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Curve Reconstruction

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Abstract

In this thesis we deal with the problem of reconstructing a curve from a finite set of points.

The general shape reconstruction problem which asks for a shape that best approximates a finite set of points got some attention in recent years. In spite of its importance there has been little success in giving a satisfactory answer to the general problem. It turned out that the main source of problems is the overwhelming variety of shape in possible application domains.

One way to cope with these problems is to restrict the shapes allowed for reconstruction. Most prominent under these restricted problems is the surface reconstruction problem. Here the requirement is that the approximating shape is a closed surface in three dimensions. This problem is much easier than the general one, but nevertheless there are still problems one has to solve:

1) It is difficult to validate the quality of the approximating surface in terms of mathematics.

2) No methods are known which give a reasonable reconstructing surface in dimensions larger than three. Almost all known approaches to surface reconstruction use a subcomplex of the Delaunay complex for reconstruction. The cost to compute this complex grows exponentially in the dimension. Finally it is not known which surfaces can faithfully be reconstructed using the Delaunay complex.

3) There has been little success in phrasing the surface reconstruction problem as an optimization problem.

In this thesis we restrict the approximating shapes to curves. In curve reconstruction we can cope with the problems listed above for surface reconstruction.

1) It is easy to validate the quality of the reconstructing curve. Assume that the given finite point set is sampled from a curve. The correct reconstruction connects the points in the same order as the points are connected along the curve.

2) We classify all curves that can be reconstructed correctly using the Delaunay complex. It is shown that none of the known curve reconstruction schemes based on the Delaunay complex can exhaust the class of curves defined by our characterization.

3) We classify all curves that can be reconstructed correctly by the Traveling salesman tour. This class coincides with the class of curves which can be reconstructed correctly using the Delaunay complex. Recently it was shown by Althaus and Mehlhorn that the Traveling Salesman tour can be computed efficiently for point sets suitably sampled from a curve. The cost of computing the Traveling salesman tour should depend only moderately on the dimension of the space in which the curve is embedded.

Zusammenfassung

Die vorliegende Doktorarbeit behandelt das Thema, eine Kurve von einem endlichen Sample zu rekonstruieren.

In den letzten Jahren wurde dem allgemeineren Gestaltrekonstruktionsproblem einiges Interesse entgegengebracht. In diesem Problem wird nach der Form gefragt wird, die eine endliche Punktemenge am besten approximiert. Trotz der praktischen Bedeutung des Gestaltrekonstruktionsproblems ist man einer befriedigenden Lösung bislang noch nicht sehr Nahe gekommen. Die Hauptursache für diesen wenig befriedigenden Zustand ist die überwältigende Anzahl von Formen in möglichen Anwendungsbereichen.

Ein Weg mit diesen Problemen umzugehen ist, die in der Rekonstruktion erlaubten Formen einzuschränken. Am bekanntesten unter diesen eingeschränkten Problemen ist das Flächenrekonstruktionsproblem, bei dem eine endliche Punktemenge im dreidimensionalen euklidischen Raum durch eine geschlossene Fläche approximiert werden soll. Dieses Problem ist natürlich einfacher als das allgemeine Problem, trotzdem gilt es auch hier ein paar Schwierigkeiten zu lösen:

1) Es ist schwierig, die Qualtität der approximierenden Fläche in einem mathematisch strengen Sinne zu bewerten.

2) Bislang sind keine Methoden bekannt, die eine vernünftige Fläche zur Rekonstruktion in Räumen mit mehr als drei Dimensionen berechnen. Fast alle bekannten Ansätze zur Flaächenrekonstruktion benutzen den Delaunaykomplex für die Rekonstruktion. Leider wächst der Aufwand, diesen Komplex zu berechnen, exponentiell in der Dimension. Ausserdem ist nicht bekannt, welches die Klasse der Flächen ist, die zuverlssig mit dem Delaunaykomplex rekonstruiert werden können.

3) Bislang war dem Versuch, das Flächenrekonstruktionsproblem als Optimierungsproblem zu formulieren, wenig Erfolg beschieden.

In dieser Doktorarbeit beschränken wir die Formen, die in der Rekonstruktion erlaubt sind, auf Kurven. Im zugehörigen Kurvenrekonstruktionsproblem können wir die Probleme lösen, die oben für Flächen aufgelistet sind.

1) Die Qualität einer approximierenden Kurve ist einfach zu beurteilen. Angenommen die gegebene endliche Punktemenge stammt von einer Kurve, dann verbindet eine korrekte Rekonstruktion die Punkte in der gleichen Reihenfolge, in der die Punkte längs der Kurve verbunden sind.

2) Wir klassifizieren diejenigen Kurven, die bei Benutzung des Delaunaykomplexes korrekt rekonstruiert werden können. Keine der bekannten Rekonstruktionsmethoden kann die durch unsere Klassifikation gegebene Klasse von Kurven korrekt rekonstruieren.

3) Wir klassifizieren alle Kurven, die korrekt durch eine sogenannte Handlungsreisenden Rundreise (Rundreise kürzester Länge, die alle Punkte besucht und zum Startpunkt zurückkehrt) rekonstruiert werden können. Diese Klasse stimmt mit der Klasse der Kurven überein, die bei Benutzung des Delaunaykomplexes korrekt rekonstruiert werden können. Kürzlich haben Althaus und Mehlhorn gezeigt, dass eine Handlungsreisenden Rundreise für gut gesampelte Kurven effizient berechnet werden kann. Der Aufwand eine Handlungsreisenden Rundreise zu berechnen, sollte nur moderat von der Dimension des Raumes abhängen, in dem die Kurve eingebettet ist.

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Contents

Introduction

Over the years enormous progress in the field of computational geometry was made in the design and development of data structures and efficient algorithms. In recent years the focus of interest shifted from the design to practical aspects of the developed data structures and algorithms. This shift of interest reflects itself in two major trends.

First, the study of actual implementations of the algorithms and data structures developed. Probably the most important work done in this direction of research is the CGAL project [CGAL]. The work on actual implementations raised a bunch of new questions, many of them requiring sophisticated theoretical tools. These new questions raised during the implementation work are still an active area of research.

Second, the attempts to apply the developed techniques to real application domains. One area to name here are geographical information systems, where lots of results from computational geometry are successfully applied. An important issue in this direction of research is to find answers to the question whether the data structures and algorithms considered are really well suited for the application.

This thesis follows the second trend. We want to apply techniques developed in computational geometry to the shape reconstruction problem. In this problem we are given a finite set of points sampled from some object and asked for the best approximating shape to this point set. Up to now it is not clear what it means that a shape approximates a point set. That is, before one tries to apply techniques developed in computational geometry one has to find an answer to the latter question. The main source of problems one has to face is the large number of shapes that appear in the world around us. It is possible that there is no reasonable collective concept for the things that we want to express by the term shape. The situation changes if one restricts the shapes allowed in reconstruction. An example of great practical importance is the case where the points are sampled from the surface of a three dimensional real world object. In this case it is reasonable to restrict the shapes allowed in reconstruction to closed surfaces in three dimensions. Even in this restricted case one is left with the question for the best approximating surface. A possible answer is, that the reconstructing surface should reflect the topology of the initial surface and should give the best approximation to the geometry of this surface. Most prominent among the geometric properties of surfaces are area and curvatures. It is a non trivial task to give a reasonable meaning to the terms area and curvatures in the case that the initial surface is non smooth. The situation is different in the case of curves. Here a well developed theory of the geometry of irregular curves exists [AR]. In curve reconstruction the term 'best reconstruction' has a concrete meaning. Assume the points are sampled from a curve. In this case the best reconstruction is the polygon that connects the points in the same order as they are connected along the curve. Hence in curve reconstruction the initial problem of shape reconstruction is reduced to the problem of computing the best reconstruction.

Probably the first article [Z] dealing with the curve reconstruction problem was published in 1971. In this article Zahn uses Euclidean minimum spanning trees to solve the problem. We will see in this thesis that this approach is quite good, but back in 1971 no proof of correctness was given. Thereafter many schemes have been developed to connect the points in perceptually justified polygons: In 1979 Fairfield discussed some heuristics. Toussaint [T] in 1981 invented the relative neighborhood graph and the β-skeleton. In 1987 O'Rourke et al. [OBW] proposed the minimal spanning Voronoi tree, which is a subgraph of the Delaunay triangulation of the sample points, for reconstruction. But only very recently the first proofs of correct reconstruction were published. In 1997 Amenta et al. [ABE] showed that the crust, another subgraph of the Delaunay triangulation, can reconstruct smooth curves. In the same article the authors also show that the β-skeleton for appropriate values of β is also well suited to reconstruct smooth curves. Also in 1997 Attali [At] showed that the normalized mesh can reconstruct smooth curves and Bernardini and Bajaj [BB] showed that one can reconstruct smooth curves with α-shapes. Both graphs, the normalized mesh and α-shapes, are subgraphs of the Delaunay triangulation.

All known provably correct reconstruction schemes make use of some subgraph of the Delaunay graph. This rises the question to what extent a curve can be reconstructed by using the Delaunay graph. In this thesis we classify the curves that can be reconstructed using the Delaunay graph. The class of these curves turns out to contain more than just the smooth curves. For example it also contains polygons, which are non smooth. There was no reconstruction scheme known that can reconstruct the whole class. Even worse there was no reconstruction scheme that can reconstruct polygons. We show that the Traveling salesman tour does it. The class of curves that can be reconstructed with the Traveling salesman tour turns out to be the same as the class of curves that can be reconstructed using the Delaunay graph. Furthermore recent work of Althaus and Mehlhorn [AM] shows that the Traveling salesman tour of reasonable sampled points from a curve can be computed efficiently.

This thesis is organized as follows. The main parts are contained in chapters one and two. The first chapter deals with the geometry of curves and the second chapter is about reconstruction schemes. In a third chapter some supplementary results are presented.

In the first chapter we give a definition of curves used throughout this thesis. We give a classification of Jordan curves in types I to III. Curves of type I and II turn out to be essentially one-manifolds. We only deal with these types of curves because there is no parameterization independent geometry for curves of type III. Furthermore we give a new definition of regularity for curves. This definition has a simple geometric interpretation, which also makes sense in metric spaces with no linear structure. This is interesting, because the usual definition of regularity relies on the linear structure. Finally we summarize some facts on length and curvature. An interesting theorem on length was given by Menger [M1]. Here we present a new proof of this theorem which is more general. We also derive some new corollaries from Menger's theorem.

In the second chapter we discuss several reconstruction schemes. We divide these schemes into two categories. Schemes of the first category are filter based. That is, they filter out edges from the Delaunay graph for the reconstruction. We present a new filter based reconstruction scheme for smooth curves, that we call λ-filter. This scheme can be generalized to surfaces. Schemes of the second category use optimization techniques for the reconstruction. In the section about filter techniques we classify the curves for which the correct reconstruction can be found in the Delaunay graph and we show that almost all known filter based reconstruction schemes cannot exhaust this class. In the section about optimization techniques we give a new proof that the minimum spanning tree is a reconstruction scheme for smooth curves. This proof is more general than the previously known proofs. Finally we show that the Traveling salesman tour is a reconstruction scheme for the class of curves for which the correct reconstruction is a subgraph of the Delaunay graph.

In the third chapter we present two new algorithms that provably can reconstruct smooth curves. Furthermore we discuss what lessons we learned from our work for the surface reconstruction problem. Finally we motivate a conjecture about the curvature of curves and the Traveling salesman tour.

Curves and their Geometry

1.1 What is a Curve?

Curves are fundamental objects in geometry. They find widespread use in almost all areas of mathematics, but it took a long time to formalize the intuitive notion of a curve. Hausdorff [H] remarked in 1914:

> ... the sets traditionally called curves are so heterogeneous, that they do not fall under any reasonable collective concept.

In the beginning of the last century there was no formal definition of a curve. The intuition was that a curve is a set that can be mapped bijectively to an interval. It was a surprise in set theory when Cantor in 1878 constructed a bijection between the unit interval and the the unit square. This showed that the unit interval and the unit square, which are quite different from a geometric viewpoint, have the same cardinality. Hence it is far to weak to demand that the map preserves the cardinality. At this time Jordan [J] realized that it is more important that the mapping preserves proximity. He defines curves as the image of a continuous mapping from an interval. About the same time Peano [P] constructed a continuous map from the unit interval to the unit square. That is, one can also map the unit interval to the unit square in such a way that proximity is preserved. The way out of this dilemma was to postulate both properties at the same time: A curve is the image of a bijective continuous function of some interval. Nowadays the elements in this class of sets are called Jordan curves. Possible types of Jordan curves are depicted in Figure 1.1.

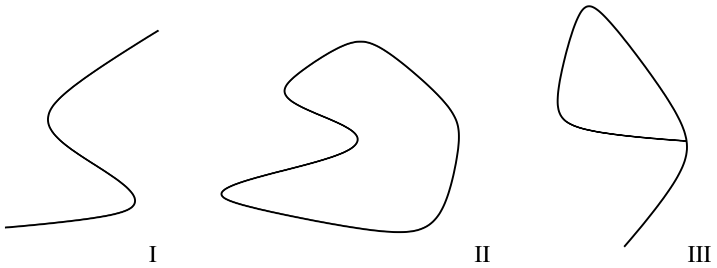

Figure 1.1: *Examples of Jordan curves of type* I*,* II *and* III

Curves of type I are images of arbitrary intervals. These curves are often called arcs. Curves of type II, called circles, must be images of semi-open intervals and curves of type III must be images of open or semi-open intervals.

A bijective continuous mapping from an interval to a curve is called a parameterization of the curve and a homeomorphism between intervals is called change of parameterization. Any reasonable geometric property of a curve should be independent of its parameterization.

Curves of type I, II and III can be characterized by the following properties: Let I be an interval and γ be a parameterization of a curve with preimage I.

1) The curve is of type I, if for any two convergent sequences (s_n) and (t_n) in closure(I), $\lim s_n \neq \lim t_n$ implies $\lim \gamma(s_n) \neq \lim \gamma(t_n)$.
2) The curve is of type II, if it is not of type I, but for any for any two monotonly convergent sequences (s_n) and (t_n) in closure(I), $\lim s_n \neq \lim t_n$ implies $\lim \gamma(s_n) \neq \lim \gamma(t_n)$.
3) The curve is of type III, if it is not of type I or II.

We will work only with curves of type I and II. These are one-manifolds. One-manifolds are completely classified:

Theorem 1.1 *Every one-manifold is homeomorphic either to* $(0, 1)$ *or to the unit circle* $\mathbb{S}^1 = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$*. Every one-manifold with boundary is homeomorphic to one of the intervals* $[0, 1]$, $(0, 1]$ *or* $[0, 1)$.

□

A proof of this theorem can be found in [FR]. Every one-manifold can be embedded in any Euclidean space $\mathbb{R}^d$ with $d > 1$. In this work we deal with such embeddings for compact one-manifolds. That is, we consider only curves in $\mathbb{R}^d$. The compact one-manifolds are $[0, 1]$ and $\mathbb{S}^1$. An important observation for this work is that an one-manifold embedding locally inherits an ordering from an interval by a parameterization.

Sometimes it is more appropriate to look at a curve as a map rather than a set. For example consider the two mappings

$$\gamma_1 : [0,1) \to \mathbb{C} \quad , \quad t \mapsto \exp(2\pi i t)$$
$$\gamma_2 : [0,1) \to \mathbb{C} \quad , \quad t \mapsto \exp(4\pi i t)$$

The image of both maps is the unit circle $\mathbb{S}^1$, but γ_2 runs through $\mathbb{S}^1$ once more than γ_2. There are situations where it is important to distinguish between γ_1 and γ_2, but in our work a curve is always a set and not a map.

1.2 Regularity

Arbitrary embeddings of one-manifolds can be very weird. One has to impose regularity conditions on the embedding to get curves that look tame. Here we introduce the notion of s(emi)-regularity. For s-regular curves left- and right tangents have to exist in every point of the curve. These tangents are independent of the actual parameterization of the curve. If nothing else is written, let γ always denote a Jordan curve of type I or II. We will refer to these curves simply as curves. Jordan curves of type II can be parameterized over $\mathbb{S}^1$. Hence there is also a homeomorphic parameterization for all Jordan curves of type II. Sometimes we use these homeomorphic parameterizations, sometimes we use a parameterization over an interval.

Definition 1.1 *Let I be an interval and*

$$T = \{(t_1, t_2) \, : \, t_1 < t_2, t_1, t_2 \in I\}$$

and

$$\tau : T \to \mathbb{S}^{d-1}, (t_1, t_2) \mapsto \frac{\gamma(t_2) - \gamma(t_1)}{|\gamma(t_2) - \gamma(t_1)|}.$$

The curve γ is called left regular at $\gamma(t_0)$ with left tangent $\mathrm{l}(\gamma(t_0))$ if for every sequence (ξ_n) in T which converges to (t_0, t_0) from the left in $\mathrm{closure}(T)$ the sequence $\tau(\xi_n)$ converges to $\mathrm{l}(\gamma(t_0))$. Right regularity and right tangents $\mathrm{r}(\gamma(t_0))$ are defined analogously.

A curve γ is called s-regular if it is left and right regular in all points $\gamma(t)$ with $t \in I$.

A point of a curve is called regular if in this point left- and right tangent coincide.

Left regularity in a point has a simple geometric interpretation:

Lemma 1.1 *Let γ be left regular in $p \in \gamma$. Let $(p_n), (q_n)$ and (r_n) be sequences of points from γ, that converge to p from the left, such that $p_n < q_n < r_n$ for all $n \in \mathbb{N}$ in an order locally around p along γ. Let α_n be the angle at q_n of the triangle with vertices p_n, q_n and r_n. Then the sequence of angles (α_n) converges to π.*

PROOF. If we make use of a homeomorphic parameterization of γ the sequences $(\gamma^{-1}(p_n)), (\gamma^{-1}(q_n))$ and $(\gamma^{-1}(r_n))$ converge to $\gamma^{-1}(p)$ from the left. Thus by our definition of regularity asymptotically the three secants

$$\mathrm{conv}\{p_n, q_n\}, \mathrm{conv}\{q_n, r_n\} \text{ and } \mathrm{conv}\{p_n, r_n\}$$

have to point in the direction of the left tangent $\mathrm{l}(p)$. Here a secant $\mathrm{conv}\{q, r\}$ is oriented from q to r. Hence

$$\lim_{n \to \infty} \alpha_n = \pi.$$

□

For right regular and regular points analogous results hold.

Semi regular curves can have an infinite number of irregular points. For example (due to Althaus and Mehlhorn [AM]) the convex hull of the points $(\cos(\pi(1 - 1/n)), \sin(\pi(1 - 1/n)))$, $n \geq 2$ has an infinite number of irregular points. But for every angle α with $0 < \alpha \leq \pi$ s-regular curves can have only a finite number of irregular points where left- and right tangent make an angle larger than α with each other.

1.3 Geometric Properties

We are interested in global geometric properties of curves and their reconstruction from finite samples. Most prominent among these properties are length and curvature. Both properties can be defined via inscribed polygons through finite samples of the curve.

Definition 1.2 *A sample S of γ is a finite sequence $S = \{p^1, \dots, p^n\}$ of points where $p^i \in \gamma$. The sample points p^i are ordered according to the order of the $\gamma^{-1}(p^i) \in I$. To every sample S its density is defined as the inverse of the following number*

$$\varepsilon(S) = \sup_{x \in \gamma} \min\{|p^i - x| \,:\, i = 0, \dots n\}.$$

Our definition of the sample density is uniform. That is, it is global and does not adopt to the local behavior of the sample. Now we can give the definition of an inscribed polygon.

Definition 1.3 *Let $S = \{p^1, \dots, p^n\}$ be a sample of the curve γ. The set*

$$P(S) = \bigcup_{i=1}^{n} \operatorname{conv}\{p^i, p^{i+1}\}$$

is called inscribed polygon of γ to S. Here p^{n+1} equals p^n if γ is of type I*, otherwise p^{n+1} equals p^1.*

In the following we sometimes make use of the convention $p^{|S|+1} = p^{|S|}$ if γ is of type I and $p^{|S|+1} = p^1$ if γ is of type II.

It is interesting to study how the geometric properties behave if a sequence of curves converges to a limit curve. To do so we have to introduce a notion of convergence for curves. The following definition is due to Fréchet [Fr].

Definition 1.4 *A sequence of curves (γ_n) in $\mathbb{R}^d$ converges to a curve γ in $\mathbb{R}^d$ iff there are parameterizations γ_n on the same unit interval I (open, closed, semi-open) and a parameterization γ on I such that*

$$\lim_{n \to \infty} \sup\{|\gamma(t) - \gamma_n(t)| \,:\, t \in I\} = 0.$$

A weaker form of convergence was defined by Hausdorff:

Definition 1.5 *A sequence of curves (γ_n) in $\mathbb{R}^d$ converges to a curve γ in $\mathbb{R}^d$ iff for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$:*

$$\max\left\{\sup_{p \in \gamma_n} \inf_{q \in \gamma} |p - q|\,, \sup_{p \in \gamma} \inf_{q \in \gamma_n} |p - q|\right\} < \varepsilon$$

Convergence in the sense of Fréchet implies convergence in the sense of Hausdorff.

