quadrilaterals in terms of *k-sets* of S. A k-set of S is a subset $T \subseteq S$ of cardinality k such that T can be separated from its complement $T \setminus S$ by a line. An i-set with $1 \leq i \leq k$ is called an $(\leq k)$-set.

The second ingredient is the following bound on the number of $(\leq k)$-sets:

Theorem 2. *Let S be a set of n points in the plane in general position, and $k < n/2$. Then the number of $(\leq k)$-sets of S is at least $3\binom{k+1}{2}$.*

This lower bound was first formulated by Edelsbrunner, Hasan, Seidel, and Chen [8]. Unfortunately, their proof seems to contain an unpluggable gap. Our proof will, however, follow the same basic approach via circular sequences (see [10]).

We note that this bound is tight for $k \leq n/3$, as shown by a construction which will also be instructive for many of the arguments that follow.

Example 3. Let r_1, r_2 and r_3 be three rays emanating from the origin with an angle of 120° between each pair. Let S_i be a set of m points in general position, all very close to r_i but at distance at least 1 from each other and from the origin. Let $S = S_1 \cup S_2 \cup S_3$.

Then for $1 \leq k \leq n/3$, every k-set of S contains the i points farthest from 0 in one S_a, for some $1 \leq i \leq k$, and the $(k-i)$ points farthest from 0 in another S_b. Hence the number of k-sets is $3k$ and the number of $(\leq k)$-sets equals $3\binom{k+1}{2}$.

However, the bound in Theorem 2 is *not* tight if k is near $n/2$. To get the tiny improvement over 3/8, we use a consequence of the results in [20]:

Theorem 4. *Let S be a set of n points in the plane in general position, and $k < n/2$. Then the number of $(\leq k)$-sets of S is at least*

$$\binom{n}{2} - n\sqrt{n^2-2n-4k^2+4k}.$$

This bound is better than $3\binom{k+1}{2}$ if $k > 0.4956n$.

2 k-Sets, j-Edges and Convex Quadrilaterals

It is known [3] that the maximum number of $(\leq k)$-sets of an n-point set in the plane is nk, which is attained for point sets in convex position (and only for those, see [21]). By contrast, despite significant progress in recent years, the problem of determining the (order of magnitude of) the maximum number of k-sets for a single k remains tantalizingly open (the currently best bounds are $O(nk^{1/3})$ and $ne^{\Omega(\log k)}$, due to Dey [7] and Tóth [18], respectively).

Often, it is technically more convenient to think about the k-set problem in terms of the following, related objects. A j-edge of S is an ordered pair uv, with $u, v \in S$ and $u \neq v$, such that there are exactly j points of S on the right hand side of the line uv. Let $e_j = e_j(S)$ denote the number of j-edges of S; it is well known and not hard to see that for $1 \leq k \leq n-1$, the number of k-sets is e_{k-1}. An i-edge with $i \leq j$ will be called a *$(\leq j)$-edge*; we denote the number of $(\leq j)$-edges by $E_j = e_0 + \ldots + e_j$.

Let $\square$ denote the number of 4-tuples of points in S that are in convex position, and let $\triangle$ denote the number of those in concave position. We are now ready to state the crucial lemma (also noted in [21]), which expresses $\square$ as a positive linear combination of the numbers e_j (one might say, as the second moment of the distribution of j-edges).

Lemma 5. *For every set of n points in the plane in general position,*

$$\square = \sum_{j < \frac{n-2}{2}} e_j \left(\frac{n-2}{2} - j\right)^2 - \frac{3}{4}\binom{n}{3}.$$

Proof. Clearly we have

$$\square + \triangle = \binom{n}{4}. \tag{1}$$

To get another equation between these quantities, let us count, in two different ways, ordered 4-tuples (u, v, w, z) such that w is on the right of the line uv and z is on the left of this line. First, if $\{u, v, w, z\}$ is in convex position, then we can order it in 4 ways to get such an ordered quadruple; if $\{u, v, w, z\}$ is in concave position, then it has 6 such orderings. Hence the number of such ordered quadruples is $4\square + 6\triangle$. On the other hand, any j-edge uv it can be completed to such a quadruple in $j(n-j-2)$ ways. So we have

$$4\square + 6\triangle = \sum_{j=0}^{n-2} e_j j(n-j-2). \tag{2}$$

From (1) and (2) we get that

$$\square = \frac{1}{2}\left(6\binom{n}{4} - \sum_{j=0}^{n-2} e_j(n-j-2)j\right).$$

