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On k-sets and their Generalizations

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Abstract

Among fundamental geometric objects k-sets are the ones which have challenged a lot of researchers in the fields of computational and combinatorial geometry. This can be partially attributed to the fact that k-sets occur in many forms in geometry. But the main reason is that obtaining matching lower and upper bounds on the maximum number of k-sets is still an open problem.

In this thesis we study k-sets and (i, j)-partitions, which generalize k-sets. The results comprise contributions to both combinatorial and computational geometry. In the first research area we refine a technique for continuous motions of points and obtain a complete characterization in all dimensions of the changes in the numbers of (i, j)-partitions during such motions. Based on these results we extend the index range for which the so-called generalized Dehn-Sommerville relations for (i, j)-partitions are valid.

We also treat the redundancy in the matrix of numbers of (i, j)-partitions of a point set in $\mathbb{R}^d$, studying the relations between the numbers of k-sets and the numbers of j-facets (i.e. the (d, j)-partitions). For $d = 3$ it turns out that the numbers of k-sets are determined by the numbers of j-facets and vice versa. We show that in higher dimensions there is no unique relation between the numbers of k-sets and the numbers of j-facets or vice versa.

Furthermore we investigate the so-called k-set polytopes, obtaining a bijection between the $(i-1)$-faces of a k-set polytope and the (i, j)-partitions of a point set. In consequence of this bijection some additional relations for the numbers of (i, j)-partitions for a point set arise.

Except for these combinatorial results, the study of k-set polytopes yields two algorithms for enumeration of k-sets of a point set in general position. They are based on the reverse search technique, which guarantees low memory requirements and a simple implementation.

Kurzfassung

Viele Forscher aus den Gebieten der kombinatorischen und der algorithmischen Geometrie wurden durch die k-Mengen betreffende Fragestellungen herausgefordert. Das kann durch die Tatsache erklärt werden, daß die k-Mengen in vielen Formen in der Geometrie auftreten. Jedoch liegt der Hauptgrund darin, daß man bis heute die Lücke zwischen den oberen und den unteren Schranken für die Anzahl der k-Mengen nicht schliessen konnte.

In dieser Doktorarbeit studieren wir k-Mengen und ihre Verallgemeinerungen, die (i, j)-Partitionen. Die Ergebnisse umfassen Beiträge zur kombinatorischen und zur algorithmischen Geometrie. In dem ersten Forschungsgebiet verfeinern wir die Technik der stetigen Punktbewegungen und erhalten eine vollständige Charakterisierung (in allen Dimensionen) der Änderung der Anzahl der (i, j)-Partitionen während solcher Bewegungen. Aufbauend auf diesen Resultaten erweitern wir den Indexbereich, in welchem die sogenannten verallgemeinerten Dehn-Sommerville Relationen für die (i, j)-Partitionen gelten.

Wir behandeln auch die Redundanz der Matrix der Anzahlen der (i, j)-Partitionen einer Punktmenge im $\mathbb{R}^d$, indem wir die Beziehungen zwischen den Anzahlen der k-Mengen und den Anzahlen der j-Facetten (d.h. den (d, j)-Partitionen) studieren. Es stellt sich heraus, daß für $d = 3$ die Anzahlen der k-Mengen durch die Anzahlen der j-Facetten bestimmt sind und umgekehrt. Wir zeigen, daß es in höheren Dimensionen keine eindeutige Beziehungen zwischen den Anzahlen der k-Mengen und den Anzahlen der j-Facetten gibt.

Weiterhin untersuchen wir k-Mengen-Polytope, und erhalten eine Bijektion zwischen den $(i - 1)$-dimensionalen Seiten eines k-Mengen-Polytopes und den (i, j)-Partitionen einer Punktmenge. Als Folge dieser Bijektion ergeben sich einige zusätzliche Beziehungen für die Anzahlen der (i, j)-Partitionen einer Punktmenge.

Außer den kombinatorischen Ergebnissen ergibt die Untersuchung der k-Mengen-Polytope zwei Algorithmen zum Aufzählen der k-Mengen einer Punktmenge in allgemeiner Lage. Diesen Algorithmen liegt die Technik der Rückwärtssuche zugrunde, die einen geringen Speicherbedarf und eine einfache Implementierung der Algorithmen garantiert.

Contents

Chapter 1

Introduction

Computational geometry is the area of research which studies efficient ways of processing large amounts of simple geometric objects. It emerged in the early seventies, when computers became powerful enough to handle graphics and large amounts of data. A typical problem in computational geometry is the question

> "*How to compute efficiently* all partitions of a point set into two cardinality-equal subsets defined by hyperplanes?"

A closely related area of research is the considerably older field of *combinatorial geometry*, which treats questions like

> "*How many* partitions of a point set into two cardinality-equal subsets defined by hyperplanes are there?"