1.3.1 Length

The most fundamental geometric property of a curve besides to dimension is the length. A very intuitive definition of length is given by the following one.

Definition 1.6 *The (Jordan) length $L(\gamma)$ of a curve γ is the following number*

$$\sup\{L(P(S))\ :\ S \textit{ is sample of } \gamma\},$$

where

$$L(P(S)) = \sum_{i=1}^{|S|} |p^{i+1} - p^i|.$$

This definition is obviously invariant under change of parameterization. The following Lemma, for a proof see [M2], gives an equivalent definition of length.

Lemma 1.2 *Let (S_n) be a sequence of samples of γ with $\lim_{n\to\infty} \varepsilon(S_n) = 0$. Then*

$$L(\gamma) = \lim_{n\to\infty} L(P(S_n)).$$

□

Almost all geometric properties are lower semi-continuous. The next Lemma, for a proof see [A], says that this is also true for the length.

Theorem 1.2 *Let (γ_n) be a sequence of curves which converges in the sense of Fréchet to a curve γ. Then*

$$L(\gamma) \leq \liminf L(\gamma_n).$$

□

Given the intuitive definition of the length of a polygon one can get another equivalent definition of curve length.

Lemma 1.3 *Every curve γ satisfies*

$$L(\gamma) = \inf\{\liminf L(P_n)\ :\ (P_n) \textit{ sequence of polygons that converges to } \gamma\}$$

□

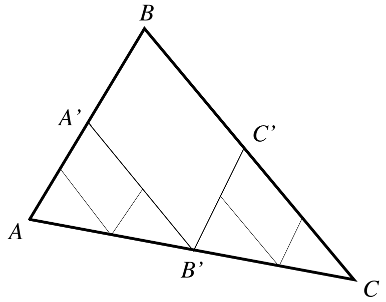

Figure 1.2: *The paradox of Lebesgue*

Lower semi-continuity is the source of some paradoxes. Best known is a paradox discovered by Lebesgue and friends when they were in high school. They found a 'proof' that the length of one side of a triangle in the plane always equals the sum of the lengths of the two other sides. Here is their 'proof': Take any triangle in the plane. Let A, B and C be its vertices.

Let A' be the midpoint of the side AB, B' be the midpoint of the side AC and C' be the midpoint of the side BC. To show that the length of AC equals the length of AB plus the length of BC replace the line ABC by the line $AA'B'C'C$. A simple calculation shows that the length of ABC and the length of $AA'B'C'C$ are equal. Now recursively do the same for the triangles $AA'B'$ and $B'C'C$. This construction gives a sequence of lines that converges to the line AC. Each of these lines has the same length as the line ABC. That concludes the 'proof'.

A modern definition of the length of a curve is via the one-dimensional Hausdorff measure $\mathfrak{H}^1$. The s-dimensional Hausdorff measure is defined as follows.

Definition 1.7 *Let M be a separable metric space, $0 \leq s < \infty$ and $A \subseteq M$. The quantity*

$$\mathfrak{H}^s(A) = \lim_{\delta \downarrow 0} \mathfrak{H}^s_\delta(A),$$

where

$$\mathfrak{H}^s_\delta(A) = \inf\left\{ \sum_i \mathrm{diam}(E_i)^s \ : \ A \subset \bigcup_i E_i,\ E_i \subseteq M,\ \mathrm{diam}(E_i) \leq \delta \right\}.$$

is called s-dimensional Hausdorff measure of A.

The following theorem, see [F] for a proof, shows that the definition of curve length via the one-dimensional Hausdorff measure is equivalent to the previous ones.

Theorem 1.3 *For every curve γ it holds $\mathfrak{H}^1(\gamma) = L(\gamma)$.* □

An analogue of Theorem 1.2 is also true for the one-dimensional Hausdorff measure, see [F] for a proof.

Theorem 1.4 *Let (γ_n) be a sequence of curves convergent in the Hausdorff sense to a compact set γ. Then*

$$\mathfrak{H}^1(\gamma) \leq \liminf \mathfrak{H}^1(\gamma_n).$$

□

An important class of curves are curves of finite length. Aleksandrov [A] shows that this class contains all regular curves.

Theorem 1.5 *Every s-regular curve has finite length.* □

The inclusion stated in this theorem is strict. Here is an example of an arc of finite length that is not s-regular.

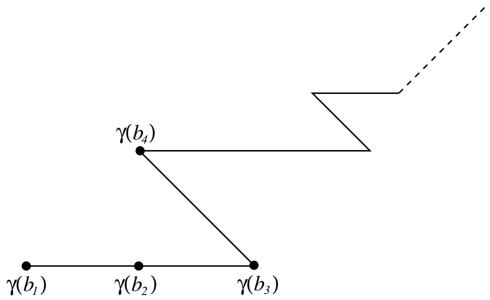

Figure 1.3: *An non s-regular arc of finite length*

Let γ be the arc given by the parameterization $\gamma : [0,1] \to \mathbb{R}^2$, where

$\gamma \mid_{[b_j, b_{j+1}]}: t \mapsto (b_{j+1} - t)/(b_{j+1} - b_j) \cdot \gamma(b_j) + (t - b_j)/(b_{j+1} - b_j) \cdot \gamma(b_{j+1})$

and $\gamma(1) = (1,2)$. For $a_i = 1 - 2^{-i},\ i \geq 0$, i.e. $[0,1] = \bigcup_{i=0}^{\infty} [a_i, a_{i+1}] \cup \{1\}$, and

$$\begin{array}{rclrcl} b_{4i} & = & a_i, & \gamma(b_{4i}) & = & (2a_i, a_i), \\ b_{4i+1} & = & a_i + 2^{-i-3}, & \gamma(b_{4i+1}) & = & (a_i + a_{i+1}, a_i), \\ b_{4i+2} & = & a_i + 2^{-i-2}, & \gamma(b_{4i+2}) & = & (2a_{i+1}, a_i) \text{ and} \\ b_{4i+3} & = & a_i + 3 \cdot 2^{-i-3}, & \gamma(b_{4i+3}) & = & (a_i + a_{i+1}, a_{i+1}). \end{array}$$

The length of γ is finite, but in the point $(1,2) \in \gamma$ no left tangent exists.

1.3.2 A Theorem of Menger

In 1930 Karl Menger [M1] proposed a new definition of arc length:

> Let the length of an arc be defined as the least upper bound of the set of all numbers that could be obtained by taking each finite set of points of the curve and determining the length of the shortest polygonal graph joining all the points.
> .....
> We call this the messenger problem (because in practice the problem has to be solved by every postman, and also by many travelers): finding the shortest path joining all of a finite set of points, whose pairwise distances are known.

This statement is one of the first references to the famous Traveling salesman problem.

In [M2] Menger proves the equivalence of his definition and the common one, theorem of Menger. His elementary proof is only valid for curves of type I, but these curves need not be embedded in some Euclidean space. Here we give a new proof that does only hold for curves embedded in some Euclidean space, but on the other hand also holds for curves of type II. For the proof we need the following lemma.

Lemma 1.4 *Let γ be a curve and let (S_n) be a sequence of samples of γ with $\lim_{n\to\infty} \varepsilon(S_n) = 0$ and $\limsup L(TSP(S_n)) < \infty$. Let*

$$\gamma_n : [0, L(TSP(S_n))] \to \mathbb{R}^d$$

be the parameterization by length of $TSP(S_n)$. Then the sequence (γ_n) has a Fréchet convergent subsequence.

PROOF. By turning to an appropriate subsequence we can assume that

$$L(TSP(S_n)) < 2 \limsup L(TSP(S_n)) =: c < \infty$$

for all $n \in \mathbb{N}$.

To prove the lemma it is sufficient to check that the premises of the theorem of Ascoli are fulfilled.

1) The sequence of reduced parameterizations

$$\gamma_n' : [0,1] \to \mathbb{R}^d,\ t \mapsto \gamma_n(tL(TSP(S_n)))$$

is equi-continuous, because

$$\begin{aligned} |\gamma_n'(t') - \gamma_n'(t)| &\leq (t'-t)L(\gamma_n') \\ &= (t'-t)L(TSP(S_n)) \\ &\leq c(t'-t) \end{aligned}$$

for all $0 \leq t < t' \leq 1$ and all $n \in \mathbb{N}$.

2) We have $\sup_{n\in\mathbb{N}} \sup\{|\gamma_n'(t)| : t \in [0,1]\} < \infty$, because γ is compact.

Now the theorem of Ascoli states that the sequence (γ_n') has a convergent subsequence (γ_m') which converges to a continuous function

$$\gamma' : [0,1] \to \mathbb{R}^d.$$

□

Note that the condition $\limsup L(TSP(S_n)) < \infty$ is always fulfilled if γ has finite length. Now we are prepared to prove Menger's theorem.

Theorem 1.6 *Every curve γ satisfies*

$$L(\gamma) = \sup\{L(TSP(S)) \ : \ S \ \textit{sample of}\ \gamma\},$$

where $TSP(S)$ denotes a shortest path through the sample points S if γ is of type I and a shortest tour otherwise.

PROOF. Assume the contrary. That is, we can assume that

$$\sup\{L(TSP(S)) \ : \ S \ \text{sample of}\ \gamma\} < L(\gamma),$$

because $TSP(S) \leq P(S) \leq L(\gamma)$ for every sample S of γ. Take any sequence (S_n) of samples from γ with $\lim_{n\to\infty} \varepsilon(S_n) = 0$. By our assumption we have

$$\limsup L(TSP(S_n)) < L(\gamma).$$

Let $\gamma_n : [0,1] \to \mathbb{R}^d$ be the reduced parameterization of $TSP(S_n)$. From Lemma 1.4 we know that there exists a Fréchet convergent subsequence (γ_m) of (γ_n). Let the continuous function $\gamma' : [0,1] \to \mathbb{R}^d$ be the limit of (γ_m). We have for the convergent subsequence

$$\limsup L(TSP(S_m)) \leq \limsup L(TSP(S_n)) < L(\gamma).$$

Next we show that $\gamma \subset \gamma'([0,1])$: Take $p \in \gamma$. Since $\lim_{n\to\infty} \varepsilon(S_n) = 0$, one finds a sequence (p_n) with $p_n \in S_n$ which converges to p. Of course the subsequence (p_m) with $p_m \in \gamma_m$ also converges to p. For all $k \in \mathbb{N}$ there exists $m(k) \in \mathbb{N}$ and $p'_{m(k)} \in \gamma'([0,1])$ such that

$$|p'_{m(k)} - p_{m(k)}| < \frac{1}{2k} \text{ and } |p_{m(k)} - p| < \frac{1}{2k}.$$

Hence by the triangle inequality we have $|p'_{m(k)} - p| < 1/k$. From the compactness of $\gamma'([0,1])$ we get $p \in \gamma'([0,1])$.

Both γ and $\gamma'([0,1])$ are as compact sets $\mathfrak{H}^1$ measurable. Using the isotony of $\mathfrak{H}^1$, Theorems 1.3 and 1.4 and our assumption we find:

$$\begin{aligned} L(\gamma) &= \mathfrak{H}^1(\gamma) \\ &\leq \mathfrak{H}^1(\gamma'([0,1])) \\ &\leq \liminf \mathfrak{H}^1(TSP(S_m)) \\ &= \liminf L(TSP(S_m)) \\ &\leq \limsup L(TSP(S_m)) \\ &< L(\gamma) \end{aligned}$$

That is a contradiction. □

The following corollary is an analogue of Lemma 1.2. It states that the length of the Traveling salesman path (tour) through a sequence of sample points converges to the length of the curve when the density of the sample goes to infinity.

Corollary 1.7 *Let γ be a curve and (S_n) be a sequence of samples of γ, with $\lim_{n\to\infty}\varepsilon(S_n) = 0$. Then*

$$L(\gamma) = \lim_{n\to\infty} L(TSP(S_n)),$$

where $TSP(S_n)$ denotes a shortest path through the sample points S_n if γ is of type I and a shortest tour otherwise.

PROOF. Assume $L(\gamma) < \infty$. For a given $\eta > 0$ consider three sets

1) From Menger's Theorem one knows that there exists a sample

$$S = \{q^1, \ldots, q^{|S|}\} \text{ with } L(TSP(S)) > L(\gamma) - \frac{\eta}{2}$$

2) Let $S' = \{p^1, \ldots, p^{|S'|}\} \in \{S_n \ : \ n \in \mathbb{N}\}$ be such that $\varepsilon(S') < \frac{\eta}{4|S|}$.

3) Let $S'' = \{r^1, \ldots, r^{|S|}\} \subset S'$ be a multi-set with $|r^i - q^i| \leq \varepsilon(S')$ for all $i = 1, \ldots, |S|$.

In the following we make use the conventions for $|S| + 1$ and $|S'| + 1$ introduced above. Let π be a permutation of $\{1, \ldots, |S|\}$ such that

$$\sum_{i=1}^{|S|} |q^{\pi(i+1)} - q^{\pi(i)}| = L(TSP(S)),$$

π' be a permutation of $\{1, \ldots, |S'|\}$ such that

$$\sum_{i=1}^{|S'|} |p^{\pi'(i+1)} - p^{\pi'(i)}| = L(TSP(S'))$$

and π'' be a permutation of $\{1, \ldots, |S|\}$ such that

$$\sum_{i=1}^{|S|} |r^{\pi''(i+1)} - r^{\pi''(i)}| \text{ is minimal.}$$

Then it follows that

1)
$$\sum_{i=1}^{|S|} |r^{\pi''(i+1)} - r^{\pi''(i)}| \leq \sum_{i=1}^{|S'|} |p^{\pi'(i+1)} - p^{\pi'(i)}|$$

by construction.

2)
$$\sum_{i=1}^{|S|} |q^{\pi(i+1)} - q^{\pi(i)}| \leq \sum_{i=1}^{|S|} |q^{\pi''(i+1)} - q^{\pi''(i)}|$$

by the definition of π.

3) From $|r^{\pi''(j)} - q^{\pi''(j)}| < \varepsilon(S')$ for all $j = 1, \ldots, |S|$ and the triangle inequality it follows:

$$\left| |r^{\pi''(i+1)} - r^{\pi''(i)}| - |q^{\pi''(i+1)} - q^{\pi''(i)}| \right| < 2\varepsilon(S')$$

Combining these inequalities leads to:

$$\begin{aligned}
L(TSP(S')) &= \sum_{i=1}^{|S'|} |p^{\pi'(i+1)} - p^{\pi'(i)}| \\
&\geq \sum_{i=1}^{|S|} |r^{\pi''(i+1)} - r^{\pi''(i)}| \\
&\geq \sum_{i=1}^{|S|} |q^{\pi''(i+1)} - q^{\pi''(i)}| - 2|S|\varepsilon(S') \\
&\geq \sum_{i=1}^{|S|} |q^{\pi(i+1)} - q^{\pi(i)}| - 2|S|\varepsilon(S') \\
&= L(TSP(S)) - 2|S|\varepsilon(S') \\
&> L(TSP(S)) - \frac{\eta}{2} \\
&> L(\gamma) - \eta
\end{aligned}$$

Since η can be arbitrary small, we are done. The proof in the case $L(\gamma) = \infty$ is quite similar. □

The next lemma states that the maximal length of a segment in a Traveling salesman path (tour) of a sequence of sample points from a curve of finite length tends to zero as the density of the sample goes to infinity. In the following we will make use of this lemma several times.

Lemma 1.5 *Assume that the curve γ has finite length and let (S_n) be a sequence of samples of γ with $\lim_{n\to\infty} \varepsilon(S_n) = 0$. For all $n \in \mathbb{N}$ let π_n be the* *permutation of* $\{1, \ldots, |S_n|\}$ *induced by* $TSP(S_n)$. *Then*

$$\lim_{n\to\infty} \max\{|p^{\pi_n(i+1)} - p^{\pi_n(i)}| \,:\, i = 1, \ldots, |S_n|\} = 0.$$

where $\pi_n(|S_n + 1|) = \pi_n(|S_n|)$ *if* γ *is of type* I *and* $\pi_n(|S_n + 1|) = \pi_n(1)$ *if* γ *is of type* II.

PROOF. Assume the contrary. That is, by choosing an appropriate subsequence one can assume that there exists $c > 0$ such that for every $n \in \mathbb{N}$ it holds

$$\max\{|p^{\pi_n(i+1)} - p^{\pi_n(i)}| \,:\, i = 1, \ldots, |S_n|\} > c.$$

Let $p_n = p^{\pi_n(i)} \in S_n$ and $q_n = p^{\pi_n(i+1)} \in S_n$ be such that $|p_n - q_n| > c$. By the compactness of γ we can turn to an appropriate subsequence such that (p_n) converges to $p \in \gamma$ and (q_n) converges to $q \in \gamma$. Of course it holds $|p - q| \geq c$. Remove the interior of $\operatorname{conv}\{p_n, q_n\}$ from $TSP(S_n)$. We consider two cases:

First, if γ is of type I then $TSP(S_n)$ decomposes into two paths P_n^1 which connects $p^{\pi_n(1)}$ with $p^{\pi_n(i)}$ and P_n^2 which connects $p^{\pi_n(i+1)}$ with $p^{\pi_n(|S_n|)}$. Let P_n denote $P_n^1 \cup P_n^2$ and let $\gamma_n^j \,:\, [0,1] \to \mathbb{R}^d$, $j = 1, 2$ be the reduced parameterization of P_n^j. As in Lemma 1.4 one can use the theorem of Ascoli to show that (γ_n^j) has a Fréchet convergent subsequence. That is, we can assume by considering a common subsequence that (γ_n^1) and (γ_n^2) converge to continuous functions $\gamma^1 \,:\, [0,1] \to \mathbb{R}^d$ and $\gamma^2 \,:\, [0,1] \to \mathbb{R}^d$. Let γ' be $\gamma^1([0,1]) \cup \gamma^2([0,1])$.

Second, if γ is of type II then $TSP(S_n)$ becomes after the removal of the interior of $\operatorname{conv}\{p_n, q_n\}$ a path denoted by P_n which connects $p^{\pi_n(i)}$ with $p^{\pi_n(i+1)}$. Let γ_n be the reduced parameterization of P_n. As in Lemma 1.4 one can show that (γ_n) has a Fréchet convergent subsequence. That is, we can assume by taking an appropriate subsequence that (γ_n) converges to a continuous function $\gamma' \,:\, [0,1] \to \mathbb{R}^d$. We use the symbol γ' also to denote the set $\gamma'([0,1])$.

Next we show that $\gamma \subset \gamma'$: Take $r \in \gamma$. Since $\lim_{n\to\infty} \varepsilon(S_n) = 0$, one finds a sequence (r_n) with $r_n \in S_n$ which converges to r. For all $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ and $r'_{n(k)} \in \gamma'$ such that

$$|r'_{n(k)} - r_{n(k)}| < \frac{1}{2k} \text{ and } |r_{n(k)} - r| < \frac{1}{2k}.$$

Hence by the triangle inequality $|r'_{n(k)} - r| < 1/k$. From the compactness of γ' we get $r \in \gamma'$.

The sets γ, γ', P_n and $\mathrm{conv}\{p_n, q_n\}$ are all as compact sets $\mathfrak{H}^1$ measurable. Using the isotony and σ-additivity of $\mathfrak{H}^1$ and Theorems 1.3 and 1.4 we find:

$$\begin{aligned}
\mathfrak{H}^1(\gamma') &\geq \mathfrak{H}^1(\gamma \cup \mathrm{conv}\{p_n, q_n\}) \\
&= \mathfrak{H}^1(\gamma) + \mathfrak{H}^1(\mathrm{conv}\{p, q\}) \\
&> \mathfrak{H}^1(\gamma) \\
&= L(\gamma) \\
&\geq \sup L(P_n) \\
&= \sup \mathfrak{H}^1(P_n) \\
&\geq \mathfrak{H}^1(\gamma')
\end{aligned}$$

That is a contradiction. □

Given a sequence of samples from a curve with finite length. The following theorem says that not only the length of the Traveling salesman paths (tours) converges but also the sequence of paths (tours) itself provided the density of the samples in the sequence of samples goes to infinity. In the proof of this theorem we make use of the notations introduced in the proof of Lemma 1.4.

Theorem 1.8 *Let γ be a curve of finite length, let (S_n) be a sequence of samples of γ with $\lim_{n\to\infty} \varepsilon(S_n) = 0$ and let $\gamma_n : [0,1] \to \mathbb{R}^d$ be the reduced parameterization of $TSP(S_n)$. Then the sequence (γ_n) converges to γ in the sense of Fréchet.*

PROOF. The claim is that for γ' from Lemma 1.4 we have $\gamma = \gamma'([0,1])$. Let (γ_m) be a subsequence that converges to the continuous function γ'. We show both inclusions.