Using that

$$\sum_{j=0}^{n-2} e_j = n(n-1), \tag{3}$$

we can write

$$6\binom{n}{4} = \sum_{j=0}^{n-2} e_j \frac{(n-2)(n-3)}{4}$$

to get

$$\square = \frac{1}{2}\left(\sum_{j=0}^{n-2} e_j \left(\frac{(n-2)(n-3)}{4} - j(n-j-2)\right)\right),$$

from which the lemma follows by simple computation, using that $e_j = e_{n-2-j}$. $\square$

Having expressed $\square$ (up to some error term) as a positive linear combination of the e_j's, we can substitute any lower estimates for the numbers e_j to obtain a lower bound for $\square$.

It is not hard to derive a sharp lower bound for each individual e_j. We will use the following theorem from [9]

Theorem 6. *Let S be a set of n points in the plane in general position, and let T be a k-set of S. Then for every $0 \leq j \leq (n-2)/2$, the number of j-edges uv with $u \in T$ and $v \in S \setminus T$ is exactly $\min(j+1, k, n-k)$.*

Proposition 7. *For every set of n points in the plane in general position and for every $j < \frac{n-2}{2}$,*

$$e_j \geq 2j + 3.$$

For every $j \geq 0$ and $n \geq 2j+3$, this bound is attained.

Proof. Consider any j-edge uv, and let e be the line obtained by shifting the line uv by a small distance so that u and v are also on the right hand side, and let T be the set of points of S on the smaller side of e. This will be the side containing u and v unless $n = 2j+3$; hence $|T| \geq j+1$. By Theorem 6, the number of j-edges xy with $x \in T$, $y \in S \setminus T$ is exactly $j+1$. Similarly, the number of j-edges xy with $y \in T$, $x \in S \setminus T$ is exactly $j+1$, and these are distinct from the others since $n \geq 2j+3$. Together with uv, this gives a total of $2j+3$ such pairs. $\square$

The following construction shows that this bound is sharp.

Example 8. Let S_0 be a regular $(2j+3)$-gon, and let S_1 be any set of $n-2j-3$ points in general position very near the center of S_0.

Every line through any point in S_1 has at least $j+1$ points of S_0 on both sides, so the j-edges are the longest diagonals of S_0, which shows that their number is $2j+3$.

Using Proposition 7 in the formula of Lemma 5, we get

$$\square \geq \sum_{j<\frac{n-2}{2}} (2j-3)\left(\frac{n-2}{2} - j\right)^2 - \frac{3}{4}\binom{n}{3} = \frac{1}{4}\binom{n}{4} + O(n^3).$$

This lower bound for $\square$ is weaker than previously known lower bounds. Its weakness rests mainly in the fact that the point set in Example 8 is highly attuned to the specific j at hand.

To obtain the stronger lower bound stated in Theorem 1, we do "integration by parts", i.e., we pass from j-facets to $(\leq j)$-facets. We substitute $e_j = E_j - E_{j-1}$ in Lemma 5 (with the notation $E_j = \sum_{i=0}^{j} e_i$ introduced above) and rearrange to get the following:

Lemma 9. *For every set of n points in the plane in general position,*

$$\square = \sum_{j<\frac{n-2}{2}} E_j(n-2j-3) - \frac{3}{4}\binom{n}{3} + c_n,$$

where

$$c_n = \begin{cases} \frac{1}{4}E_{\frac{n-3}{2}}, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Note that the last two terms in the formula of this lemma are $O(n^3)$.

If we substitute the lower bound from Theorem 2 in Lemma 9, we obtain the bound

$$\square \geq \frac{3}{8}\binom{n}{4} + O(n^3).$$

for the number of convex quadrilaterals of n points in general position. Using the bound in Theorem 4 whenever it is better, we get Theorem 1.

3 A Lower Bound for the Number of $(\leq k)$-Sets

We now proceed to prove the sharp lower bound stated in Theorem 2. Let $\mathbf{\Pi}$ be (a halfperiod of) a *circular sequence* of $\{1\ldots n\}$. That is, $\mathbf{\Pi} = (\Pi_0, \ldots, \Pi_{\binom{n}{2}})$ is a sequence of permutations of $\{1\ldots n\}$ such that Π_0 is the identity permutation $(1, 2, \ldots, n)$, $\Pi_{\binom{n}{2}}$ is the reverse permutation $(n, n-1, \ldots, 1)$, and any two consecutive permutations differ by exactly one transposition of two elements in adjacent positions.