Frequently, as in the above example, the combinatorial structure of a problem decides which algorithm solves it most efficiently. On the other hand, computational geometry brings up questions which initiate new research areas in combinatorial geometry.

In this thesis we consider geometric objects which are a perfect example of this symbiosis: k-sets and their variations. In the following, we illustrate that k-sets are of considerable interest in both areas of research. Given a set S of n points in $\mathbb{R}^d$, a *k-set* is a subset of k points of S which can be strictly separated from the remaining points by a hyperplane. A useful generalization of k-sets are so-called *(i, j)-partitions*: they are pairs (A, B) of subsets of S such that $|A| = i$, $|B| = j$ and there is a hyperplane h with $h \cap S = A$, $h^+ \cap S = B$.

1.1 k-sets and combinatorial geometry

In combinatorial geometry, the research related to k-sets is focused on the *k-set problem*:

> find matching lower and upper bounds on the maximum number of k-sets of point sets with n points in $\mathbb{R}^d$.

Studies of the k-set problem began in 1970, when ERDŐS, LOVÁSZ, SIMMONS and STRAUS published first lower and upper bounds on the maximum number on k-sets in the plane [ELSS73]. The same bounds have been independently confirmed 1985 by EDELSBRUNNER and WELZL in [EW85]. An improvement of the upper bound in the plane has been obtained by PACH, STEIGER and SZEMERÉDI [PSS92], but only by a factor of $O(1/\log^*(k+1))$. It took further 6 years until this upper bound could be more significantly improved to $O(nk^{1/3})$ by DEY [DEY98]. Only few months ago TÓTH [Tót99] proved a better lower bound than the one given in [ELSS73, EW85] for the plane; it can be even extended to higher dimensions. For $\mathbb{R}^3$, DEY and EDELSBRUNNER [DE94] obtained 1994 an upper bound $O(n^{8/3})$ on the number of $(n/2)$-sets. This bound has been improved 1998 by AGARWAL, ARONOV, CHAN and SHARIR [AACS98] to $O(nk^{5/3})$ for arbitrary k-sets in $\mathbb{R}^3$. For dimensions higher 3, the only known upper bound is $O(n^{d-\varepsilon})$, where $\varepsilon > 0$ is a small constant [ABFK92, ŽV92]. The last chapter of this thesis gives a more detailed account on the historical development of the research on the k-set problem.

In this thesis we study two research fields which are linked to the k-set problem. First, we refine the technique of continuous motion of points which already yielded some interesting results [AAH$^+$98, Lin94, GH97]. This technique exploits the fact that while a point set is moving under some conditions, the number of (i,j)-partitions changes only if the general position condition imposed on the point set is violated. We characterize in all dimensions the changes in the numbers of (i,j)-partitions when the general position condition is violated. We also study the cumulative change of the numbers of (i,j)-partitions during the movement of the point set dependent on the numbers of violations of the general position condition.

The second field of research related to the k-set problem studied here is the analysis of the combinatorics between (i,j)-partitions and their two extreme special cases: k-sets and *j-facets*, which are exactly the (d,j)-partitions. We obtain a set of relations for the numbers of (i,j)-partitions by studying the so-called k-set polytopes. Another set of such relations is provided by the so-called generalized Dehn-Sommerville relations for the numbers $D_{i,j}$ of (i, j)-partitions. Such relations have been obtained by MULMULEY for arrangements of hyperplanes and can be translated for (i, j)-partitions. Another approach to showing the generalized Dehn-Sommerville relations is given by results on continuous motions of points (mentioned above). By these results we can increase the index range for which these relations hold. One interesting aspect of these results is the emphasis of the link between the k-set problem and polytope theory. In particular, it is known that the k-set problem is closely related to the Upper Bound Theorem [Mul93, Wel99].

In order to further analyse the combinatorics of (i, j)-partitions we investigate the redundancy in the matrix $D = (D_{i,j})$ and rise the question: which rows of D are determined by other rows? For $\mathbb{R}^3$ we obtain the answer that the numbers of k-sets determine the numbers of j-facets and vice versa, and we show that in higher dimensions more than one row of D is necessary to determine other rows of D. While such relations between the numbers of k-sets and numbers of j-facets have been previously known in an asymptotic form, the goal here was to obtain the *exact* relations. This is important, as the tight upper bounds on the numbers of so-called $(\leq j)$-facets (i.e. i-facets for $i = 0, \ldots, j$) have been refined recently and hence they are now known in the exact form for dimensions 3 [AAH$^+$98, Wel99]; see also [Lin94].

1.2 k-sets and computational geometry

k-sets are linked to several notions in computational geometry: k-levels of hyperplane arrangements [AW97], higher order Voronoi Diagrams [Aur91], orthogonal L_1-line fitting [YKII88], halfspace range searching [CP86], weak separators misclassifying at most k points [ERv93], and some other objects [AW97]. A special case of k-sets are so-called corner cuts [OS99], which occur in various contexts, for example in computer vision.