1) $\gamma \subset \gamma'([0,1])$: Take $p \in \gamma$. Since $\lim_{n\to\infty} \varepsilon(S_n) = 0$, one finds a sequence (p_n) with $p_n \in S_n$ which converges to p. The subsequence (p_m) with $p_m \in \gamma'_m$ of course also converges to p. For all $k \in \mathbb{N}$ there exists $m(k) \in \mathbb{N}$ and $p'_{m(k)} \in \gamma'([0,1])$ such that

$$|p'_{m(k)} - p_{m(k)}| < \frac{1}{2k} \text{ and } |p_{m(k)} - p| < \frac{1}{2k}.$$

Hence by the triangle inequality $|p'_{m(k)} - p| < 1/k$. From the compactness of $\gamma'([0,1])$ we get $p \in \gamma'([0,1])$.

2) $\gamma'([0,1]) \subset \gamma$: Take $p \in \gamma'([0,1])$. By the convergence of (γ_m) we have

$$B_{1/2k}(p) \cap \gamma'_{m(k)} \neq \emptyset$$

and from Lemma 1.5 it follows that for all $k \in \mathbb{N}$ there exists $m(k) \in \mathbb{N}$ such that

$$\max\{|p^{\pi_{m(k)}(i+1)} - p^{\pi_{m(k)}(i)}| \,:\, i = 1, \ldots, |S_{m(k)}| - 1\} < \frac{1}{2k},$$

where $\pi_{m(k)}$ is the permutation of $\{1, \ldots, |S_{m(k)}|\}$ which is induced by the ordering along $TSP(S_{m(k)})$. From the triangle inequality it follows that there exists $p_k \in S_{m(k)}$ with $|p - p_k| < \frac{1}{k}$. It is $p_k \in \gamma$, because $S_{m(k)} \subset \gamma$. The sequence (p_k) converges to p. Hence $p \in \gamma$, by the compactness of γ.

Since these considerations hold for every subsequence of (γ'_n). It follows

$$\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \gamma'_n = \gamma.$$

□

1.3.3 Curvature

Another important geometric property of curves is curvature. Here we deal with total absolute curvature, in the following curvature for short. The definition of total absolute curvature is similar to the definition of Jordan length.

Definition 1.8 *The curvature $C(\gamma)$ of a curve γ is defined as the number*

$$\sup\{C(P(S)) \,:\, S \text{ is sample of } \gamma\},$$

where

$$C(P(S)) = \sum_{i=1}^{|S|-2} |\angle(p^{i+1} - p^i, p^{i+2} - p^{i+1})|,$$

if γ is of type I*, and*

$$C(P(S)) = \sum_{i=1}^{|S|-1} |\angle(p^{i+1} - p^i, p^{i+2} - p^{i+1})|,$$

where $p^{|S|+1} = p^1$, if γ is of type II.

This definition is like the definition of length invariant under change of parameterization. Furthermore, the analog statement to Lemma 1.2 holds, for a proof see [A]. From this we get an equivalent definition of curvature.

Lemma 1.6 *Let (S_n) be a sequence of samples of γ with $\lim_{n\to\infty} \varepsilon(S_n) = 0$. Then*

$$C(\gamma) = \lim_{n\to\infty} C(P(S_n)).$$

□

The next Lemma shows that the curvature is also lower semi-continuous, for a proof see [A].

Theorem 1.9 *Let (γ_n) be a sequence of curves which converges to a curve γ. Then*

$$C(\gamma) \leq \liminf C(\gamma_n).$$

□

Given the intuitive notion of the curvature of a polygon we can get another equivalent definition of curvature from this theorem.

Lemma 1.7 *For every curve γ it holds*

$$C(\gamma) = \inf\{\liminf C(P_n) : (P_n) \text{ sequence of polygons that converges to } \gamma\}$$

□

The class of curves of curves of finite curvature is included in the class of s-regular curves. Aleksandrov [A] proves:

Theorem 1.10 *Every curve of finite curvature is s-regular.* □

Reconstruction

2.1 The Reconstruction Problem

Following Amenta et al. [ABE] the curve reconstruction problem can be stated as follows:

> Given a curve γ and a finite set of sample points $S \subset \gamma$. The polygonal reconstruction $P(S)$ of γ from S is a graph that connects all points adjacent along γ. Find the polygonal reconstruction of γ from S.

Of course there is no algorithm that can reconstruct any curve from any set of sample points. Conditions have to be imposed on the curve and on the sample. We study conditions which have to be imposed on the curve such that a certain method can reconstruct the curve from a sufficiently dense sample. Such conditions define a class of curves (e.g. polygons). A method that works for all curves of a given class and sufficiently dense samples is called a reconstruction scheme.

Basically we can distinguish two approaches to the curve reconstruction problem:

1) Filter: The reconstruction filters out edges from the complete straight line graph on the sample points.

2) Optimization: The reconstruction is some extremal straight line graph on the sample points.

Reconstruction schemes can be further distinguished. Some need uniform sampling to give the correct reconstruction, some do not. That is, for a reconstruction scheme that does not need uniform sampling the curve can be sampled sparser in 'uninteresting' regions.

2.2 Filter Techniques

All known filter based reconstruction schemes do not select edges from the complete straight line graph on the sample points but consider only edges of the Delaunay graph. The Delaunay graph is not a reconstruction scheme itself, because it has far too much edges. But the class of curves for which these filter based reconstruction schemes work cannot be larger than the class for which the correct polygonal reconstruction is a subgraph of the Delaunay graph for sufficiently dense samples. That is the reason why we start our discussion of filter techniques with the Delaunay graph.

2.2.1 Delaunay Graph

The Delaunay graph on a set sample points is the one-dimensional subcomplex of the Delaunay complex of these points. The fact that all known filter based reconstruction schemes select edges from the Delaunay graph motivates the question: For which curves and sufficiently dense samples is the polygonal reconstruction a subgraph of the Delaunay graph. We will prove the following theorem:

Theorem 2.1 *Let γ be an s-regular curve with*

$$\alpha = \sup\{\angle(\mathrm{l}(q), \mathrm{r}(q)) \,:\, q \in \gamma\} < \pi.$$

Then there exists an $\varepsilon > 0$ such that for all samples S with $\varepsilon(S) < \varepsilon$ the correct polygonal reconstruction $P(S)$ of γ is a subgraph of the Delaunay graph of S.

This theorem implies that for sufficiently dense samples of a curve of type II the Delaunay graph contains a Hamiltonian cycle. This is not true for all Delaunay graphs.

Let (S_n) be a sequence of samples with $\lim_{n\to\infty} \varepsilon(S_n) = 0$. Choose pairs $p_n, q_n \in S_n$ adjacent along γ. We have to show, that for large n the edges $\mathrm{conv}\{p_n, q_n\}$ are edges in the Delaunay graph. In order to show this we start with two simple observations:

Lemma 2.1 *Let (p_n) and (q_n) as above. Then*

1) $\lim\limits_{n\to\infty} |p_n - q_n| = 0.$

2) $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = p \in \gamma$ *for appropriate subsequences.*

PROOF. 1) Assume the contrary. Then we find a subsequence and a $c > 0$ such that $|p_n - q_n| > 2c$ for all $n \in \mathbb{N}$. By the compactness of γ we can assume that (p_n) converges to $p \in \gamma$ and (q_n) converges to $q \in \gamma$, otherwise we choose an appropriate subsequence. On γ we find a point r between p and q along γ with $|r-p| \geq c$ and $|r-q| \geq c$. For every $n \in \mathbb{N}$ we find $r_n \in S_n$ with $|r - r_n| < \varepsilon(S_n)$. Because of $\lim_{n\to\infty} \varepsilon(S_n)$ the sequence (r_n) converges to r, which is impossible for the curves that we consider.

2) Direct consequence from 1) and the compactness of γ. □

In order to prove Theorem 2.1 we need a simple lemma.

Lemma 2.2 *Let* $(p_n), (q_n)$ *and* p *be as above and let* (r_n) *be a sequence of points in* $S_n - \{p_n, q_n\}$ *with* $\lim_{n\to\infty} r_n = p$*. Let* d_n *be the diameter of the unique circle through the points* p_n, q_n, r_n*. If there are no subsequences of* $(p_n), (q_n), (r_n)$ *such that* p_n, q_n, r_n *are asymptotically collinear then*

$$\lim_{n\to\infty} d_n = 0.$$

PROOF. Consider the triangle with vertices p_n, q_n, r_n.

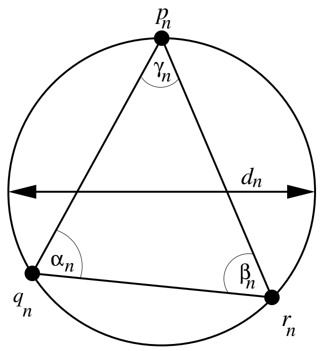

Figure 2.1: *The triangle with vertices* p_n, q_n, r_n *and its circumcircle*

Let $\delta_n = \max\{\alpha_n, \beta_n, \gamma_n\}$ and let e_n be the edge opposite to δ_n. The diameter d_n can be computed as follows:

$$d_n = \frac{\mathrm{length}(e_n)}{\sin(\delta_n)}$$

By our assumptions we find,

$$\lim_{n\to\infty} \operatorname{length}(e_n) \leq \lim_{n\to\infty} \max\{|p_n - r_n|, |p_n - q_n|, |q_n - r_n|\} = 0$$

and

$$\limsup \ \delta_n < \pi.$$

By construction we have $\delta_n \geq \frac{\pi}{3}$ for all $n \in \mathbb{N}$. This together implies that $\lim_{n\to\infty} d_n = 0$. □

PROOF. (of Theorem 2.1) Let $(S_n), (p_n)$ and (q_n) be as above. We assume that there are subsequences of $(S_n), (p_n)$ and (q_n) such that $\operatorname{conv}\{p_n, q_n\}$ is not an edge of the Delaunay graph for all $n \in \mathbb{N}$. Our goal is to show that this assumption leads to a contradiction.

Without loss of generality we assume in the following $p_n < q_n$ along γ. In the point p which is the limit point of the sequences (p_n) and (q_n) both left and right tangent exist.

First assume that in p left and right tangent coincide. That is, in p exists a unique tangent. Let B_n be the open ball with diameter $d_n = |p_n - q_n|$ and with $\{p_n, q_n\} \subseteq S_n \cap \partial B_n$. For large n one has $S_n \cap \operatorname{closure}(B_n) = \{p_n, q_n\}$. Otherwise assume that there exists $r_n \in \operatorname{closure}(B_n) \cap (S_n - \{p_n, q_n\})$ for all $n \in \mathbb{N}$. Without loss of generality assume that $q_n < r_n$ along γ. From $\lim_{n\to\infty} d_n = \lim_{n\to\infty} |p_n - q_n| = 0$ it follows $\lim_{n\to\infty} r_n = p$. Consider the triangle with vertices p_n, q_n and r_n. From the law of cosines together with $|p_n - r_n| \leq |p_n - q_n|$, one finds for the angle α_n at q_n

$$\cos(\alpha_n) = -\frac{|p_n - r_n|^2 - |p_n - q_n|^2 - |q_n - r_n|^2}{2|p_n - q_n||q_n - r_n|} \geq 0.$$

Thus α_n has to be smaller or equal than $\frac{\pi}{2}$, but that is a contradiction to Lemma 1.1. Hence it suffices to consider the case that in p left and right tangent do not coincide.

By turning to appropriate subsequences one can consider without loss of generality two cases:

1. $p_n < p < q_n$ along γ for all $n \in \mathbb{N}$.
2. $p_n, q_n \leq p$ or $p \leq p_n, q_n$ along γ for all $n \in \mathbb{N}$.

First case. Let B_n be the smallest open ball which contains the unique circle through the points p_n, q_n and p. Let d_n be the diameter of B_n. Since in p

left and right tangent do not coincide there are no subsequences of (p_n) and (q_n) such that p_n, q_n and p become collinear. Using Lemma 2.2 this implies $\lim_{n\to\infty} d_n = 0$. If B_n is asymptotically empty of sample points we are done by the empty ball criterion for Delaunay edges. So assume that there exists $r_n \in \mathrm{closure}(B_n) \cap (S_n - \{p_n, q_n\})$ for all $n \in \mathbb{N}$. Without loss of generality one can also assume that $q_n < r_n$ along γ and by moving r_n a little bit along γ that $r_n \in \partial B_n$. From $\lim_{n\to\infty} d_n = 0$ it follows $\lim_{n\to\infty} r_n = p$. Now consider the quadrilateral with vertices p_n, p, q_n and r_n.

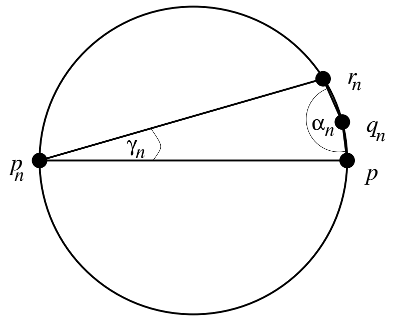

Figure 2.2: *The quadrilateral with vertices p_n, p, q_n, r_n which lie asymptotically on a circle*

Let α_n be the angle at q_n and γ_n be the angle at p_n. It is an old theorem in geometry that $\alpha_n + \gamma_n = \pi$ if p_n, p, q_n and r_n are collinear. From the right regularity of γ in p one has $\lim_{n\to\infty} \alpha_n = \pi$. Since p_n, p, q_n and r_n are asymptotically collinear we find

$$\lim_{n\to\infty} \gamma_n = 0 \text{ and } \lim_{n\to\infty} \angle(\mathrm{conv}\{p_n, q_n\}, \mathrm{l}(p)) = 0.$$

Now consider the following ball B'_n: It has p_n and q_n in its boundary and the tangent hyperplane at q_n is orthogonal to the line l in $\mathrm{span}\{\mathrm{l}(p), \mathrm{r}(p)\}$ that halves the angle $\angle(\mathrm{l}(p), \mathrm{r}(p))$. That is, for large n the situation is similar to the one depicted in Figure 2.3.

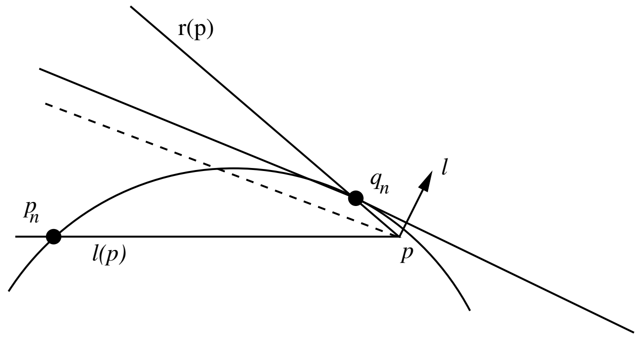

Figure 2.3: *Situation near the irregular point* p

For the diameters d_n' of B_n'

$$\lim_{n\to\infty} d_n' = 0$$

holds, because from the construction it holds

$$0 < \cos(\frac{\alpha}{2}) \leq \frac{|p_n - q_n|}{d_n'}$$

and $\lim_{n\to\infty} |p_n - q_n| = 0$. Recall that

$$\alpha = \sup\{\angle(\mathrm{l}(q), \mathrm{r}(q)) \ : \ q \in \gamma\} < \pi.$$

For large n we have $S_n \cap \mathrm{closure}(B_n') = \{p_n, q_n\}$. Otherwise, one can assume that there exist

$$s_n \in \mathrm{closure}(B_n') \cap (S_n - \{p_n, q_n\}) \text{ for all } n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} d_n' = 0$ one finds $\lim_{n\to\infty} s_n = p$. Consider the triangle with vertices p_n, q_n and s_n. Without loss of generality we can restrict ourselves to two cases: First, $q_n < s_n$ along γ for all $n \in \mathbb{N}$, and second, $s_n < p_n$ along γ for all $n \in \mathbb{N}$.

If $q_n < s_n$ then consider the triangle with vertices p, q_n and s_n. One finds for the angle α_n at q_n:

$$\lim_{n\to\infty} \alpha_n \leq \pi - \frac{\pi - \alpha}{2} = \frac{\pi + \alpha}{2} < \pi$$

By Lemma 1.1 that is a contradiction to the right regularity of γ in p.

If $s_n < p_n$ then consider the triangle with vertices s_n, p_n and p. One finds for the angle γ_n at p_n:

$$\lim_{n\to\infty} \gamma_n \le \pi - \frac{\pi - \alpha}{2} = \frac{\pi + \alpha}{2} < \pi$$

By Lemma 1.1 that is a contradiction to the left regularity of γ in p.

Hence $S_n \cap \mathrm{closure}(B'_n) = \{p_n, q_n\}$ for large n and $\mathrm{conv}\{p_n, q_n\}$ is an edge of the Delaunay graph for large n.

Second case. Assume $p_n, q_n \le p$. The proof for $p_n, q_n \ge p$ follows the same lines. Let l be the line in $\mathrm{span}\{\mathrm{r}(p), \mathrm{l}(p)\}$ that halves the angle $\angle(\mathrm{l}(p), \mathrm{r}(p))$. Let h be the hyperplane orthogonal to l that passes through p and let B_n be the open ball with $\{p_n, q_n\} \subset \partial B_n$ and with diameter $d_n = |p_n - q_n|$. There are three possibilities:

a) $\partial B_n \cap h = \emptyset$.

b) $\partial B_n \cap h$ is a single point.

c) $\partial B_n \cap h$ is a $(d-1)$-dimensional ball B'_n.

In case c) replace B_n with another ball. By a continuity argument there exists a point $r_n \in B'_n$ such that in this point h is tangent to the smallest ball which contains the unique circle through the points p_n, q_n and r_n. Replace B_n by this ball. Since p_n, q_n and r_n cannot have subsequences that become collinear by Lemma 2.2 it still holds that $\lim_{n\to\infty} d_n = 0$. The situation is similar to the one depicted in Figure 2.4

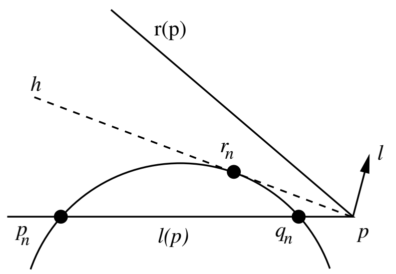

Figure 2.4: *Situation near the irregular point p*

For large n, $S_n \cap \mathrm{closure}(B_n) = \{p_n, q_n\}$, because otherwise one can assume without loss of generality that there exists $s_n \in \mathrm{closure}(B_n) \cap (S_n - \{p_n, q_n\})$ for all $n \in \mathbb{N}$. From $\lim_{n\to\infty} d_n = 0$ one has $\lim_{n\to\infty} s_n = p$. Without loss of generality we can restrict ourselves to three cases:
First, $p \le s_n$ along γ for all $n \in \mathbb{N}$. Second, $q_n < s_n < p$ along γ for all $n \in \mathbb{N}$, and third, $s_n < p_n$ along γ for all $n \in \mathbb{N}$.

If $p \le s_n$ then

$$\lim_{n\to\infty} \angle(\mathrm{conv}\{p, s_n\}, \mathrm{l}(p)) \ge \frac{\pi - \alpha}{2}$$

in contradiction to the right regularity of γ in p.

If $q_n < s_n < p$ then consider the triangle with vertices p_n, q_n and s_n. One has for the angle α_n at q_n:

$$\lim_{n\to\infty} \alpha_n \le \pi - \frac{\pi - \alpha}{2} = \frac{\pi + \alpha}{2} < \pi$$

By Lemma 1.1 that is a contradiction to the left regularity of γ in p.

If $s_n < p_n$ then consider the triangle with vertices s_n, p_n and q_n. One has for the angle γ_n at p_n:

$$\lim_{n\to\infty} \gamma_n \le \pi - \frac{\pi - \alpha}{2} = \frac{\pi + \alpha}{2} < \pi$$

By Lemma 1.1 that is a contradiction to the left regularity of γ in p.