Circular sequences (which were introduced by Goodman and Pollack [10]) can be used to encode any planar point set. For our purposes and for simplicity, we only consider the case of a point set S in general position. Moreover, we will make the additional assumption that no two segments spanned by points from S are parallel (we can assume this without loss of generality, since it can be ensured by sufficiently small perturbations of the points, and this will not affect the number of convex quadrilaterals or the number of k-sets).

Let ℓ be a directed line which is not orthogonal to any of the lines spanned by points from S, and assume that $S = \{p_1, \ldots, p_n\}$, where the points are labeled according to the order in which their orthogonal projections appear along the line. Now suppose that we start rotating ℓ counterclockwise, say. Then the ordering of the projections changes whenever ℓ passes through a position where it is orthogonal to a segment uv, with $u, v \in S$. When such a change occurs, u

and v are adjacent in the ordering, and the ordering changes by u and v being transposed. Thus, if we keep track of all permutations of the projections as the line ℓ is rotated by $180°$, we obtain a circular sequence $\mathbf{\Pi} = \mathbf{\Pi}(S)$. (The sequence also depends on the initial choice of ℓ, which for sake of definiteness, we can assume to be vertical and directed upwards).

Observe that if a circular sequence arises in this fashion from a point set, then the $(i-1)$-edges (and hence the i-sets) of the point set correspond to transpositions between elements in positions i and $i+1$, or in positions $n-i$ and $n-i+1$. These will be referred to as *i-critical* transpositions of the circular sequence.

For $k \leq n/2$, we consider the number of *$(\leq k)$-critical transpositions*, i.e., the number of transpositions that are i-critical for some $i \leq k$.

Theorem 10. *For any circular sequence $\mathbf{\Pi}$ on n elements and any $k \leq n/2$, the number of $(\leq k)$-critical transpositions is at least $3\binom{k+1}{2}$.*

(To be completely precise, for $k = n/2$, the lower bound is $\min\{\binom{3(k+1)}{2}, \binom{n}{2}\}$, but we ignore this since it is only relevant for $n \leq 6$.) If the sequence arises from a set S of n points in general position in the plane as the list of the combinatorially different orthogonal projections of S onto a rotating directed line, then the i-critical swaps are in one-to-one correspondence with the i-sets of S, and hence with the $(i-1)$-edges of S. Thus, the number $E_j = \sum_{i=0}^{j} e_i$ of $(\leq j)$-edges of S is at least $3\binom{j+2}{2}$, which will prove Theorem 2.

Proof. Fix k and let $m := n - 2k$. It will be convenient to label the points so that the starting permutation is

$$\Pi_0 = (a_k, a_{k-1}, \ldots, a_1, b_1, b_2, \ldots, b_m, c_1, c_2, \ldots, c_k).$$

We introduce some terminology. We say that an element x *exits* (respectively, *enters*) *through the ith A-gate* if it moves from position $k-i+1$ to position $k-i+2$ (respectively, from position $k-i+2$ to position $k-i+1$) during a transposition with another element. Similarly, x *exits* (respectively, *enters*) *through the ith C-gate* if it moves from position $m+k+i$ to position $m+k+i-1$ (respectively, from $m+k+i-1$ to $m+k+i$) during a transposition. Observe that for $1 \leq i \leq j \leq k$, a_j has to exit through the ith A-gate and to enter through the i-th C-gate at least once, and analogously for c_j.

Further, we say that $a \in \{a_1, \ldots, a_k\}$ (respectively, $c \in \{c_1, \ldots, c_k\}$) is *confined* until the first time it exits through the 1st A-gate (respectively, C-gate); then it becomes *free*. Elements $b \in \{b_1, \ldots, b_m\}$ are always free.

Simplifying Observation. For every circular sequence $\mathbf{\Pi'}$, there is another sequence $\mathbf{\Pi}$ with the same number of $(\leq k)$-critical transpositions and without transpositions between confined elements. Thus, we may restrict our attention to sequences without such *confined transpositions*.

Proof of the observation. To see why this is so, consider the first confined transposition in $\mathbf{\Pi'}$ (if there isn't any, we are done). Clearly, this first transposition must be either between two a's or between two c's. But before a_i and a_j, say, can be transposed, every a_s with $i < s < j$ has to be transposed with either a_i

or a_j. And as long as a_j is confined, every element a_s, $s < j$ which has not yet been transposed with a_j is also confined.