Hence $S_n \cap \mathrm{closure}(B'_n) = \{p_n, q_n\}$ for large n and $\mathrm{conv}\{p_n, q_n\}$ is an edge of the Delaunay graph for large n. $\square$

The following example shows that the conditions in Theorem 2.1 are necessary. Let γ be the arc which consists of the unit interval on the x-axis and the graph of $y = x^2$ on this interval. That is,

$$\gamma : [0,1] \to \mathbb{R}^2, t \mapsto \begin{cases} (1-2t, 0) & : \ t \le \frac{1}{2} \\ (2t-1, (2t-1)^2) & : \ t > \frac{1}{2} \end{cases}$$

In every point of γ left and right tangents exist, but

$$\alpha = \sup\{\angle(\mathrm{l}(q), \mathrm{r}(q)) \ : \ q \in \gamma\} = \pi,$$

because in the point $(0,0)$ the left and right tangent make an angle of π with each other. For large n the samples

$$S_n = \{p_n^1, p_n^2, p_n^3, p_n^4\} \cup \bigcup_{i=2}^{n} \left\{ \left(\frac{i}{n}, 0\right), \left(\frac{i}{n}, \frac{i^2}{n^2}\right) \right\}$$

with

$$p_n^1 = \left(\frac{1}{n}, 0\right) \ , \ p_n^2 = \left(\frac{1}{n^3}, \frac{1}{n^6}\right) \ , \ p_n^3 = \left(\frac{2}{n^3}, \frac{4}{n^6}\right) \ , \ p_n^4 = \left(\frac{1}{n^2}, \frac{1}{n^4}\right)$$

become arbitrary dense in γ. Let R_n be the radius of the unique circle through the points p_n^1, p_n^2 and p_n^3. One can calculate that:

$$\lim_{n\to\infty} R_n = \infty.$$

That is, if one extends γ to the half space $\{(x, y) \ : \ y < 0\}$ and takes sample points in this half space one finds for large n by the open ball criterion for Delaunay edges that the edge $\operatorname{conv}\{p_n^1, p_n^2\}$ cannot be a Delaunay edge. Hence the conditions in Theorem 2.1 are sufficient and necessary.

2.2.2 The Crust

The crust is a subgraph of the Delaunay graph on point sets in the plane.

Definition 2.1 *Given a set S of points in the plane and let S' be the union of S with the vertices of its Voronoi diagram. An edge in the Delaunay triangulation of S' belongs to the crust of S if both of its endpoints belong to S.*

Amenta et. al. [ABE] prove that the crust is a reconstruction scheme for smooth curves of type II. Furthermore they show that uniform sampling is not needed for the correct reconstruction.

The following example shows that the crust is not a reconstruction scheme for polygons.

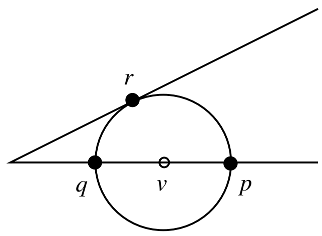

Figure 2.5: *The crust is not a reconstruction scheme for polygons*

In the situation pictured in Figure 2.5 the edge $\mathrm{conv}\{p, q\}$ cannot belong to crust, because every ball which has p and q in its boundary has to contain the Voronoi vertex v defined by p, q and r. Because this situation can be scaled down to arbitrarily dense samples the crust cannot be a reconstruction scheme for polygons.

2.2.3 β-Skeleton

In their paper on the crust [ABE] Amenta et. al. also show that the β-skeleton for a certain value of β is a reconstruction scheme for smooth curves of type II. Again uniform sampling is not needed for the correct reconstruction. Here we just review the definition of β-skeletons.

Definition 2.2 *Let $\beta \geq 1$ be a constant. Let p and q be a pair of points in the plane at distance $|p - q|$. The edge $\mathrm{conv}\{p, q\}$ is in the β-skeleton if the union of the two balls of diameter $\beta|p - q|$ tangent to p and q is empty.*

The following example shows that the β-skeleton is not a reconstruction scheme for polygons.

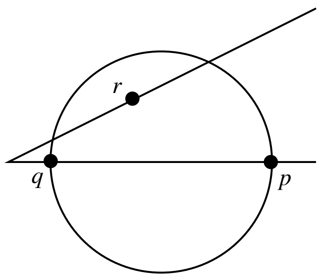

Figure 2.6: *The β-skeleton is not a reconstruction scheme for polygons*

The circle drawn in Figure 2.6 corresponds to $\beta = 1$, in this case the two balls coincide. This shows that $\mathrm{conv}\{p, q\}$ cannot be an edge of the β-skeleton for all $\beta \geq 1$, because the forbidden region of an edge grows with growing β. Again this situation can be scaled down to arbitrarily dense samples.

2.2.4 Gabriel Graph

The Gabriel graph is the β-skeleton with $\beta = 1$. Dey et. al. [DMR] consider a filter algorithm, called conservative crust, which filters edges from the Gabriel graph. The purpose of this algorithm is to detect from a sample if a curve is closed or not. Dey et. al. prove that the conservative crust is a reconstruction scheme for smooth curves. For the conservative crust uniform sampling is not needed to give the correct reconstruction. The counterexample for β-skeletons shows that the conservative crust neither can reconstruct polygons in general.

2.2.5 λ-Filter

The λ-Filter generalizes the ideas behind the Gabriel graph of a point set. It makes use of a more flexible definition of the forbidden region used in its definition.

Definition 2.3 *Given an edge* $\operatorname{conv}\{p, q\} \subset \mathbb{R}^d$ *and* $\lambda \in [0, 1]$. *A* λ*-ball* $B_\lambda(p, q)$ *of this edge is an open ball with diameter* $|p - q|/\lambda$ *and* $p, q \in \partial B_\lambda(p, q)$.

A Gabriel ball is a λ-ball with $\lambda = 1$.

Definition 2.4 *Let* S *be a point set in* $\mathbb{R}^d$. *For every two points* $p, q \in S$ *consider the following set:*

$$L(p, q) = \{\lambda \, : \, B_\lambda(p, q) \cap S = \emptyset\}$$

The λ*-interval assigned to the edge* $\operatorname{conv}\{p, q\}$ *is:*

$$[\inf L(p, q), \sup L(p, q)]$$

In the following assume that S is a finite subset of $\mathbb{R}^2$. The λ-interval of an edge $\operatorname{conv}\{p, q\}$ of the Delaunay graph of S can be easily computed:
The diameters of the Delaunay balls associated to $\operatorname{conv}\{p, q\}$ are

$$\frac{|p - q|}{\sin \alpha} \text{ and } \frac{|p - q|}{\sin \beta},$$

where α and β are as in Figure 2.7.

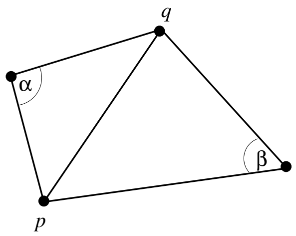

Figure 2.7: *The λ-interval of an edge in the Delaunay graph can be easily computed*

Therefrom we derive $\inf L(p,q) = \min\{\sin\alpha, \sin\beta\}$ and

$$\sup L(p,q) = \begin{cases} 1 & : \ \alpha \leq \frac{\pi}{2} \text{ and } \beta \leq \frac{\pi}{2} \\ \max\{\sin\alpha, \sin\beta\} & : \ \text{otherwise} \end{cases}$$

Next we want to prove some properties of λ-intervals in curve reconstruction.

Theorem 2.2 *Let γ be a smooth curve in $\mathbb{R}^2$ and (S_n) be a sequence of samples from γ with $\lim_{n\to\infty} \varepsilon(S_n) = 0$. For every two sequences (p_n), (q_n) with $p_n, q_n \in S_n$ and $\operatorname{conv}\{p_n, q_n\} \subset P(S_n)$ one finds:*

$$\lim_{n\to\infty} \sup L(p_n, q_n) = 1.$$

Proof. In [DMR] it is shown that for sufficiently dense samples the correct reconstruction of a smooth curve is a subgraph of the Gabriel graph. □

Theorem 2.3 *Let γ be a s-regular curve in $\mathbb{R}^2$ with*

$$\alpha = \sup\{\angle(\mathrm{l}(q), \mathrm{r}(q)) \ : \ q \in \gamma\} < \pi$$

and let (S_n) be a sequence of samples from γ with $\lim_{n\to\infty} \varepsilon(S_n) = 0$. For every two sequences (p_n), (q_n) with $p_n, q_n \in S_n$ and $\operatorname{conv}\{p_n, q_n\} \subset P(S_n)$ one finds:

$$\lim_{n\to\infty} \inf L(p_n, q_n) = 0.$$

Proof. Assume the contrary. That is, there exist two sequences (p_n), (q_n) with $p_n, q_n \in S_n$ and $\operatorname{conv}\{p_n, q_n\} \subset P(S_n)$ such that

$$\limsup \ \inf L(p_n, q_n) > 0.$$

By turning to an appropriate subsequence we can assume that

$$\lim_{n\to\infty} \inf L(p_n, q_n) = c > 0.$$

Observe that the edges $\mathrm{conv}\{p_n, q_n\}$ cannot be edges of the convex hull of S_n, because we have $\inf L(e) = 0$ for convex hull edges e. Furthermore we can assume, using the compactness of γ and $\lim_{n\to\infty} \varepsilon(S) = 0$, that

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = p \in \gamma.$$

From Theorem 2.1 we know that for sufficiently dense samples the correct reconstruction is a subgraph of the Delaunay graph of the sample points. Hence for large n the edge $\mathrm{conv}\{p_n, q_n\}$ is an edge of the Delaunay graph, which has two incident triangles with all vertices in S_n. Let r_n and s_n be the vertices opposite to the edge $\mathrm{conv}\{p_n, q_n\}$ in these triangles. From our assumptions we find for the sequence (α_n) of angles α_n opposite to the edges $\mathrm{conv}\{p_n, q_n\}$ in the triangles with vertices p_n, q_n and r_n:

$$\limsup\ \sin \alpha_n \geq c$$

Analogously we find for the sequence (β_n) of angles β_n opposite to the edges $\mathrm{conv}\{p_n, q_n\}$ in the triangles with vertices p_n, q_n and s_n:

$$\limsup\ \sin \beta_n \geq c$$

By turning to appropriate subsequences we can assume that

$$\lim_{n\to\infty} \sin \alpha_n \geq c \text{ and } \lim_{n\to\infty} \sin \beta_n \geq c \tag{2.1}$$

We find for the diameters $d(p_n, q_n, r_n)$ and $d(p_n, q_n, s_n)$ of the unique balls through the points p_n, q_n and r_n resp. p_n, q_n and s_n:

$$\lim_{n\to\infty} d(p_n, q_n, r_n) = \lim_{n\to\infty} \frac{|p_n - q_n|}{\sin \alpha_n} = 0$$

and

$$\lim_{n\to\infty} d(p_n, q_n, s_n) = \lim_{n\to\infty} \frac{|p_n - q_n|}{\sin \beta_n} = 0$$

That is,

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n = p.$$

Assume that in p left- and right tangent coincide. That is, in p exists a unique tangent. Without loss of generality we can assume that $p_n < q_n < r_n$ for all $n \in \mathbb{N}$ in the order along γ locally around p. Consider the triangle with vertices p_n, q_n and r_n. From the definition of regularity the sequence of angles (α_n) converges to zero. That is a contradiction to one of the inequalities (2.1). Hence it suffices to consider the case that in p left- and right tangent do not coincide.

By turning to appropriate subsequences one can consider without loss of generality two cases:

1. $p_n < p < q_n$ along γ for all $n \in \mathbb{N}$.
2. $p_n, q_n \leq p$ or $p \leq p_n, q_n$ along γ for all $n \in \mathbb{N}$.

First case. Let (H_n) be the sequence of hyperplanes through the points p_n and q_n. Since the sequences (r_n) and (s_n) both converge to p we find for sufficiently large $n \in \mathbb{N}$ that the points r_n and s_n lie both on the same side of H_n. But that is impossible in a Delaunay triangulation.

Second case. Without loss of generality we assume that $p \leq p_n, q_n$ along γ for all $n \in \mathbb{N}$.

Now assume that $p \leq r_n$ or $p \leq s_n$ along γ for arbitrary large n. From the existence of the right tangent in p we find that the sequence of angles (α_n) or the sequence (β_n) has a subsequence that has to converge to zero. That is a contradiction to inequalities (2.1).

We are left with the case that there exists $N \in \mathbb{N}$ such that $r_n, s_n < p$ for all $n \geq N$. Consider the sequence (H_n) of hyperplanes through the points p_n and q_n. This sequence converges to the hyperplane defined by the right tangent at p. Since $\angle(\mathrm{l}(p), \mathrm{r}(p)) < \pi$ and the sequences (r_n) and (s_n) both converge to p we find for large n that the points r_n and s_n have to lie both on the same side of the hyperplane H_n. That is impossible in a Delaunay triangulation. □

Theorems 2.2 and 2.3 together with the following theorem can be used to derive a reconstruction scheme for smooth curves in the plane.

Theorem 2.4 *Let γ be a smooth curve in $\mathbb{R}^2$ and (S_n) be a sequence of samples from γ with $\lim_{n\to\infty} \varepsilon(S_n) = 0$. For every two sequences (p_n), (q_n) with $p_n, q_n \in S_n$ and $\operatorname{conv}\{p_n, q_n\} \not\subset P(S_n)$ one finds:*

$$\limsup \sup L(p_n, q_n) < 1$$

or

$$\liminf \inf L(p_n, q_n) > 0.$$

PROOF. Assume the contrary. That is, without loss of generality we can assume that there exist two sequences (p_n) and (q_n) with

a) $\operatorname{conv}\{p_n, q_n\} \not\subset P(S_n)$ for all $n \in \mathbb{N}$.

b) $\lim_{n\to\infty} \sup L(p_n, q_n) = 1$.

c) $\lim_{n\to\infty} \inf L(p_n, q_n) = 0$.

d) $\lim_{n\to\infty} p_n = p \in \gamma$ and $\lim_{n\to\infty} q_n = q \in \gamma$.

We distinguish two cases: 1) $p = q$ and 2) $p \neq q$.

First case. Assume $p_n < q_n$ locally around p for all $n \in \mathbb{N}$ in the order along γ. By condition a) there has to exist a sequence (r_n) of sample points $r_n \in S_n$ with $p_n < r_n < q_n$ locally around p. Consider the triangles with vertices p_n, q_n and r_n. By the regularity of γ in p, see Lemma 1.1 the sequence of angles α_n at the vertices r_n has to converge to π. Hence

$$\lim_{n\to\infty} \sup L(p_n, q_n) = 0,$$

which contradicts condition b).

Second case. We show that in this case γ cannot be regular in the points p and q. Let H be the hyperplane defined by p and q. This hyperplane partitions $\mathbb{R}^2$ in two halfspaces. By condition c) the curve γ has to lie completely in one of these halfspaces. Let B be the open Ball with diameter $|p - q|$ which has p and q in its boundary. By condition b) B cannot contain any point of γ. Let (r_n) and (s_n) be sequences of sample points $r_n, s_n \in S_n$ with

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n = p$$

and $r_n < p < s_n$ locally around p in the order along γ. Consider the triangles with vertices r_n, p and s_n and the sequence (α_n) of angles at the vertex p. By construction we have

$$\limsup \alpha_n \leq \frac{\pi}{2},$$

which is a contradiction to Lemma 1.1, since γ is regular in p. An analogues reasoning shows that γ cannot be regular in q neither. □

2.2.6 Normalized Mesh

Attali [At] studies the normalized mesh for curve and surface reconstruction.

Definition 2.5 *The normalized mesh of γ associated to a finite sample $S \subset \gamma$ is the set of edges $\mathrm{conv}\{p, q\}$ with $p, q \in S$ for which there exists a point $m \in \gamma$ such that*

$$|p - m| = |q - m| = \mathrm{dist}(m, S).$$

The normalized mesh is a subgraph of the Delaunay graph, because for an edge $\mathrm{conv}\{p, q\}$ in the normalized mesh there exists a point $m \in \gamma$ such that the open ball with center m and radius $|p - m|$ is empty of sample points and has p and q in its boundary. That is, by the empty ball criterion the edge $\mathrm{conv}\{p, q\}$ is in the Delaunay graph. Attali proves that the normalized mesh is a reconstruction scheme for smooth curves. In her proof she needs uniform sampling.

The following example shows that the normalized mesh can in general not reconstruct polygons.

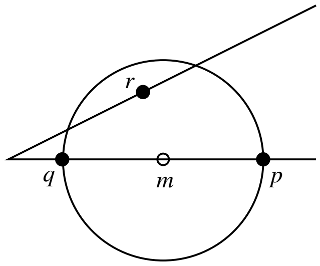

Figure 2.8: *The normalized mesh is not a reconstruction scheme for polygons*

In the situation depicted in Figure 2.8 the edge $\mathrm{conv}\{p, q\}$ is not an edge of the normalized mesh, because one has for the point $m \in \gamma - S$:

$$|p - m| \; = \; |q - m| \; > \; \mathrm{dist}(m, S) \; = \; |r - m|$$

Again this situation can be scaled down to arbitrarily dense samples.

2.2.7 α-Shape

Bernardini and Bajaj [BB] show that α-shapes are a reconstruction scheme for smooth curves. α-shapes were introduced by Edelsbrunner et al. [EKS] to describe the shape of point sets. For reconstruction with α-shapes uniform sampling is needed. This is a main disadvantage of α-shapes. But it is still possible that α-shapes are a reconstruction scheme for a larger class of curves than just smooth curves.

2.3 Optimization Techniques

The motivation to study optimization techniques in curve reconstruction originates from surface reconstruction. In contrast to the curve reconstruction problem one does not know what the correct surface reconstruction is, because one lacks of an ordering of the sample points. That is, for surfaces one cannot say whether the points are connected in the right order.

What can be demanded from a reconstruction scheme for surfaces is that it gives good approximations for geometric properties like area and curvatures. Most of these geometric properties are lower semi-continuous, which suggests some minimization criterion.

We remarked that the points on a curve always can be locally ordered. Now we want to introduce a global ordering.

Definition 2.6 *Let $S = \{p^1, \dots p^{|S|}\}$ be a sample of γ. Assume that γ is parameterized such that $\gamma(t^i) = p^i$. If γ is of type* I *one writes $i \lhd j$ if $t^i < t^j$. If γ is of type* II *and has finite length choose an orientation along γ and write $i \lhd j$ if $i \neq j$ and $L(\gamma(p^i : p^j)) \leq L(\gamma(p^j : p^i))$, where $\gamma(p^i : p^j) \subset \gamma$ is the arc connecting p^i and p^j in the orientation along γ. One writes $i \unlhd j$ if the possibility that $i = j$ is included.*

2.3.1 Minimum Spanning Tree

Here we give a proof that minimum spanning trees are a reconstruction scheme for regular curves of type I. Euclidean minimum spanning trees are always subgraphs of the Delaunay graph [E]. Hence one cannot expect to reconstruct a larger class of curves with a minimum spanning tree than the class characterized in Theorem 2.1.

Another proof of Theorem 2.5 is given by Figueiredo and Gomes [FG]. We were not aware of this proof when we started our work. Our proof has some advantages compared to the proof in [FG]. First it shows that uniform sampling is not needed for the correct reconstruction and second it can be generalized to curves in abstract metric spaces.

For our proof we need two lemmas which we will prove first.

Lemma 2.3 *Let γ be a regular curve. Then there exists an $\varepsilon > 0$ such that*

1) $|p^i - p^{i-1}| < |p^i - p^k|$ *if* $k \lhd i - 1$

2) $|p^i - p^{i+1}| < |p^i - p^k|$ *if* $k \rhd i + 1$

for all samples $S = \{p^1, \ldots, p^{|S|}\}$ with $\varepsilon(S) < \varepsilon$ and all $i \in \{1, \ldots, |S|\}$.

PROOF. Assume the contrary. Then there exists a sequence (S_n) of samples with $\lim_{n\to\infty} \varepsilon(S_n) = 0$ and $p_n^i, p_n^k \in S_n$ such that

$$k \lhd i - 1 \text{ and } |p_n^i - p_n^{i-1}| \geq |p_n^i - p_n^k|$$

or

$$k \rhd i + 1 \text{ and } |p_n^i - p_n^{i+1}| \geq |p_n^i - p_n^k|.$$

By choosing a subsequence one can always assume that for all $n \in \mathbb{N}$ one of the above possibilities holds. Without loss of generality assume that this is the first one. Since γ is compact, one can also assume by choosing a subsequence that (p_n^i) converges to $p \in \gamma$. From Lemma 2.1 it follows

$$\lim_{n\to\infty} |p_n^i - p_n^k| \leq \lim_{n\to\infty} |p_n^i - p_n^{i-1}| = 0.$$

Hence also (p_n^{i-1}) and (p_n^k) converge to p. Now look at the triangle with vertices p_n^{i-1}, p_n^i and p_n^k. From the law of cosines together with $|p_n^i - p_n^{i-1}| \geq |p_n^i - p_n^k|$, it follows for the angle α_n at p_n^{i-1}

$$\cos(\alpha_n) = -\frac{|p_n^i - p_n^k|^2 - |p_n^i - p_n^{i-1}|^2 - |p_n^{i-1} - p_n^k|^2}{2|p_n^i - p_n^{i-1}||p_n^{i-1} - p_n^k|} \geq 0.$$

Thus α_n has to be smaller or equal than $\frac{\pi}{2}$, but that is a contradiction to Lemma 1.1. □

The proof of the second lemma is similar to the proof of the first one, so we omit it here.

Lemma 2.4 *Let γ be a regular curve. Then there exists an $\varepsilon > 0$ such that*

1) $|p^i - p^m| < |p^i - p^k|$ *for* $k \lhd m \lhd i$,
if $k \lhd i - 1$ *and* $|p^i - p^k| \leq |p^l - p^{l+1}|$ *for some* $l \in \{1, \ldots, |S|\}$

2) $|p^i - p^m| < |p^i - p^k|$ *for* $i \lhd m < \lhd k$,
if $k \rhd i + 1$ *and* $|p^i - p^k| \leq |p^l - p^{l+1}|$ *for some* $l \in \{1, \ldots, |S|\}$

for all samples $S = \{p^1, \dots, p^{|S|}\}$ *with* $\varepsilon(S) < \varepsilon$ *and all* $i \in \{1, \dots, |S|\}$.

□

Now we are ready to prove that minimum spanning trees are a reconstruction scheme for regular curves. With $MST(S)$ we denote a minimum spanning tree on a set S of sample points. Note that for curves of type I the Traveling salesman path cannot be shorter than a minimum spanning tree. Thus $TSP(S) = MST(S)$ if $MST(S)$ is a path. Here TSP shall denote a Traveling salesman path and not a tour.

Theorem 2.5 *Let* γ *be a regular curve of type* I. *Then there exists an* $\varepsilon > 0$ *such that* $MST(S) = TSP(S) = P(S)$ *for all samples* S *of* γ *with* $\varepsilon(S) < \varepsilon$. *In particular* $MST(S)$ *and* $TSP(S)$ *are unique, if* $\varepsilon(S) < \varepsilon$.

PROOF. We are done if we can show that there exists an $\varepsilon > 0$ such that the polygonal reconstruction $P(S_n)$ is a minimum spanning tree on the points S and that there is only one minimum spanning tree for all samples S of γ with $\varepsilon(S) < \varepsilon$. Assume the contrary. Then there exists a sequence (S_n) of samples $S_n = \{p_n^1, \dots, p_n^{|S_n|}\}$ with $\lim_{n\to\infty} \varepsilon(S_n) = 0$ and $MST(S_n) \neq P(S_n)$. That means, $P(S_n)$ has to contain an edge $e_n = \mathrm{conv}\{p_n^i, p_n^{i+1}\}$ with $e_n \notin MST(S_n)$. Adding e_n to $MST(S_n)$ induces a cycle on $MST(S_n)$. All edges in this cycle different from e_n have to be of smaller or equal length than e_n, because of the minimality property of minimum spanning trees. Hence there is an edge $e'_n = \mathrm{conv}\{p_n^k, p_n^l\} \neq e_n$ in this cycle with $k \trianglelefteq i, i+1 \trianglelefteq l$ and $|p_n^k - p_n^l| \leq |p_n^i - p_n^{i+1}|$. Lemma 2.4 tells us that there exists $N \in \mathbb{N}$ with

1) $|p_n^i - p_n^l| \leq |p_n^k - p_n^l|$ for all $n \geq N$. That means

$$|p_n^i - p_n^l| \leq |p_n^i - p_n^{i+1}| \ \text{ for all } \ n \geq N.$$

2) $|p_n^k - p_n^{i+1}| \leq |p_n^k - p_n^l|$ for all $n \geq N$. That means

$$|p_n^k - p_n^{i+1}| \leq |p_n^i - p_n^{i+1}| \ \text{ for all } \ n \geq N.$$

That is a contradiction to Lemma 2.3. □

Minimum spanning trees are also useful in the reconstruction of curves of type II as the following theorem shows.

Theorem 2.6 *Let* γ *be a regular curve of type* II. *Then there exists an* $\varepsilon > 0$ *such that* $MST(S) \subset P(S)$ *for all samples* S *of* γ *with* $\varepsilon(S) < \varepsilon$.

PROOF. Assume the contrary. Then there exists a sequence (S_n) of samples $S_n = \{p_n^1, \ldots, p_n^{|S_n|}\}$ with $\varepsilon(S_n) \to 0$ as $n \to \infty$ and $MST(S_n)$ with $MST(S_n) \not\subset P(S_n)$. Denote by E_n the set of edges of $P(S_n)$, by E'_n the set of edges of $MST(S_n)$ and

$$R = E_n - (E'_n \cap E_n) \text{ and } M = E'_n - (E'_n \cap E_n).$$

From the assumptions it holds $M \neq \emptyset$. Choose $\operatorname{conv}\{p_n^k, p_n^l\} \in M$ with $k \lhd l$. Then there exists $\operatorname{conv}\{p_n^i, p_n^{i+1}\} \in R$ with $k \unlhd i, i+1 \unlhd l$, because otherwise $MST(S_n)$ has to contain a cycle, which contradicts the definition of a tree. Assume $|p_n^i - p_n^{i+1}| < |p_n^k - p_n^l|$. Removing the edge $\operatorname{conv}\{p_n^k, p_n^l\}$ from $MST(S_n)$ decomposes $MST(S_n)$ in two connected components. Both p_n^i and p_n^{i+1} have to belong to the same component, because otherwise one gets a shorter minimum spanning tree by replacing the edge $\operatorname{conv}\{p_n^k, p_n^l\}$ by the edge $\operatorname{conv}\{p_n^i, p_n^{i+1}\}$ in $MST(S_n)$. Without loss of generality assume that $\operatorname{conv}\{p_n^k, p_n^i\}$ connects the two components, because p_n^k and p_n^l have to belong to different components. Lemma 2.4 says that for sufficiently large n, one has $|p_n^k - p_n^i| < |p_n^k - p_n^l|$. That means, one gets a shorter minimum spanning tree by replacing the edge $\operatorname{conv}\{p_n^k, p_n^l\}$ by the edge $\operatorname{conv}\{p_n^k, p_n^i\}$ in $MST(S_n)$, which is a contradiction.
Hence one can assume by choosing an appropriate subsequence that $|p_n^k - p_n^l| \leq |p_n^i, p_n^{i+1}|$ for all $n \in \mathbb{N}$. Lemma 2.4 tells us that there exists $N \in \mathbb{N}$ with

1) $|p_n^i - p_n^l| \leq |p_n^k - p_n^l|$ for all $n \geq N$. That means
$$|p_n^i - p_n^l| \leq |p_n^i - p_n^{i+1}| \text{ for all } n \geq N.$$

2) $|p_n^k - p_n^{i+1}| \leq |p_n^k - p_n^l|$ for all $n \geq N$. That means
$$|p_n^k - p_n^{i+1}| \leq |p_n^i - p_n^{i+1}| \text{ for all } n \geq N.$$

That is a contradiction to Lemma 2.3. □

Uniform sampling is not needed to prove Theorems 2.5 and 2.6. The following definition of a good sample only takes the local behavior of a curve into account.

Definition 2.7 *Let S be a sample of a regular curve γ and $B_r(p)$ be the ball with radius $r > 0$ centered at $p \in \gamma$. The set S is called a good sample, if $B_r(p^i) \cap \gamma$ is homeomorphic to the unit interval $(0, 1)$ for all*

$$r < (1 + \eta) \max\{|p^{i+1} - p^i|, |p^i - p^{i-1}|\}, \textit{ for all } i \in \{1, \ldots, |S|\},$$

where $\eta > 0$ is an arbitrarily small constant.

The scaling with $(1 + \eta)$ prevents problems with degenerate samples, where two or more sample points exist which all have the same distance to one and the same sample point.

Good samples fulfill the properties stated in our key Lemmas 2.3 and 2.4. In the proofs of Theorems 2.5 and 2.6 we construct contradictions to these lemmas. Thus for regular curves of type I and good samples S it holds $P(S) = TSP(S) = MST(S)$ and for regular curves of type II and good samples S one has $MST(S) \subset P(S)$.

Figueiredo and Gomes [FG] already recognized that the minimum spanning tree is not a reconstruction scheme for polygons. They considered the following example, which can be scaled down to arbitrarily dense samples.

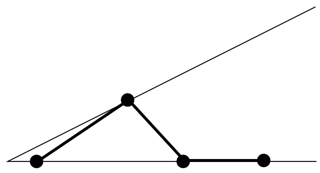

Figure 2.9: *Minimum spanning trees are not a reconstruction scheme for polygons*

2.3.2 Angular Metric Traveling Salesman Path (Tour)

The angular metric Traveling salesman path (tour) is the path (tour) through the sample points that minimizes the total curvature. It is the analog of a tight embedding which was discussed in surface reconstruction by Alboul and van Damme [AvD]. To study a minimum curvature criterion in reconstruction is motivated by the lower semi-continuity of curvature.

The following example shows that the angular metric Traveling salesman path is not a reconstruction scheme for polygons, because the situation depicted in Figure 2.10 can be scaled down to arbitrarily dense samples.

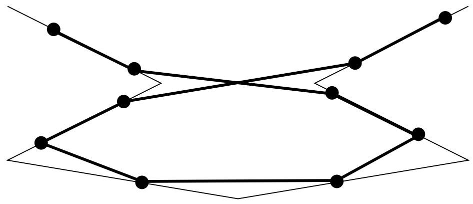

Figure 2.10: *The angular metric Traveling salesman tour is not a reconstruction scheme for polygons*

If the polygon is closed we have an example that shows that also the angular metric Traveling salesman tour is not a reconstruction scheme for polygons. The example also shows that one cannot expect to get good approximations of the length and the total curvature of the curve.

2.3.3 Traveling Salesman Path (Tour)

The discussion of minimum spanning trees shows that the Traveling salesman path is a reconstruction scheme for regular arcs. Here we will prove that the Traveling salesman path is a reconstruction scheme for curves of type I and the Traveling salesman tour is a reconstruction scheme for curves of type II, provided the curves fulfill the regularity assumptions of Theorem 2.1.

First we give several reformulations of our notion of regularity and will end up with a reformulation which is a local version of the theorem we want to prove in this section.

In the following let γ be a s-regular curve in $\mathbb{R}^{d+1}$ and let $p \in \gamma$ be a fixed point. We want to introduce the following notions. Let $\eta > 0$. The connected component of

$$\{q \in \gamma \;:\; |p - q| < \eta\}$$

which contains p is denoted by $B_\eta(p)$. The left tangent of γ in p is denoted by $\mathrm{l}(p)$ and the right tangent in p is denoted by $\mathrm{r}(p)$. Assume that every one-dimensional subspace ℓ of $\mathbb{R}^{d+1}$ not perpendicular to $\mathrm{l}(p)$ is oriented according to the orientation induced by the orthogonal projection $\pi_\ell(\mathrm{l}(p))$ of $\mathrm{l}(p)$ on ℓ and that every one-dimensional subspace ℓ perpendicular to $\mathrm{l}(p)$ carries an arbitrary orientation.

We want to compare the ordering of a set of sample points close to p on γ with the ordering of the projections of these points on one-dimensional subspaces of $\mathbb{R}^{d+1}$. The following reformulation of regularity states to which extent these orderings can be different:

Lemma 2.5 *Assume there exists a sequence (ℓ_n) of one-dimensional subspaces of $\mathbb{R}^{d+1}$ and sequences $(p_n), (q_n), (r_n) \in B_{1/n}(p)$ with $p_n < q_n < r_n \leq p$ in the order along γ, but*

$$\pi_{\ell_n}(q_n) < \pi_{\ell_n}(p_n) < \pi_{\ell_n}(r_n) \;\; \textit{or} \;\; \pi_{\ell_n}(p_n) < \pi_{\ell_n}(r_n) < \pi_{\ell_n}(q_n)$$

in the order on ℓ_n. Then the limit of every convergent subsequence of (ℓ_n) has to be perpendicular to $\mathrm{l}(p)$.

PROOF. The proof is done by contradiction. That is, one can assume that (ℓ_n) converges and that

$$0 \leq \alpha := \lim_{n\to\infty} \angle(\mathrm{l}(p), \ell_n) < \frac{\pi}{2},$$

where $\angle(\mathrm{l}(p), \ell_n)$ denotes the smaller of the two angles defined by the subspace ℓ_n and the subspace spanned by $\mathrm{l}(p)$. For the proof assume without loss of generality that $\pi_{\ell_n}(q_n) < \pi_{\ell_n}(p_n) < \pi_{\ell_n}(r_n)$, and by the continuity of γ and the continuity of the projection maps π_{ℓ_n} one can assume that there exists $0 < \lambda < 1$ (independent of n) such that

$$|\pi_{\ell_n}(p_n) - \pi_{\ell_n}(r_n)| = \lambda \cos(\alpha)\, |\pi_{\ell_n}(q_n) - \pi_{\ell_n}(r_n)|$$

if one moves q_n or r_n a little bit on γ. From the regularity of γ it follows that

$$\lim_{n\to\infty} \angle(\mathrm{conv}\{q_n, r_n\}, \mathrm{l}(p)) = \lim_{n\to\infty} \angle(\mathrm{conv}\{p_n, r_n\}, \mathrm{l}(p)) = 0$$

and together with the triangle inequality for spherical triangles

$$\begin{aligned}
&\lim_{n\to\infty} \angle(\mathrm{conv}\{q_n, r_n\}, \pi_{\ell_n}(\mathrm{conv}\{q_n, r_n\})) \\
&\leq \lim_{n\to\infty} \Big(\angle(\mathrm{conv}\{q_n, r_n\}, \mathrm{l}(p)) + \angle(\mathrm{l}(p), \pi_{\ell_n}(\mathrm{conv}\{q_n, r_n\}))\Big) \\
&= \lim_{n\to\infty} \angle(\mathrm{conv}\{q_n, r_n\}, \mathrm{l}(p)) + \lim_{n\to\infty} \angle(\mathrm{l}(p), \pi_{\ell_n}(\mathrm{conv}\{q_n, r_n\})) \\
&= 0 + \lim_{n\to\infty} \angle(\mathrm{l}(p), \ell_n) \\
&= \alpha < \frac{\pi}{2}
\end{aligned}$$

and analogously

$$\lim_{n\to\infty} \angle(\mathrm{conv}\{p_n, r_n\}, \pi_{\ell_n}(\mathrm{conv}\{p_n, r_n\})) \leq \alpha < \frac{\pi}{2}.$$

Hence,

$$0 < \cos(\alpha) \leq \lim_{n\to\infty} \frac{|\pi_{\ell_n}(q_n) - \pi_{\ell_n}(r_n)|}{|q_n - r_n|} \leq 1$$

and

$$0 < \cos(\alpha) \leq \lim_{n\to\infty} \frac{|\pi_{\ell_n}(p_n) - \pi_{\ell_n}(r_n)|}{|p_n - r_n|} \leq 1.$$

Therefore

$$\begin{aligned}
\frac{1}{\cos(\alpha)} &\geq \lim_{n\to\infty} \frac{|\pi_{\ell_n}(q_n) - \pi_{\ell_n}(r_n)|}{|q_n - r_n|} \frac{|p_n - r_n|}{|\pi_{\ell_n}(p_n) - \pi_{\ell_n}(r_n)|} \\
&= \lim_{n\to\infty} \frac{|\pi_{\ell_n}(q_n) - \pi_{\ell_n}(r_n)|}{|\pi_{\ell_n}(p_n) - \pi_{\ell_n}(r_n)|} \frac{|p_n - r_n|}{|q_n - r_n|} \\
&= \lim_{n\to\infty} \frac{|\pi_{\ell_n}(q_n) - \pi_{\ell_n}(r_n)|}{|\pi_{\ell_n}(p_n) - \pi_{\ell_n}(r_n)|} \lim_{n\to\infty} \frac{|p_n - r_n|}{|q_n - r_n|} \\
&= \frac{1}{\lambda \cos(\alpha)} \lim_{n\to\infty} \frac{|p_n - r_n|}{|q_n - r_n|}
\end{aligned}$$

and thus

$$\lim_{n\to\infty} \frac{|p_n - r_n|}{|q_n - r_n|} \leq \lambda < 1.$$

Hence, there exists $N \in \mathbb{N}$ such that $|p_n - r_n| \leq |q_n - r_n|$ for all $n \geq N$. Now consider the triangles with vertices p_n, q_n and r_n. Let α_n be the angle at q_n. From the law of cosines together with $|p_n - r_n| \leq |q_n - r_n|$ we have

$$\cos(\alpha_n) = -\frac{|p_n - r_n|^2 - |p_n - q_n|^2 - |q_n - r_n|^2}{2|p_n - q_n||q_n - r_n|} > 0.$$

Thus α_n, has to be smaller or equal than $\frac{\pi}{2}$ for all $n \geq N$. Since γ is left regular in p that is a contradiction to Lemma 1.1. □

The collection of all one-dimensional subspaces of $\mathbb{R}^{d+1}$ forms the d-dimensional projective space $\mathbb{P}^d$. In the following the elements of $\mathbb{P}^d$ are called lines. From a standard construction in Integral Geometry [R] one gets a probability measure μ_d on $\mathbb{P}^d$ from the surface area measure on the d-dimensional sphere $\mathbb{S}^d$:

The continuous and locally homeomorphic mapping

$$\phi : \mathbb{S}^d = \{x \in \mathbb{R}^{d+1} \,:\, |x| = 1\} \to \mathbb{P}^d,\; x \mapsto \{\lambda x \,:\, \lambda \in \mathbb{R}\}$$

is a double cover. Since ϕ is continuous, it is also measurable. That is the inverse image $\phi^{-1}(B)$ of a Borel set $B \in \mathbb{P}^d$ is a Borel set of $\mathbb{S}^d$. On $\mathbb{S}^d$ we have the usual surface area measure ν_d. Therefrom we get a measure μ'_d on $\mathbb{P}^d$ as $\mu'_d(B) = \nu_d(\phi^{-1}(B))$ for all Borel sets $B \subset \mathbb{P}^d$. We have

$$\int_{\mathbb{P}^d} \mathrm{d}\mu'_d = \mathrm{vol}(\mathbb{S}^d) = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)},$$

which we use to normalize μ'_d. That is, we set

$$\mu_d(B) = \frac{1}{\mathrm{vol}(\mathbb{S}^d)} \mu'_d(B) \tag{2.2}$$

for all Borel sets $B \subset \mathbb{P}^d$.

The next reformulation of regularity makes use of this measure:

Lemma 2.6 *For all $n \in \mathbb{N}$ let $p_n, q_n, r_n \in B_{1/n}(p)$ with $p_n < q_n < r_n \leq p$ in the order along γ. Let*

$$\begin{aligned} L_n \quad = \quad & \{\ell \in \mathbb{P}^d \,:\, \pi_\ell(q_n) < \pi_\ell(p_n) < \pi_\ell(r_n) \text{ or} \\ & \pi_\ell(p_n) < \pi_\ell(r_n) < \pi_\ell(q_n) \text{ in the order on } \ell\}, \end{aligned}$$

where π_ℓ denotes the orthogonal projection on the line ℓ. Then

$$\lim_{n\to\infty} \mu_d(L_n) = 0.$$

PROOF. In [R] Reshetnyak shows: Let V be a d-dimensional subspace of $\mathbb{R}^{d+1}$ and $E \subset \mathbb{P}^d$ be the set of all one-dimensional subspaces contained in V. Then $\mu_d(E) = 0$. Let $E \subset \mathbb{P}^d$ be the set of all one-dimensional subspaces of $\mathbb{R}^{d+1}$ perpendicular to $\mathrm{l}(p)$. Let (E_m) be a sequence of open sets $E_m \subset \mathbb{P}^d$ with $E_{m+1} \subset E_m$ and

$$\bigcap_{m=1}^{\infty} E_m = E.$$

Such a sequence can always be constructed. We find that $\mathbb{P}^d - E_m$ is closed and thus compact by the compactness of $\mathbb{P}^d$. There exists $n(m) \in \mathbf{N}$ with $n(m) < n(m+1)$ such that $(\mathbb{P}^d - E_m) \cap L_n = \emptyset$ for all $n \geq n(m)$. Assume the contrary. That is, we can assume without loss of generality that $(\mathbb{P}^d - E_m) \cap L_n \neq \emptyset$ for all $n \in \mathbb{N}$. Hence we find a sequence (ℓ_n) with $\ell_n \in (\mathbb{P}^d - E_m) \cap L_n$, which has by the compactness of $\mathbb{P}^d - E_m$ a convergent subsequence. The limit of this subsequence cannot be perpendicular to $\mathrm{l}(p)$ by construction. That is a contradiction to Lemma 2.5. So we have $L_n \subset E_m$

for all m and all $n \geq n(m)$. Using the isotony and continuity of the measure μ_d we get

$$\begin{aligned} \limsup \mu_d(L_n) &= \lim_{m\to\infty} \sup\{\mu_d(L_n) \,:\, n \geq n(m)\} \\ &\leq \lim_{m\to\infty} \mu_d(E_m) = \mu_d(E) \end{aligned}$$

and from Theorem of Reshetnyak $\mu_d(E) = 0$. That is, $\lim_{n\to\infty} \mu_d(L_n) = 0$. □

Obviously an analogous result for sample points larger than p in the order along γ also holds.