Therefore, the first confined transposition has to happen between two a's (or between two c's) that are adjacent in the starting permutation Π'_0, say between a_i and a_{i+1}. Now we can modify $\mathbf{\Pi'}$ as follows: Instead of transposing a_i and a_{i+1} when it happens in $\mathbf{\Pi'}$, let a_{i+1} follow the "path" of a_i in $\mathbf{\Pi'}$ and vice versa, and only transpose a_i and a_{i+1} in the end. (Observe that for this to be feasible, it is crucial that a_i and a_{i+1} are adjacent in Π'_0.) This does not affect the number of $(\leq k)$-critical transpositions and deletes one confined transposition without generating any new ones, which (by induction, say) proves the observation.

So we may assume that the circular sequence $\mathbf{\Pi}$ does not contain any confined transpositions. Now, let us write down the *liberation sequence* σ of all a's and c's in the the order in which they become free. Since $\mathbf{\Pi}$ does not contain any confined transpositions, the a's appear in σ in increasing order (i.e., a_i precedes a_j in σ if $i < j$) and the same holds for the c's.

We are now ready to estimate the number of $(\leq k)$-critical transpositions. As observed above, for $1 \leq i \leq j \leq k$, a_j has to exit through the ith A-gate and to enter through the i-th C-gate at least once, and c_j has to exit through the ith C-gate and to enter through the ith A-gate at least once. For each of these inevitable events, we count the first time it happens. This gives a total count of $4\binom{k+1}{2}$ transpositions, all of which are $(\leq k)$-critical. (In the case $k = n/2$, i.e. $m = 0$, the 1st A-gate and the 1st C-gate coincide, so we obtain $2k = n$ fewer critical transpositions by our way of counting, but we can ignore this case since for $k = n/2$, all $\binom{n}{2}$ transpositions are critical.)

What is the overcount? The transpositions that are counted twice are precisely the transpositions between some a_j and some c_l during which, for some $i \leq \min\{j, l\}$,

1. either a_j enters and c_l exits through the ith C-gate (both for the first time),

2. or a_j exits and c_l enters through the ith A-gate (both for the first time).

In order to estimate the number of such transpositions, we "credit" each transposition to the entering element. More precisely, we define a *savings digraph* D with vertex set $\{a_1, \ldots, a_k\} \cup \{c_1, \ldots, c_k\}$ and the following edges: In Case 1, we put in a directed edge from c_l to a_j, and in Case 2 a directed edge from a_j to c_l.

Thus, the number of $(\leq k)$-critical transpositions is at least $4\binom{k+1}{2}$ minus the number of edges in D, and it suffices to show that the latter is at most $\binom{k+1}{2}$.

For this, we estimate the in-degree of each vertex. On the one hand, observe that the in-degree of a_j is at most j (there is at most one incoming edge for each ith C-gate, $1 \leq i \leq j$, since we only count the first time that a_j enters through a gate). On the other hand, we observe that if there is a directed edge from c_l to a_j, then a_j precedes c_l in the liberation sequence σ (observe that a_j must have become free before entering through any C-gates, while c_l is still confined when it exits through a C-gate for the first time; again, we may ignore the complication that the first A-gate and the first C-gate coincide if $k = n/2$). Thus, since any two elements are transposed at most (in fact, exactly) once, the in-degree of a_j is also at most the number of c's that come after it in the sequence σ. Hence, the in-degree of a_j is at most the minimum $\mu_\sigma(a_j)$ of j and the number of c's that come after a_j in the sequence σ. Similarly, the in-degree of c_l is at at most minimum $\mu_\sigma(c_l)$ of l and the number of a's which come after c_l in the sequence σ.

The proof is concluded by the following observation: For all σ (subject to the constraint that the a's and the c's appear in increasing order),

$$\sum_{j=1}^{k} \big(\mu_\sigma(a_j) + \mu_\sigma(c_j)\big) = \binom{k+1}{2}. \tag{4}$$

To prove this, first note that it obviously holds true for the sequence $\langle a_1, a_2, \ldots, a_k, c_1, c_2, \ldots, c_k\rangle$. So it suffices to show that the sum is invariant under swaps of adjacent a's and c's. Suppose then that $\sigma = \rho \circ \langle a_j, c_l\rangle \circ \tau$ and that $\sigma' = \rho \circ \langle c_l, a_j\rangle \circ \tau$ (where "$\circ$" denotes concatenation of sequences). First observe that $\mu_\sigma(x) = \mu_{\sigma'}(x)$ for all $x \neq a_j, c_l$. Moreover,

$$\begin{array}{rclcrcl} \mu_\sigma(a_j) & = & \min\{j, k-l+1\}, & \qquad & \mu_\sigma(c_l) & = & \min\{l, k-j\}, \\ \mu_{\sigma'}(a_j) & = & \min\{j, k-l\}, & & \mu_{\sigma'}(c_l) & = & \min\{l, k-j+1\}. \end{array}$$