Now we are prepared to prove a local version of a reconstruction theorem. For a given sample S of a neighborhood of p we fix the smallest and the largest sample point along γ and consider paths through S which connect these points. One such path is the polygonal reconstruction $P(S)$ which connects the sample points in their order along γ. We distinguish three types of lines $\ell \in \mathbb{P}^d$:

1) There exists a path through the sample points different from $P(S)$ which has a shorter projection on ℓ than $P(S)$.
2) Every path through the sample points different from $P(S)$ has a larger projection on ℓ than $P(S)$.
3) There exist paths through the sample points different from $P(S)$ which projections on ℓ have the same length as the projection of $P(S)$, but there is no path with shorter projection.

The proof is subdivided in three parts. First, we show that the measure of the first set of lines tends to zero as the neighborhood of p shrinks to p itself. Second, there exists a constant larger than zero such that the measure of the second set of lines is larger than this constant for arbitrary small neighborhoods of p. Finally we conclude from the first two steps by integrating over the length of all projections that for small neighborhoods of p the polygonal reconstruction $P(S)$ has to be the shortest path through the sample points.

Theorem 2.7 *Let $p \in \gamma$ with $\angle(\mathrm{l}(p), \mathrm{r}(p)) < \pi$ and let (S_n) be a sequence of samples of $B_{1/n}(p)$. Let $TSP^*(S_n)$ be a path of minimal length through the sample points S_n with fixed startpoint* $\min S_n$ *and fixed endpoint* $\max S_n$.

Here $\min$ *and* $\max$ *are taken with respect to the order induced on* S_n *by* γ. *Then there exists* $N \in \mathbb{N}$ *such that*

$$TSP^*(S_n) = P(S_n),$$

for all $n \geq N$. *Furthermore* $TSP^*(S_n)$ *is unique for all* $n \geq N$.

PROOF. **First Step.** Let $L_n \subset \mathbb{P}^d$ be the set of lines for which the projection $\pi_\ell(P(S_n))$ is not a shortest path through the projected sample points $\pi_\ell(S_n)$ and let μ_d be the probability measure on $\mathbb{P}^d$ defined in equation (2.2).

We show that

$$\lim_{n\to\infty} \mu_d(L_n) = 0.$$

In the proof we use the following abbreviations:

$$\begin{aligned} m_1 &= \pi_\ell(\min S_n), & m_2 &= \min_\ell\{\pi_\ell(S_n)\}, \\ m_3 &= \max_\ell\{\pi_\ell(S_n)\}, & m_4 &= \pi_\ell(\max S_n) \end{aligned}$$

Take $\ell \in \mathbb{P}^d$ together with its orientation. A path of minimal length through the points $\pi_\ell(S_n)$ which connects m_1 with m_4 consists of

$$\mathrm{conv}\{m_1, m_2\} \cup \mathrm{conv}\{m_2, m_3\} \cup \mathrm{conv}\{m_3, m_4\}$$

Figure 2.11: *Path of minimal length*

Here we assume $m_1 \leq m_2$ on ℓ, otherwise just invert the orientation of ℓ. Hence the points of ℓ between m_1 and m_2 are covered twice by a path of minimal length through the points $\pi_\ell(S_n)$, the points between m_1 and m_4 are covered once and the points between m_3 and m_4 are covered twice again. If $\pi_\ell(P(S_n))$ is not a path of minimal length through the points $\pi_\ell(S_n)$, then there has to exists an interval

$$I = [\min_\ell\{\pi_\ell(p_n^i), \pi_\ell(p_n^{i+1})\}, \max_\ell\{\pi_\ell(p_n^i), \pi_\ell(p_n^{i+1})\}]$$

on ℓ with $p_n^i, p_n^{i+1} \in S_n$, which is covered by $\pi_\ell(P(S_n))$

1) $2 + 2k$ times, $k \geq 1$, if

$$m_2 \leq \min_\ell\{\pi_\ell(p_n^i), \pi_\ell(p_n^{i+1})\} < \max_\ell\{\pi_\ell(p_n^i), \pi_\ell(p_n^{i+1})\} \leq m_1$$

or

$$m_4 \leq \min_\ell\{\pi_\ell(p_n^i), \pi_\ell(p_n^{i+1})\} < \max_\ell\{\pi_\ell(p_n^i), \pi_\ell(p_n^{i+1})\} \leq m_3$$

2) $1 + 2k$ times, $k \geq 1$, if

$$m_1 \leq \min_\ell\{\pi_\ell(p_n^i), \pi_\ell(p_n^{i+1})\} < \max_\ell\{\pi_\ell(p_n^i), \pi_\ell(p_n^{i+1})\} \leq m_4.$$

For all $p_n^j \in S_n$, $j \in \{2, \ldots, |S_n| - 2\}$ the interval

$$I_j = [\min_\ell\{\pi_\ell(p_n^j), \pi_\ell(p_n^{j+1})\}, \max_\ell\{\pi_\ell(p_n^j), \pi_\ell(p_n^{j+1})\}]$$

is called positive if $\pi_\ell(p_n^j) < \pi_\ell(p_n^{j+1})$ in the order on ℓ and negative otherwise. The intervals I_j which cover the interval I must have alternating signs. That is, if $\pi_\ell(P(S_n))$ is not a shortest path through the points $\pi_\ell(S_n)$ one finds, using that $P(S_n)$ connects the $p_n^j \in S_n$ in their order along γ and using the continuity of γ and of the projection map π_ℓ, three points $p_n, q_n, r_n \in B_{1/n}(p)$ with $p_n < q_n < r_n \leq p$ in the order along γ, but

$$\pi_\ell(q_n) < \pi_\ell(p_n) < \pi_\ell(r_n) \text{ or } \pi_\ell(p_n) < \pi_\ell(r_n) < \pi_\ell(q_n) \text{ in the order on } \ell.$$

Or one finds three points $p_n, q_n, r_n \in B_{1/n}(p)$ with $p_n > q_n > r_n \geq p$ in the order along γ, but

$$\pi_\ell(q_n) > \pi_\ell(p_n) > \pi_\ell(r_n) \text{ or } \pi_\ell(p_n) > \pi_\ell(r_n) > \pi_\ell(q_n) \text{ in the order on } \ell.$$

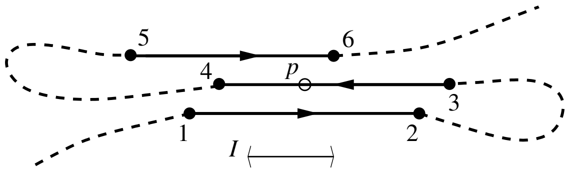

Figure 2.12: *An example where* $4, 5, 6 > p$ *are not projected in the right order*

In general the points p_n, q_n and r_n need not be sample points. From Lemma 2.5 one can conclude that

$$\lim_{n\to\infty} \mu_d(L_n) = 0.$$

Second Step. Let $M_n \subset \mathbb{P}^d$ be the set of lines ℓ for which:

1) The order of S_n along γ and the order of S_n induced by the order of $\pi_\ell(S_n)$ on ℓ coincide.
2) For all $\operatorname{conv}\{p^i, p^{i+1}\} \subset P(S_n)$ with $p^i, p^{i+1} \in S_n$ it holds
$$|\pi_\ell(p^{i+1}) - \pi_\ell(p^i)| \geq \frac{1}{2}\cos(\frac{\pi + \alpha}{4})|p^{i+1} - p^i|.$$

We show that there exist $c > 0$ and $N' \in \mathbb{N}$ such that
$$\mu_d(M_n) \geq c \text{ for all } n \geq N'.$$

For the proof we construct a set of lines $C \subset \mathbb{P}^d$ with $\mu_d(C) > 0$ and show that there exists $N' \in \mathbb{N}$ such that $C \subset M_n$ for all $n \geq N'$. The set C is defined as follows: Let ℓ_0 be the line, $\ell_0 \subset \operatorname{span}\{\mathrm{l}(p), \mathrm{r}(p)\}$ such that $\ell_0^\perp \subset \operatorname{span}\{\mathrm{l}(p), \mathrm{r}(p)\}$ halves the angle $\angle(\mathrm{l}(p), -\mathrm{r}(p))$ between the lines determined by $\mathrm{l}(p)$ and $\mathrm{r}(p)$. Now define
$$C = \left\{ \ell \in \mathbb{P}^d \,:\, \angle(\ell, \ell_0) \leq \frac{1}{4}(\pi - \alpha) \right\}.$$

By $\angle(\ell, \ell')$ for $\ell, \ell' \in \mathbb{P}^d$ denote the value of of the minimum of the two angles determined by ℓ and ℓ'. The set $\bigcup_{\ell \in C}\{x \in \ell\} \subset \mathbb{R}^{d+1}$ is a double cone.

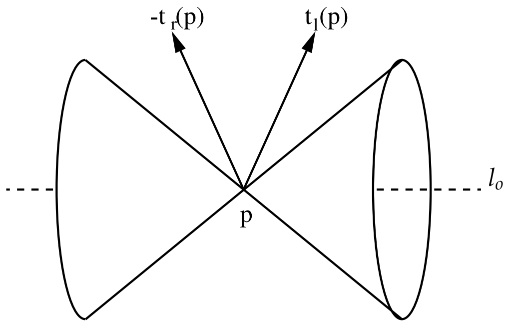

Figure 2.13: *The double cone C*

By construction $\pi_\ell(\mathrm{l}(p))$ and $\pi_\ell(\mathrm{r}(p))$ point in the same direction on every $\ell \in C$. Since $\alpha < \pi$ we have $\mu_d(C) > 0$. It remains to check conditions 1) and 2) to prove that $C \subset M_n$ for sufficiently large n.

Condition 1) Assume that for arbitrarily large n one can find $\ell_n \in C$ such that the first condition is violated. Then one can find three points $p_n, q_n, r_n \in B_{1/n}(p)$ with $p_n < q_n < r_n \leq p$ in the order along γ, but

$$\pi_{\ell_n}(q_n) < \pi_{\ell_n}(p_n) < \pi_{\ell_n}(r_n) \text{ or } \pi_{\ell_n}(p_n) < \pi_{\ell_n}(r_n) < \pi_{\ell_n}(q_n)$$

in the order on ℓ_n. Or one can find three points $p_n, q_n, r_n \in B_{1/n}(p)$ with $p_n > q_n > r_n \geq p$ in the order along γ, but

$$\pi_{\ell_n}(q_n) > \pi_{\ell_n}(p_n) > \pi_{\ell_n}(r_n) \text{ or } \pi_{\ell_n}(p_n) > \pi_{\ell_n}(r_n) > \pi_{\ell_n}(q_n)$$

in the order on ℓ_n. By Lemma 2.5 the limit of every convergent subsequence of (ℓ_n) has to be perpendicular to either $\mathrm{l}(p)$ or $\mathrm{r}(p)$. Since C is compact one finds $\ell \in C$ as the limit of a convergent subsequence of (ℓ_n) which is perpendicular to either $\mathrm{l}(p)$ or $\mathrm{r}(p)$. On the other hand one has for all $\ell \in C$ using the triangle inequality for spherical triangles:

$$\begin{aligned}\angle(\ell, \mathrm{l}(p)) &\leq \angle(\ell, \ell_0) + \angle(\ell_0, \mathrm{l}(p)) \\ &< \frac{1}{4}(\pi - \alpha) + \left(\frac{\pi}{2} - \frac{\pi - \alpha}{2}\right) \\ &= \frac{\pi + \alpha}{4} < \frac{\pi}{2}\end{aligned}$$

and analogously

$$\angle(\ell, \mathrm{r}(p)) < \frac{\pi}{2}$$

Thus every $\ell \in C$ is neither perpendicular to $\mathrm{l}(p)$ nor to $\mathrm{r}(p)$ – a contradiction. That means, for all sufficiently large n the first condition cannot be violated.

Condition 2) From the s-regularity of γ and the continuity of the angle function $\angle$ one has for $p_n^i, p_n^{i+1} \in S_n$ with $p_n^i < p_n^{i+1} \leq p$ and all $\ell \in C$,

$$\begin{aligned}\lim_{n\to\infty} \angle(\ell, \mathrm{conv}\{p_n^i, p_n^{i+1}\}) &= \angle(\ell, \lim_{n\to\infty} \mathrm{conv}\{p_n^i, p_n^{i+1}\}) \\ &= \angle(\ell, \mathrm{l}(p)) \leq \frac{\pi + \alpha}{4}.\end{aligned}$$

That is, for all sufficiently large n and all $\ell \in C$ one has

$$|\pi_\ell(p^i) - \pi_\ell(p^{i+1})| \geq \frac{1}{2}\cos\left(\frac{\pi + \alpha}{4}\right)|p^i - p^{i+1}|.$$

For $p_n^i, p_n^{i+1} \in S_n$ with $p \leq p_n^i < p_n^{i+1}$ one gets the same result using $\mathrm{r}(p)$ instead of $\mathrm{l}(p)$ in the triangle inequality for spherical triangles. It remains to consider the case that $p_n^i < p < p_n^{i+1}$. Choose $\eta > 0$ such that

$$\cos\left(\frac{\pi+\alpha}{4}+\eta\right) > \frac{1}{2}\cos\left(\frac{\pi+\alpha}{4}\right)$$

and show that for all $\ell \in C$ and all sufficiently large n it holds

$$\angle(\ell, \operatorname{conv}\{p_n^i, p_n^{i+1}\}) \quad \leq \quad \frac{\pi+\alpha}{4}+\eta.$$

To do so let π be the projection on $\operatorname{span}\{\mathrm{l}(p), \mathrm{r}(p)\}$. The s-regularity of γ implies

$$\begin{aligned} &\limsup\left(\angle(\ell_0, \pi(\operatorname{conv}\{p_n^i, p_n^{i+1}\}))\right. \\ &\leq \quad \angle(\ell_0, \mathrm{l}(p)) \;=\; \angle(\ell_0, \mathrm{r}(p)). \end{aligned} \tag{2.3}$$

Let the distance between two two-dimensional subspaces of $\mathbb{R}^{d+1}$ be the smaller of the two dihedral angles defined by them. With this metric the span seen as a mapping becomes continuous. From the continuity of $\angle$ and the continuity of span one has:

$$\begin{aligned} 0 &= \lim_{n\to\infty} \angle\left(\operatorname{span}\left\{\frac{p_n^i - p}{|p_n^i - p|}, \frac{p - p_n^{i+1}}{|p - p_n^{i+1}|}\right\}, \operatorname{conv}\{p_n^i, p_n^{i+1}\}\right) \\ &= \angle\left(\lim_{n\to\infty} \operatorname{span}\left\{\frac{p_n^i - p}{|p_n^i - p|}, \frac{p - p_n^{i+1}}{|p - p_n^{i+1}|}\right\}, \lim_{n\to\infty} \operatorname{conv}\{p_n^i, p_n^{i+1}\}\right) \\ &= \angle\left(\operatorname{span}\{\mathrm{l}(p), \mathrm{r}(p)\}, \lim_{n\to\infty} \operatorname{conv}\{p_n^i, p_n^{i+1}\}\right) \\ &= \lim_{n\to\infty} \angle(\operatorname{span}\{\mathrm{l}(p), \mathrm{r}(p)\}, \operatorname{conv}\{p_n^i, p_n^{i+1}\}) \end{aligned}$$

That is, for all sufficiently large n we have

$$\angle(\pi(\operatorname{conv}\{p_n^i, p_n^{i+1}\}), \operatorname{conv}\{p_n^i, p_n^{i+1}\}) \leq \eta. \tag{2.4}$$

Finally it holds for all sufficiently large n using the inequalities (2.3), (2.4) and the triangle inequality for spherical triangles:

$$\begin{aligned} &\angle(\ell, \operatorname{conv}\{p_n^i, p_n^{i+1}\}) \\ &\quad\leq \angle(\ell, \ell_0) + \angle(\ell_0, \operatorname{conv}\{p_n^i, p_n^{i+1}\}) \\ &\quad\leq \angle(\ell, \ell_0) + \angle(\ell_0, \pi(\operatorname{conv}\{p_n^i, p_n^{i+1}\})) + \\ &\qquad\quad \angle(\pi(\operatorname{conv}\{p_n^i, p_n^{i+1}\}), \operatorname{conv}\{p_n^i, p_n^{i+1}\}) \\ &\quad\leq \angle(\ell, \ell_0) + \angle(\ell_0, \mathrm{l}(p)) + \eta \\ &\quad\leq \frac{\pi-\alpha}{4} + \frac{\alpha}{2} + \eta \\ &\quad= \frac{\pi+\alpha}{4} + \eta \end{aligned}$$

Hence for all sufficiently large n, all $\ell \in C$ and all $p_n^i, p_n^{i+1} \in S_n$ we have

$$|\pi_\ell(p^i) - \pi_\ell(p^{i+1})| \geq \frac{1}{2}\cos(\frac{\pi+\alpha}{4})|p^i - p^{i+1}|.$$

Since condition 1) and condition 2) are fulfilled, there exists $N' \in \mathbb{N}$ such that for all $n \geq N'$ one has $C \subset M_n$. Hence

$$\mu_d(M_n) \geq \mu_d(C) > 0.$$

Third step. In the first two steps the measures of the subspaces L_n and M_n of $\mathbb{P}^d$, the spaces on which one wants to integrate, were considered. In this step the integrands are compared.

Let $\tilde{P}(S_n)$ be a path different from $P(S_n)$ through the points S_n, which connects p_n^1 with $p_n^{|S_n|}$. Define the rank $\mathrm{rk}(p_n^i, \tilde{P}(S_n))$ of a sample point $p_n^i \in S_n$ in the path $\tilde{P}(S_n)$ as the number of sample points the path $\tilde{P}(S_n)$ meets on its way from p_n^1 to $p_n^{|S_n|}$ before it meets p_n^i.
Let I_n be the following set of indices:

$$P_n = \left\{ i \in \{2, \ldots, |S_n| - 2\} \ : \ \mathrm{rk}(p_n^i, \tilde{P}(S_n)) > \mathrm{rk}(p_n^{i+1}, \tilde{P}(S_n)) \right\}$$

If one compares the length of the projections $\pi_\ell(P(S_n))$ and $\pi_\ell(\tilde{P}(S_n))$ on $\ell \in L_n$ one finds:

$$L(\pi_\ell(P(S_n))) - L(\pi_\ell(\tilde{P}(S_n))) \leq 2 \sum_{i \in I_n} |\pi_\ell(p_n^i) - \pi_\ell(p_n^{i+1})| \qquad (2.5)$$

On the other hand one finds for the length of projections on $\ell \in M_n$:

$$L(\pi_\ell(\tilde{P}(S_n))) - L(\pi_\ell(P(S_n))) \geq 2 \sum_{i \in I_n} |\pi_\ell(p_n^i) - \pi_\ell(p_n^{i+1})| \qquad (2.6)$$

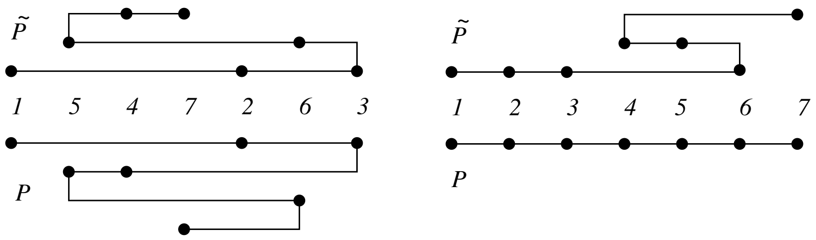

Figure 2.14: *Example of projections on $\ell \in L_n$ (left) and on $\ell \in M_n$ (right)*

Furthermore we have for all $p_n^i \in S_n$, $i \in \{1, \ldots, |S_n| - 1\}$ and all $\ell \in M_n$:

$$|\pi_\ell(p^{i+1}) - \pi_\ell(p^{i+1})| \quad \geq \quad \frac{1}{2}\cos(\frac{\pi + \alpha}{4})|p^{i+1} - p^i|$$

From this inequality it follows another inequality which is valid for all $\ell \in M_n$ and all $\ell' \in L_n$,

$$\sum_{i \in I_n} |\pi_\ell(p_n^i) - \pi_\ell(p_n^{i+1})| \geq \frac{1}{2}\cos(\frac{\pi + \alpha}{4}) \sum_{i \in I_n} |\pi_{\ell'}(p_n^i) - \pi_{\ell'}(p_n^{i+1})|$$

which implies together with (2.5) and (2.6):

$$\begin{aligned} & L(\pi_\ell(\tilde{P}(S_n))) - L(\pi_\ell(P(S_n))) \\ \geq \quad & \frac{1}{2}\cos(\frac{\pi + \alpha}{4})\Big(L(\pi_{\ell'}(P(S_n))) - L(\pi_{\ell'}(\tilde{P}(S_n)))\Big) \end{aligned}$$

Since this inequality is valid for all $\ell \in M_n$ and all $\ell' \in L_n$ one has for the increase of length on M_n

$$\begin{aligned} & \int_{M_n} \Big(L(\pi_\ell(\tilde{P}(S_n))) - L(\pi_\ell P(S_n)))\Big) \mathrm{d}\mu_d(\ell) \\ \geq \quad & \mu_d(M_n)\,\frac{1}{2}\cos(\frac{\pi + \alpha}{4}) \sup_{\ell \in L_n} \Big(L(\pi_\ell(P(S_n))) - L(\pi_\ell(\tilde{P}(S_n)))\Big) \end{aligned}$$

and for the decrease of length on L_n

$$\begin{aligned} & \int_{L_n} \Big(L(\pi_\ell(P(S_n))) - L(\pi_\ell(\tilde{P}(S_n)))\Big) \mathrm{d}\mu_d(\ell) \\ \leq \quad & \mu_d(L_n) \sup_{\ell \in L_n} \Big(L(\pi_\ell(P(S_n))) - L(\pi_\ell(\tilde{P}(S_n)))\Big). \end{aligned}$$

Because of $\lim_{n\to\infty} \mu_d(L_n) = 0$ there exists $N \geq N'$ such for all $n \geq N$ it holds

$$\mu_d(L_n) < \mu_d(C)\,\frac{1}{2}\cos(\frac{\pi+\alpha}{4}) \leq \mu_d(M_n)\,\frac{1}{2}\cos(\frac{\pi+\alpha}{4}).$$

Using another theorem of Reshetnyak [R] which states for s-regular curves γ:

$$\int_{\mathbb{P}^d} L(\pi_\ell(\gamma))\mathrm{d}\mu_d(\ell) = c_d L(\gamma) \text{ with } c_d = \frac{\Gamma((d+1)/2)}{\sqrt{\pi}\,\Gamma((d+2)/2)}$$

one has that for all $n \geq N$ there is no shortcut possible and that $P(S_n)$ is the unique path of minimal length through the points S_n with fixed start- and endpoint, because the polygon connecting the points in the order induced by γ has a shorter projection on all $\ell \in C$ than every other polygon through the points S_n with fixed start- and endpoint. $\square$

It remains to achieve the transition from the local to the global. To do so we introduce the notion of a return point.

Definition 2.8 *Given a curve γ and a sample S from γ. We call $r \in S$ a return point, if r is connected to $p, q \in S$ in a Traveling salesman path (tour) and $r \lhd p, q$ or $r \rhd p, q$ in the order along γ. In the first case we call the return point positive and in the second case we call it negative. For curves of type* I *we also call the start- and endpoint of $TSP(S)$ return point.*

First we prove the global result for curves of type II.

Theorem 2.8 *Let γ be a curve of type* II. *Assume*

$$\alpha = \sup\{\angle(\mathrm{l}(q), \mathrm{r}(q)) \,:\, q \in \gamma\} < \pi$$

and let (S_n) be a sequence of samples of γ with $\lim_{n\to\infty} \varepsilon(S_n) = 0$. Then there exists $N \in \mathbb{N}$ such that $TSP(S_n) = P(S_n)$ for all $n \geq N$. Here $TSP(S_n)$ is a shortest tour through the points S_n. Furthermore $TSP(S_n)$ is unique for all $n \geq N$.

PROOF. The proof is done by contradiction. Assume without loss of generality that $TSP(S_n) \neq P(S_n)$ for all $n \in \mathbb{N}$.

First Step. We show that for large n there exist at least four return points. First we show that there exists at least one return point for large n. Assume the contrary. That is, there does not exist a return point in S_n for arbitrary large n. By turning to a subsequence one can assume without loss of generality that there does not exist a return point for all $n \in \mathbb{N}$. Since $TSP(S_n) \neq P(S_n)$ there exists $p_n^i \in S_n$ which is not connected to p_n^{i-1} in $TSP(S_n)$. Cut $TSP(S_n)$ in two polygonal arcs P_n^1, with startpoint p_n^i and endpoint p_n^{i-1}, and P_n^2, with startpoint p_n^{i-1} and endpoint p_n^i. By our assumption that there does not exist a return point in S_n the sample points in both polygonal arcs are connected in their order along γ. From Lemma 1.5 and the Definition 1.6 of the Jordan length we can conclude that

$$\lim_{n\to\infty} L(P_n^1),\ \lim_{n\to\infty} L(P_n^2) = L(\gamma).$$

That is, $\lim_{n\to\infty} L(TSP(S_n)) = 2L(\gamma)$. Which is a contradiction. Hence there has to exist a return point for large n.

Observe that the signs (Definition 2.8) of the return points in S_n always sum up to zero and that return points incident along $TSP(S_n)$ always have different signs. So one can conclude from that for sufficiently large n there exist at least two return points. Now assume that we have only two return points p_n and q_n for arbitrary large n. Cut $TSP(S_n)$ in two polygonal arcs P_n^1 and P_n^2 that connect p_n with q_n. The points along these arcs, running from p_n to q_n, are ordered in the same way as they are ordered along γ. From Lemma 1.5 and the Definition 1.6 of the Jordan length we can conclude that

$$\lim_{n\to\infty} L(P_n^1),\ \lim_{n\to\infty} L(P_n^2) = L(\gamma).$$

That is, $\lim_{n\to\infty} L(TSP(S_n)) = 2L(\gamma)$. Which is a contradiction to Corollary 1.7. Hence for large n there are at least four return points in S_n.

Second Step. We show that for large n there must exist two return points $r_n^1 \lhd r_n^2$ neighborly along $TSP(S_n)$, i.e. r_n^1 and r_n^2 are consequent return points along $TSP(S_n)$, such that the other return points $\tilde{r}_n^1$ neighborly to r_n^1 and $\tilde{r}_n^2$ neighborly to r_n^2 along $TSP(S_n)$ are not in between r_n^1 and r_n^2. That is, one cannot find the following situation

$$r_n^1 \lhd \tilde{r}_n^1 \lhd r_n^2 \ \text{ or } \ r_n^1 \lhd \tilde{r}_n^2 \lhd r_n^2. \tag{2.7}$$

But it is possible that $\tilde{r}_n^1 = r_n^2$ and $\tilde{r}_n^2 = r_n^1$.

Assume that for all neighborly return points one finds situation (2.7) then all return points have to accumulate in between two return points.

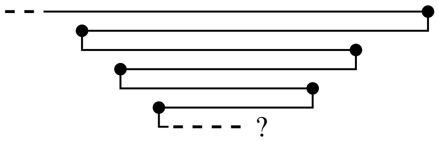

Figure 2.15: *Accumulating return points*

That is impossible since $TSP(S_n)$ is closed.

Third Step. In this step the transition from the local version of the theorem to the global one is achieved. In this transition it is made use of the return points r_n^1 and r_n^2 found in the second step. Choose the orientation of $TSP(S_n)$ such that $r_n^2 \lhd r_n^1$ along $TSP(S_n)$. Let $r_n^0 \in S_n$ be the last sample point one encounters running through $TSP(S_n)$ with

$$r_n^0 \lhd r_n^1 \ \text{ along } \gamma \ \text{ and } \ r_n^0 \lhd r_n^2 \ \text{ along } TSP(S_n)$$

and let $r_n^3 \in S_n$ be the first sample point one encounters running through $TSP(S_n)$ with

$$r_n^2 \lhd r_n^3 \ \text{ along } \gamma \ \text{ and } \ r_n^1 \lhd r_n^3 \text{ along } TSP(S_n).$$

That is, one has the following situation:

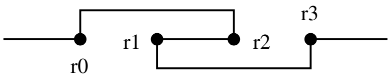

Figure 2.16: *Shortcut through return points*

By the compactness of γ we can assume by turning to convergent subsequences that $(r_n^0), (r_n^1), (r_n^2)$ and (r_n^3) converge to $r^0, r^1, r^2, r^3 \in \gamma$. Let $s_n \in S_n$ be the successor of r_n^0 and let $p_n \in S_n$ be the predecessor of r_n^3 along $TSP(S_n)$. By construction we have

$$r_n^0 \lhd r_n^1 \lhd s_n \ \text{ and } \ p_n \lhd r_n^2 \lhd r_n^3 \ \text{ along } \gamma.$$

From Lemma 1.5 we can conclude that

$$\lim_{n\to\infty} |r_n^0 - s_n| = \lim_{n\to\infty} |p_n - r_n^3| = 0.$$

That is, $r^0 = r^1$ and $r^2 = r^3$. Now assume $r^1 \lhd r^2$. Consider three sets of sample points

$$\begin{aligned} M_n^1 &= \{ p \in S_n \ : \ p \unlhd r_n^2 \text{ along } TSP(S_n) \} \\ M_n^2 &= \{ p \in S_n \ : \ r_n^2 \unlhd p \unlhd r_n^1 \text{ along } TSP(S_n) \} \\ M_n^3 &= \{ p \in S_n \ : \ p \unrhd r_n^1 \text{ along } TSP(S_n) \}. \end{aligned}$$

Using Corollary 1.7 and Theorem 1.3 one has

$$\begin{aligned} &\lim_{n\to\infty} L(TSP(S_n)) \\ &\geq \lim_{n\to\infty} \Big(L(TSP(M_n^1)) + L(TSP(M_n^2)) + L(TSP(M_n^3)) \Big) \\ &= \lim_{n\to\infty} L(TSP(M_n^1)) + \lim_{n\to\infty} L(TSP(M_n^2)) + \lim_{n\to\infty} L(TSP(M_n^3)) \\ &= L(\gamma|_{[0,\gamma^{-1}(r^2)]}) + L(\gamma|_{[\gamma^{-1}(r^1),\gamma^{-1}(r^2)]}) + L(\gamma|_{[\gamma^{-1}(r^1),1]}) \\ &= L(\gamma) + 2L(\gamma|_{[\gamma^{-1}(r^1),\gamma^{-1}(r^2)]}) \\ &> L(\gamma). \end{aligned}$$

That is a contradiction to Corollary 1.7. Hence we have

$$r^0 = r^1 = r^2 = r^3 =: r \in \gamma.$$

By turning to an appropriate subsequence of (S_n) we can assume without loss of generality that

$$r_n^0, r_n^1, r_n^2, r_n^3 \in S_n \cap B_{1/n}(r).$$

That is a contradiction to Theorem 2.7, which is the local version of this theorem. □

Using this theorem together with Theorem 2.1 the following corollary can be derived:

Corollary 2.9 *Let γ be a s-regular curve of type* II *with*

$$\alpha = \sup\{\angle(\mathrm{l}(q), \mathrm{r}(q)) \ : \ q \in \gamma\} < \pi$$

then there exists an $\varepsilon > 0$ such that for all samples S with $\varepsilon(S) < \varepsilon$ the Traveling salesman tour through the points S is a subgraph of the Delaunay graph.

Next we want prove the global result for curves of type I. To do so we start with a simple lemma.

Lemma 2.7 *Let S be a sample of a curve γ of type* I, *$E(S)$ be the set of edges of $P(S)$ and $E'(S)$ be the set of edges of $P'(S)$, where $P'(S)$ is another polygonal path on S. Let π be the permutation of $\{1, \ldots, |S|\}$ induced by $P'(S)$. Then there exists a bijection $f : E'(S) \to E(S)$ with*

1) $f : \mathrm{conv}\{p^{\pi(i)}, p^{\pi(i+1)}\} \mapsto \mathrm{conv}\{p^j, p^{j+1}\}$ *with* $\pi(i) \leq j$ *and* $j+1 \leq \pi(i+1)$.

2) $f \mid_{E(S) \cap E'(S)} = \mathrm{id}$.

PROOF. The proof is done by induction on the cardinality of S. If $|S| = 2$ we must have $P(S) = P'(S)$ and set $f = \mathrm{id}$. Now assume the theorem is proven for $|S| < n$. Now assume that it holds $|S| = n$. We distinguish two cases, which are depicted graphically below:

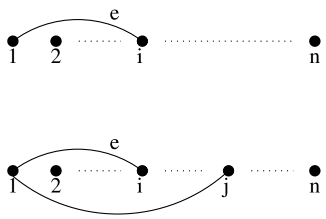

Figure 2.17: *Two cases in the induction*

The lines in this figure shall denote edges of $P_{min}(S)$. In both cases map the edge e to $\mathrm{conv}\{p^1, p^2\}$. Now we remove p^1 and consider the induced polygon $TSP(S - \{p^1\})$ on the vertex set $\{p^2, \ldots, p^n\}$. In the second case add $\mathrm{conv}\{p^i, p^j\}$ to the induced polygon $TSP(S - \{p^1\})$. We are left with the problem to find a suitable bijection on edge sets with $n - 1$ elements. That is, we have reduced the problem to find an appropriate bijection to the case $|S| = n - 1$. □

Now we are prepared to prove the global result for curves of type I.

Theorem 2.10 *Let γ be a curve of type* I. *Assume*

$$\alpha = \sup\{\angle(\mathrm{l}(q), \mathrm{r}(q)) : q \in \gamma\} < \pi$$

and let (S_n) be a sequence of samples of γ with $\lim_{n\to\infty} \varepsilon(S_n) = 0$. Then there exists $N \in \mathbb{N}$ such that $TSP(S_n) = P(S_n)$ for all $n \geq N$. Here

$TSP(S_n)$ is a shortest path through the points S_n. Furthermore $TSP(S_n)$ is unique for all $n \geq N$.

PROOF. The proof is done by contradiction. Assume without loss of generality that $TSP(S_n) \neq P(S_n)$ for all $n \in \mathbb{N}$.

First Step. We show that for large n there exist at least four return points. Because of $TSP(S_n) \neq P(S_n)$ there have to exist at least three return points. If in S_n there is only one return point besides the start- and endpoint of $TSP(S_n)$, then this point is either $\min S_n$ or $\max S_n$ along γ. Assume that there exists $N \in \mathbb{N}$ such that the number of return points in S_n is three for all $n \geq N$. Without loss of generality we can assume that the only return point besides the endpoints of $TSP(S_n)$ is $p_n^1 = \min S_n$.
In $TSP(S_n)$ we also have that p_n^1 is connected to p_n^2, because otherwise S_n has more than three return points. Cut $TSP(S_n)$ in two polygonal arcs P_n^1 and P_n^2 with endpoint p_n^1. Since $TSP(S)$ has only three return points one of these arcs has as its second endpoint $p_n^{|S_n|}$. Assume without loss of generality that this arc always is P_n^1. From Lemma 1.5 and the definition of Jordan length we can conclude that $\lim_{n\to\infty} L(P_n^1) = L(\gamma)$. This implies

$$\lim_{n\to\infty} L(P_n^2) = 0.$$

Let p_n^i, $i \in \{3, \ldots, |S_n|\}$ be the second endpoint of P_n^2. We observe two things:

1) $\lim_{n\to\infty} |p_n^1 - p_n^i| = 0$, because $|p_n^1 - p_n^i| \leq L(P_n^2)$ for large n.

2) $\operatorname{conv}\{p_n^j, p_n^{j+1}\} \subset TSP(S_n)$ for all $i < j < |S_n|$ and large n. That is, the shortcuts take place on the first i indices. Using that $TSP(S_n) \neq P(S_n)$ and the statement and notions of Lemma 2.7, with $P'(S) = TSP(S)$, we find that there exists

$$\operatorname{conv}\{p_n^{\pi(j)}, p_n^{\pi(j+1)}\} \in E'(S_n) \text{ and } \operatorname{conv}\{p_n^k, p_n^{k+1}\} \in E(S_n)$$

with

$$|p_n^{\pi(j)} - p_n^{\pi(j+1)}| \leq |p_n^k - p_n^{k+1}|$$

$j, k < i$, $\pi(j) \leq k$, $k+1 \leq \pi(j+1)$ and $\pi(j) \neq k$ or $k+1 \neq \pi(j+1)$.

From $\lim_{n\to\infty} \varepsilon(S_n) = 0$ we can conclude that p_n^1 converges to $\gamma(0)$. Since $\gamma(0)$ is a regular point of γ we can apply here Lemma 2.3 and Lemma 2.4. Now Lemma 2.4 tells us that there exists $n \in \mathbb{N}$ with

1) $|p_n^k - p_n^{\pi(j+1)}| \leq |p_n^{\pi(j)} - p_n^{\pi(j+1)}|$ for all $n \geq N$. That means

$$|p_n^k - p_n^{\pi(j+1)}| < |p_n^k - p_n^{k+1}| \text{ for all } n \geq N.$$

2) $|p_n^{k+1} - p_n^{\pi(j)}| \leq |p_n^{\pi(j)} - p_n^{\pi(j+1)}|$ for all $n \geq N$. That means

$$|p_n^{k+1} - p_n^{\pi(j)}| \leq |p_n^k - p_n^{k+1}| \text{ for all } n \geq N.$$

That is a contradiction to Lemma 2.3. Hence for large n there are at least four return points in S_n.

Second Step. We show that for large n there must exist two return points $r_n^1 \lhd r_n^2$ incident along $TSP(S_n)$ such that the other return points $\tilde{r}_n^1$ incident to r_n^1 and $\tilde{r}_n^2$ incident to r_n^2 along $TSP(S_n)$ are not in between r_n^1 and r_n^2. That is, one cannot find the following situation

$$r_n^1 \lhd \tilde{r}_n^1 \lhd r_n^2 \text{ or } r_n^1 \lhd \tilde{r}_n^2 \lhd r_n^2. \tag{2.8}$$

But it is possible that $\tilde{r}_n^1 = r_n^2$ and $\tilde{r}_n^2 = r_n^1$.

The path $P_n^1 \subset TSP(S_n)$ connecting p_n^1 with $p_n^{|S_n|}$ either contains two return points as we are looking for or it does not contain any return point besides p_n^1 and $p_n^{|S_n|}$. So assume that P_n^1 does not contain any return point besides p_n^1 and $p_n^{|S_n|}$. Cut $TSP(S_n)$ in three paths P_n^1, P_n^2 with endpoint p_n^1 and P_n^3 with endpoint $p_n^{|S_n|}$. At least one of the the paths P_n^2, P_n^3 cannot be empty since S_n has at least four return points. From Lemma 1.5 and the definition of Jordan length we can conclude that $\lim_{n\to\infty} L(P_n^1) = L(\gamma)$. This implies

$$\lim_{n\to\infty} L(P_n^2) = \lim_{n\to\infty} L(P_n^2) = 0.$$

Without loss of generality we can assume that P_n^2 is not empty for all sufficiently large $n \in \mathbb{N}$. The same reasoning as in the end of the first step shows that this leads to a contradiction. Hence for large n the sets S_n have to contain two return as we are looking for.

Third Step. In this step the transition from the local version of the theorem to the global one is done. This step is the same as the third step in the proof of Theorem 2.8. □

The conditions in Theorems 2.8 and 2.10 are necessary. Consider the same example that we used to show that the conditions in Theorem 2.1 are necessary. That is, let γ be the arc which consists of the unit interval on the x-axis and the graph of $y = x^2$ on this interval. Parameterize γ as follows:

$$\gamma : [0,1] \to \mathbb{R}^2, t \mapsto \begin{cases} (1-2t, 0) & : \ t \leq \frac{1}{2} \\ (2t-1, (2t-1)^2) & : \ t > \frac{1}{2} \end{cases}$$

For large n the samples

$$S_n = \{p_n^1, p_n^2, p_n^3, p_n^4\} \cup \bigcup_{i=2}^{n} \left\{ \left(\frac{i}{n}, 0\right), \left(\frac{i}{n}, \frac{i^2}{n^2}\right) \right\}$$

with

$$p_n^1 = \left(\frac{1}{n}, 0\right) \ , \ p_n^2 = \left(\frac{1}{n^3}, \frac{1}{n^6}\right) \ , \ p_n^3 = \left(\frac{2}{n^3}, \frac{4}{n^6}\right) \ , \ p_n^4 = \left(\frac{1}{n^2}, \frac{1}{n^4}\right)$$

become arbitrary dense in γ. But the Traveling Salesman path through S_n is different from the polygonal reconstruction from S_n because

$$|p_n^1 - p_n^2| + |p_n^2 - p_n^3| + |p_n^3 - p_n^4| \quad > \quad |p_n^1 - p_n^3| + |p_n^3 - p_n^2| + |p_n^2 - p_n^4|.$$

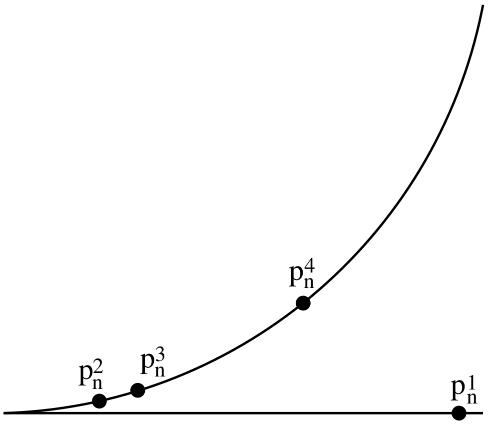

Figure 2.18: $p_n^1 - p_n^3 - p_n^2 - p_n^4$ *is shorter than* $p_n^1 - p_n^2 - p_n^3 - p_n^4$

Hence the conditions in Theorem 2.8 are necessary and sufficient. If we close γ in a suitable way we see that also the conditions in Theorem 2.10 are necessary and sufficient

Miscellaneous

3.1 Practical Reconstruction

Using Lemmas 2.3 and 2.4 we can prove the correctness of two simple algorithms which can reconstruct regular curves. The first algorithm computes a path or a tour. The algorithm holds in every iteration of its main loop a path until all sample points are vertices of this path. Again γ denotes a regular curve and $S = \{p^1, \ldots, p^{|S|}\}$ a sample of γ.