We distinguish two cases: On the one hand, if $j + l \leq k$, then $\mu_\sigma(a_j) = j = \mu_{\sigma'}(a_j)$ and $\mu_\sigma(c_l) = l = \mu_{\sigma'}(c_l)$, i.e. nothing changes. On the other hand, if $j+l > k$, then $\mu_\sigma(a_j) = k-l+1 = \mu_{\sigma'}(a_j)+1$ and $\mu_\sigma(c_l) = k-j = \mu_{\sigma'}(c_l)-1$, so the sum remains unaffected. This proves (4) and hence the theorem. $\square$

4 j-Edges for Large j

The goal of this section is to show that, as stated in Theorem 1, the constant factor for $\binom{n}{4}$ is strictly larger than 3/8. We will exploit the fact that while the lower bound $E_j \geq 3\binom{j+2}{2}$ is sharp for $j < n/3$, it is no longer tight for j close to $n/2$ (in particular, observe that for odd n, $E_{(n-3)/2} = \binom{n}{2} \sim 4\binom{(n-3)/2}{2}$). Specifically, we will use the following result from [20]:

Theorem 11. *Let S be a set of n points in the plane, and consider a (not necessarily contiguous) index set $K \subseteq \{1, 2, \ldots, \lfloor n/2\rfloor\}$. Then the total number of k-sets with $k \in K$ is at most*

$$2n\sqrt{2\textstyle\sum_{k\in K} k}.$$

In particular, let $m = \lfloor n/2\rfloor$, and apply this theorem to the intervals of the form $\{j+2, j+3, \ldots, m\}$. Observing that e_i is precisely the number of $(i+1)$-sets, we obtain that for all $j \leq m-1$,

$$E_{m-1} - E_j \leq 2n\sqrt{2\textstyle\sum_{i=j+2}^{m} i} = 2n\sqrt{m^2 + m - j^2 - 3j - 2},$$

and since $E_{m-1} \geq \binom{n}{2}$,

$$E_j \;\geq\; \binom{n}{2} - 2n\sqrt{m^2 + m - j^2 - 3j - 2},$$

which proves Theorem 4.

Combining the two theorems, we get

$$\begin{aligned} E_j &\geq 3\binom{j+2}{2} + \max\left(0, \binom{n}{2} - 3\binom{j+2}{2} - n\sqrt{n^2 + 2n - 4j^2 - 12j - 8}\right) \\ &\geq 3\binom{j+2}{2} + n^2 \max\left(0, \frac{1 - 3(j/n)^2}{2} - \sqrt{1 - 4(j/n)^2}\right) + O(n). \end{aligned}$$

The "max" term is positive for $j/n \geq t_0 = \sqrt{(2\sqrt{13} - 5)/9} \approx 0.4956$, so we do gain when j is very near $n/2$. Using Lemma 9, we get

$$\begin{aligned} \square &= \sum_{j \leq m-1} (n - 2j - 3)E_j + O(n^3) \\ &\geq \sum_{j \leq m-1} 3(n - 2j - 3)\binom{j+2}{2} \\ &\quad + n^3 \sum_{t_0 n \leq j \leq m} (1 - 2(j/n))\left(\frac{1 - 3(j/n)^2}{2} - \sqrt{1 - 4(j/n)^2}\right) + O(n^3) \\ &= \frac{3}{8}\binom{n}{4} + n^4 \int_{t_0}^{1/2} (1 - 2t)\left(\frac{1 - 3t^2}{2} - \sqrt{1 - 4t^2}\right) dt + O(n^3). \end{aligned}$$

Thus,

$$\square \geq (3/8 + \varepsilon)\binom{n}{4} + O(n^3),$$

with

$$\varepsilon = 24 \int_{t_0}^{1/2} (1 - 2t)\left(\frac{1 - 3t^2}{2} - \sqrt{1 - 4t^2}\right) dt \approx 1.0887 \cdot 10^{-5}.$$

This completes the proof of Theorem 1. We remark that in the set-up of Theorem 11, an asymptotically stronger lower bound of $O(n\left(|K| \sum_{k \in K} k\right)^{1/3})$ can be proved [4, 7]. This, in turn, can be used for a further tiny improvement in the ε. We omit the details.

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