Algorithm 3.1

1) *Choose an arbitrary point $p^i \in S$ and connect it to one of its nearest neighbors in $S - \{p^i\}$. This defines a path with two vertices v^1 and v^2.*

2) *Consider the path with vertex set $V = \{v^1, \ldots, v^j\}$, s.t. v^k is connected to v^{k+1} for $k = 1, \ldots, j-1$. Calculate nearest neighbors v^- of $v^1 \in V$ and v^+ of $v^j \in V$ in $S - V$. Connect v^1 to v^- if $|v^1 - v^-| \leq |v^j - v^+|$. Otherwise connect v^j to v^+.*

3) *Repeat step 2 until $j = |S|$.*

4) *If γ is closed then connect v^1 with $v^{|S|}$.*

Let $A(S)$ denote the polygon which is computed by the algorithm. Next we prove that for dense samples $A(S)$ equals the polygonal reconstruction $P(S)$

Theorem 3.1 *There exists an $\varepsilon > 0$ such that $A(S) = P(S)$ for all samples S of γ with $\varepsilon(S) < \varepsilon$.*

PROOF. Look at the construction of $A(S)$. In the first step one connects an arbitrarily chosen p^i to one of its nearest neighbors. From Lemma 2.3 one

knows that this neighbor has to be one of p^{i-1}, p^{i+1} if $\varepsilon(S)$ is sufficiently small. Hence p^i is connected to p^{i-1} or p^{i+1} for sufficiently small $\varepsilon(S)$.
In the second step one of the endpoints of the polygon with vertex set $V = \{v^1, \dots, v^j\}$ is connected with one of its nearest neighbors. Without loss of generality let $v^1 = p^k$ be this endpoint and let $p^{k+1} = v^2 \in V$. Assume that $v^- = p^l$, $l \neq k-1$. That is $|p^k - p^l| \leq |p^k - p^{k-1}|$. Then there are the following possibilities:

1) $l \lhd k-1$. From Lemma 2.3 one knows that this cannot happen if $\varepsilon(S)$ is sufficiently small.

2) $l \rhd k+1$. By induction one can assume that (here the indices are taken modulo n)

$$v^j = p^{k+j}, v^{j-1} = p^{k+j-1}, \dots, v^1 = p^k.$$

That means $l \notin \{k, \dots, k+j\}$ and therefore $k+j \lhd l$. Distinguish two cases

   a) $l = k+j+1$: It holds $|p^{k+j+1} - p^k| < |p^{k+j+1} - p^{k+j}|$.
   That contradicts Lemma 2.3 in the point p^{k+j+1} for sufficiently small $\varepsilon(S)$.

   b) $l \rhd k+j+1$: It holds $|p^l - p^k| < |p^{k+j+1} - p^k|$.
   That contradicts Lemma 2.4 in the point p^k for sufficiently small $\varepsilon(S)$.

That means p^k is connected to $v^- = p^{k-1}$ for sufficiently small $\varepsilon(S)$. If the last step is repeated until $j = |S|$ one gets a path in which every vertex v^j, $j = 1, \dots, |S|-1$ is connected to the same neighbors as in $P(S)$. This remains valid if γ is closed and one connects $v^{|S|}$ with v^1. □

Since the algorithm starts in an arbitrary sample point it is natural to ask whether one can start in all sample points simultaneously. This leads to a variant of Algorithm 3.1 which does not hold just one path but a collection of vertex distinct paths in every iteration of its main loop. Again let $S = \{p^1, \dots, p^{|S|}\}$ be a sample of a regular curve γ.

Algorithm 3.2

1) Let $\mathcal{P} := \Big\{\{p^1\}, \ldots, \{p^{|S|}\}\Big\}$ be a set of trivial paths. That is, all paths in P consist of just one vertex.

2) Connect two paths $P_i, P_j \in \mathcal{P}$ if there exist two endpoints, one of each path, such that these endpoints are nearest neighbors of each other. That is, the set P of paths is updated as follows*

$$\mathcal{P} := \Big(\mathcal{P} - \{P_i, P_j\}\Big) \cup \{P_i \cup P_j\}$$

3) Repeat step 2 as long as connections are possible.

4) For every path in $\mathcal{P}$ connect its endpoints, if these endpoints are adjacent via a Delaunay edge.

** A nearest neighbor of an endpoint p of a path is another endpoint of a different path which has the shortest distance to p among all such endpoints.*

This algorithm also computes the polygonal reconstruction for sufficiently dense sampling. The output of Algorithm 3.2 need not consist of exactly one component, in general it is a collection of simple polygons which can be open or closed. But in contrast to Algorithm 3.1 the output cannot contain crossing edges. Thus Algorithm 3.2 is better suited for practical purposes. Using Definition 2.7 of a good sample one finds that both algorithms do not need uniform sampling.

Althaus and Mehlhorn [AM] give another proof of Theorem 2.8 which has two advantages. First, it is practical. That is, it shows that the Traveling salesman tour actually can be computed in polynomial time. Second, it shows that uniform sampling is not needed. They study the subtour elimination linear program to solve the Traveling salesman problem:

$$\min \sum_e \mathrm{length}(e) x(e)$$

1) $0 \le x(e) \le 1$

2) $x(\delta(\{p\}, S - \{p\}) = 2, \ \forall p \in S$

3) $x(\delta(S', S - S')) \ge 2, \ \forall S' \subset S, S \neq \emptyset, S$

Here $x(e)$ is an indicator variable, which is 1 if the edge e is in the tour, and $x(\delta(S', S - S'))$ is the sum over all indicator variables of edges in the cut $(S', S - S')$. Althaus and Mehlhorn can prove the following theorem:

Theorem 3.2 *Let $\gamma : [0,1] \to \mathbf{R}^2$ be a simple s-regular curve with*

$$\alpha = \sup\{\angle(\mathrm{l}(q), \mathrm{r}(q)) \,:\, q \in \gamma\} < \pi$$

then there exists $\varepsilon > 0$ such that the subtour elimination LP has an unique optimal integer solution for all samples S with $\varepsilon(S) < \varepsilon$.

That is, in the plane the Traveling salesman problem can be solved in polynomial time provided the points are sampled sufficiently dense from some curve. Corollary 2.9 can be used to speed up the LP computation, because it reduces the set of candidate edges.

It seems that even in practice the Traveling Salesman tour is a very good reconstruction scheme [AMNS]. It is fast and always outputs a simple tour. On simple curves that might be sampled quite sparse, but without noise, the Traveling salesman tour does the reconstruction favorably better than any other known reconstruction scheme [AMNS]. But reconstruction with the Traveling salesman tour also has its drawbacks. It cannot cope with several connected components and is not able to do the correct reconstruction for samples with noise, see Figure 3.1.

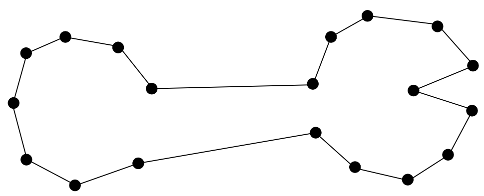

Figure 3.1: *The Traveling salesman tour consists of only one component*

In contrast the crust does well with several connected components and it can cope with some noise, see Figure 3.2.

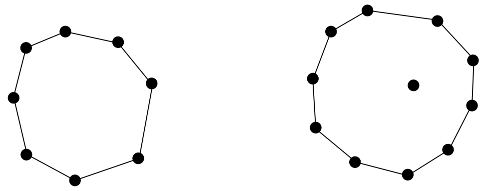

Figure 3.2: *The crust can cope with some noise and several connected components*

On the other hand the crust cannot cope with sharp corners. For practical purposes it seems to be good to combine both techniques. First identify the connected components with the crust or the λ-filter. Afterwards do the reconstruction of every connected component with the Traveling salesman tour.

Althaus et al. [AMNS] examine the practical behavior of several known curve reconstruction algorithms. Their results can be summarized as follows: Although in theory the running time of all filter based algorithms and the time to compute a minimum spanning tree is bounded by the time to compute a Delaunay triangulation, which is $O(|S| \log |S|)$ for a sample S in the plane, the running times of the diverse algorithms differ substantially in practice. Besides the TSP, which gives the best results but takes longest to compute, the algorithm of Dey and Kumar [DK], which is similar to our Algorithm 3.1, computes the best reconstruction. Furthermore it turns out to be the fastest algorithm of the algorithms known so far.

3.2 Reconstruction in Metric Spaces without Linear Structure

In this section we consider curves $\gamma : I \to M$ in some metric space M. We claim that even such curves can be faithfully reconstructed by the Traveling Salesman path (tour) provided the curves are regular and the samples are sufficiently dense. Here we only curves of type I. To do so we need a definition of regularity that does not rely on a linear structure on M.

Regularity has two aspects. The first one is algorithmical in a certain sense: We can approximate a regular function locally by a linear one, which allows us to make use of the apparatus of Linear Algebra. The second aspect is tameness, which means that a regular function cannot look very weird. This aspect is independent of a underlying linear structure on the range space of the function. It only makes use of the metric structure. Here we have to concentrate on the second aspect since we lack a linear structure on M.

Our definition of regularity is motivated by Definition 1.1 and Lemma 1.1.

Definition 3.1 *A curve $\gamma : I \to M$ is called regular in $p \in \gamma$ iff for any three sequences $(p_n), (q_n)$ and (r_n) in γ that converge to p such that $p_n < q_n < r_n$ in the order along γ locally around p it is*

$$\lim_{n\to\infty} -\frac{d(p_n, r_n)^2 - d(p_n, q_n)^2 - d(q_n, r_n)^2}{2d(p_n, q_n)d(q_n, r_n)} = -1.$$

Here d denotes the metric on M. A curve γ is called regular if it is regular in every point.

Left-, right and semi regularity are similarly defined. The only difference is that we make use of sequences that converge to p from the left or from the right respectively.

Using Definition 3.1 instead of Definition 1.1 the proof of Theorem 2.5 remains valid for curves in arbitrary metric spaces. That is another advantage of our proof compared to the proof in [FG]. Furthermore Algorithms 3.1 and 3.2 work for such curves. We should remark that in the case of abstract metric spaces the graphs $P(S), MST(S), TSP(S)$ and $A(S)$ for some sample S are no more geometric graphs but subgraphs the complete graph on $|S|$ vertices weighted with the distances induced by the metric d.

It is an interesting question if Theorems 2.5 and 2.6 can be generalized in the abstract metric setting like in the case of curves in $\mathbb{R}^d$.

3.3 Surface Reconstruction

3.3.1 Geometry of Surfaces

It is not only very difficult to define what the correct reconstruction of a surface from a finite sample is. The lack of a canonical inscribed triangulation makes it also very difficult to define the geometry of a surface. The first who realized this was Schwartz. He studied a family of triangulations inscribed in a cylinder which converges to the cylinder, but the area of the triangulations does not converge to the area of the cylinder.

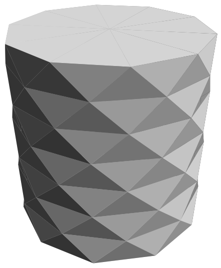

Figure 3.3: *Schwartz cylinder*

Assume that the cylinder has height h and circumference r. Divide the height in m and the circumference in n equal parts such that the points on each circumference lie in the middle of two points on the circumference that lies below. Triangulate the points as in Figure 3.3. This triangulation consists of $2mn$ triangles. Each of this triangles has has a base of length $2r\sin(\pi/n)$ and height

$$\sqrt{\left(\frac{h}{m}\right)^2 + r^2\left(1-\cos(\frac{\pi}{n})\right)^2}.$$

Hence one gets for the surface area S_{mn} of the triangulation:

$$S_{mn} = 2nr\sin(\pi/n)\sqrt{h^2 + m^2r^2\left(1-\cos(\frac{\pi}{n})\right)^2}$$

Suppose that $\lim m/n^2 = q$. Then it holds:

$$\lim_{n\to\infty} S_{qn^2,n} = 2\pi r\sqrt{\frac{\pi^2 r^2}{4}q^2 + h^2}$$

Hence the area of the cylinder is well approximated by the inscribed triangulation only for $q = 0$. Since $q \geq 0$ the surface area of the cylinder can never be underestimated by a family of inscribed triangulations. Analogous results hold for surface curvatures. This is the main motivation to study optimization criterions in reconstruction.

3.3.2 Extending Reconstruction Schemes

Amenta et. al. [AB] extend the crust to three dimensions and use it for surface reconstruction of smooth surfaces. There they have the problem that Voronoi vertices can lie very close to the surface. Such vertices can filter out triangles which belong to the ‘correct’ reconstruction. That is, in three dimensions problems occur that also occur for irregular curves.

Attali [At] considers the normalized mesh also for surface reconstruction. She cannot generalize her proof for curves to this case. The proof in the case of curves makes use of a regularity assumption which implies that the Delaunay spheres of the sample points become tangential to the curve for large sampling densities. That is no longer the case for irregular curves and smooth surfaces.

This shows that it is interesting to study irregular curves, because problems with these curves can show up also for smooth surfaces. At this point we should say that Amenta et. al. developed techniques to cope with this problems. They can prove that the output of their algorithm approximates the surface normals of smooth surfaces well. This implies that the surface area of smooth surfaces is approximated well.

In both cases, the crust and the normalized mesh, we have the problem of missing triangles. When we constructed examples that show that some reconstruction schemes cannot reconstruct polygons we always constructed examples that show that some edges from the correct reconstruction are missing. In all these cases it is also possible to find examples that contain edges that are not in the correct reconstruction. But missing edges are worse, because in many cases auxiliary edges can be removed by topological reasons. For example a reconstruction with auxiliary edges, but without missing edges, can be used as input for a TSP computation. The situation is similar in surface reconstruction. It is difficult to fill holes, but additional triangles can be removed by topological reasons. Based on this observation we suggest a two step procedure for surface reconstruction. First, a conservative filter step. Here the generalization of the λ-filter to surfaces can be used. Second, an optimization step based on a linear program that ensures the topology of a surface. Like in the curve case the topology of piecewise linear surfaces can be encoded in linear inequalities.

3.4 A Conjecture

The following counterexample shows that the analogue of Menger's theorem 1.6 for the total absolute curvature does not hold. Let γ be the following arc in the plane:

$$\operatorname{conv}\{(0,0),(2,0)\} \cup \operatorname{conv}\{(2,0),(1,1-\delta)\} \cup \operatorname{conv}\{(1,1-\delta),(3,1-\delta)\},$$

for some positive $\delta \ll 1$. The curvature of γ is $\sim 3\pi/2$. Now consider the following sample

$$S = \{(0,0),(1,0),(2,0),(1,1-\delta),(2,1-\delta),(3,1-\delta)\}.$$

A shortest path through the points S is

$$\begin{aligned} & \operatorname{conv}\{(0,0),(1,0)\} \\ \cup \quad & \operatorname{conv}\{(1,0),(1,1-\delta)\} \\ \cup \quad & \operatorname{conv}\{(1,1-\delta),(2,1-\delta)\} \\ \cup \quad & \operatorname{conv}\{(2,1-\delta),(2,0)\} \\ \cup \quad & \operatorname{conv}\{(2,0),(3,1-\delta)\}. \end{aligned}$$

The curvature of this path is $\sim 9\pi/4$ which is larger than the curvature of γ. A similar counterexample can be given for curves of type II.

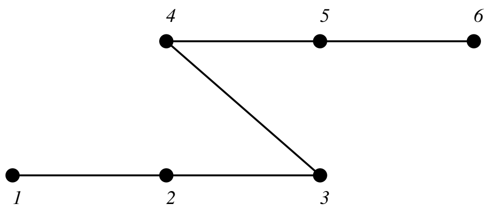

Figure 3.4: *1-2-4-5-3-6 is shorter than 1-2-3-4-5-6*

But nevertheless we conjecture that the analogue of Corollary 1.7 of Menger's theorem for the total absolute curvature holds.

Conjecture 3.1 *Let γ be a curve and let (S_n) be a sequence of samples of γ, with $\lim_{n\to\infty} \varepsilon(S_n) = 0$. Then*

$$C(\gamma) = \lim_{n\to\infty} C(TSP(S_n)) = \infty.$$

From Theorems 2.8 and 2.10 we know that this conjecture holds for s-regular curves, where in no point left- and right tangent make an angle of π with each other. Next we show that Conjecture 3.1 holds for curves of infinite curvature. For the proof we need a theorem first proved, in the special case of regular arcs, by E. Schmidt [S]. A proof for the general case can be found in [AR].

Theorem 3.3 *Let γ be a curve of type I, such that $C(\gamma) < \pi$. Let d be the distance of the endpoints of γ, then*

$$d \geq L(\gamma) \cos\left(C(\gamma)/2\right).$$

□

Now we can prove that conjecture 3.1 holds for curves of infinite curvature.

Lemma 3.1 *Let γ be a curve of infinite curvature and let (S_n) be a sequence of samples of γ, with $\lim_{n\to\infty} \varepsilon(S_n) = 0$. Then*

$$C(\gamma) = \lim_{n\to\infty} C(TSP(S_n)) = \infty.$$

PROOF. First consider the case that γ has finite length. From Theorem 1.8 and the lower semi-continuity of the curvature (Theorem 1.9) we can conclude that

$$\infty = C(\gamma) \leq \liminf C(TSP(S_n)) \leq \limsup C(TSP(S_n)).$$

That is, $\lim_{n\to\infty} C(TSP(S_n)) = \infty$.

Now consider the case that γ has infinite length and assume that γ is a curve in $\mathbb{R}^{d+1}$. We want to do the proof by contradiction. To do so we assume that there exists $0 \leq c < \infty$ such that

$$C(TSP(S_n)) \leq c$$

for infinitely many $n \in \mathbb{N}$. By turning to an appropriate subsequence we can assume that this is the case for all $n \in \mathbb{N}$. Furthermore we can assume that $c = m\frac{\pi}{2}$ for some $m \in \mathbb{N}$. The compactness of γ and the continuity of the projection maps π_ℓ imply

$$\sup_{n\in\mathbb{N}} \sup_{\ell\in\mathbb{P}^d} \max\left\{|\pi_\ell(p_n) - \pi_\ell(q_n)| \,:\, (p_n, q_n) \in S_n \times S_n\right\} < \infty. \tag{3.1}$$

We cut all paths (tours) $TSP(S_n)$ in m paths P_n^i, $i = 1, \dots m$, of equal length. Using the additivity of length and the subadditivity of curvature, see [AR] for a proof, we find

$$L(TSP(S_n)) = \sum_{i=1}^{m} L(P_n^i)$$

and

$$C(TSP(S_n)) \geq \sum_{i=1}^{m} C(P_n^i).$$

That is, for all $n \in \mathbb{N}$ we find an $i(n) \in \{1, \dots, m\}$ with

$$L(P_n^{i(n)})) = \frac{1}{m} L(TSP(S_n)) \text{ and } C(P_n^{i(n)}) \leq \frac{\pi}{2}.$$

Let $\ell_n \in \mathbb{P}^d$ be the line passing through the endpoints p_n and q_n of $P_n^{i(n)}$. Using Theorem 3.3 we find

$$|\pi_{\ell_n}(p_n) - \pi_{\ell_n}(q_n)| \geq \frac{1}{\sqrt{2}m} L(TSP(S_n)).$$

Together with Corollary 1.7 it follows,

$$\lim_{n\to\infty} |\pi_{\ell_n}(p_n) - \pi_{\ell_n}(q_n)| \geq \lim_{n\to\infty} \frac{1}{\sqrt{2}m} L(TSP(S_n)) = \infty,$$

which is a contradiction to inequality 3.1. □

To prove Conjecture 3.1 it remains to consider curves of finite curvature where in some points left- and right tangents make an angle of π with each other.

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Curriculum Vitae

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