As a tribute to this fact we discuss objects related to k-sets known in the literature, especially j-facets, k-levels and k-set polytopes. We derive several new results regarding k-set polytopes, in particular an interesting bijection between the $(i-1)$-faces of a k-set polytope and the (i, j)-partitions for $j \in \{k-i+1, \ldots, k-1\}$.

The fact that k-sets occur in many problems of computational geometry makes an efficient algorithm for the enumeration of k-sets of particular value. While the known algorithms [EW86, CSY87, AM95] for enumeration of k-sets rely on sophisticated data structures and are difficult to implement, the two algorithms presented here are free of these disadvantages. They are based on the technique of reverse search introduced by AVIS and FUKUDA [AF96]. This technique allows the creation of algorithms which are easy to implement, can be parallelized and have low memory requirements.

We also study the enumeration of so-called affine k-sets, which are k-sets such that there is a hyperplane which strictly separates a k-set and the origin of $\mathbb{R}^d$. These objects can be applied for a heuristic solution of the important problem MAX FLS of finding the maximum number of linear inequalities which can be simultaneously satisfied in an inequality system. Here we provide an example which indicates that the affine k-sets cannot be enumerated in an output-sensitive manner by a graph-search technique (the reverse search is such a technique).

1.3 Outline of the thesis and research contributions

Chapter 2 is devoted to geometric objects related to k-sets. In the first two sections of this chapter we overview the most widely known among these objects: $(\leq k)$-sets, j-facets and $(\leq j)$-facets. In Section 2.2 we discuss a bijection between the vertices of level j in a hyperplane arrangement and the j-facets of the dual point set. We also study k-levels and their relations to k-sets; among others, we restate the proofs of some known complexity results on k-levels.

Chapter 3 and Chapter 4 contain the main research contributions of this thesis. In Chapter 3 we study the (i, j)-partitions, especially their numbers. This chapter can be subdivided in three parts: in Sections 3.1 - 3.5 we study basic properties of (i, j)-partitions and obtain results on them using various techniques. In Sections 3.6 - 3.8 we develop a technique to study the changes of the numbers of (i, j)-partitions during continuous motions of points. Finally, Sections 3.9 - 3.11 are devoted to results on the numbers of (i, j)-partitions obtained by the technique developed in Sections 3.6 - 3.8.

In Section 3.1 we discuss the geometric duals of (i, j)-partitions in a hyperplane arrangement. Section 3.2 examines the properties of k-set polytopes. The most important result of this section is a bijection between the $(i-1)$-faces of a k-set polytope $Q_k(S)$ of a point set S and the (i, j)-partitions of S for $j \in \{k-i+1, \ldots, k-1\}$. As a conclusion, we obtain a set of relations among the numbers of (i, j)-partitions (by an application of the Euler-Poincaré formula and the Dehn-Sommerville relations for convex polytopes).

We also develop formulas for the numbers of (i, j)-partitions in terms of the numbers of $(i-1)$-faces of k-set polytopes (for certain k). In the subsequent Section 3.3 we derive the exact formulas for the numbers of (i, j)-partitions of a cyclic polytope with n vertices in $\mathbb{R}^d$ (these numbers depend only on n, d, i and j). It was known that such formulas must exist, but except for two special cases they have not been stated. Section 3.4 focuses on the estimates of the numbers of k-sets of a point set S in terms of the numbers of j-facets of S, and vice versa. We state the proofs of estimations and asymptotic relations which have been frequently used but have never been worked out carefully. In Section 3.5 we use duality to translate the results of MULMULEY [Mul93] obtained for hyperplane arrangements to results on the (i, j)-partitions of a dual point set. We discuss the numbers $h_{i,j}$ which play a similar role for the numbers $D_{i,j}$ as the h-vector plays for the f-vector of a convex polytope. Among the "translated" results stated in this section are the so-called generalized Dehn-Sommerville relations, which read $h_{i,j} = h_{d-i,j}$ for appropriate i and j. (In Section 3.9 we extend the index range of j for which these relations are valid). Our contribution in this section is the "translation" of the results from the dual setting and a geometric interpretation of the numbers $h_{i,j}$.

In Section 3.6 we introduce the notion of a mutation of an oriented matroid and a mutation of a point set, and classify them according to two integers m and l, obtaining so-called (m, l)-mutations. In addition, we discuss some basic properties of such mutations. In Section 3.7 we completely characterize the changes of the numbers $D_{i,j}$ of the (i, j)-partitions due to a single (m, l)-mutation (in any dimension) and elaborate a more simple formula for the cases of the k-sets and the j-facets. The next Section 3.8 studies a sequence of such changes of the matrix D during a continuous motion of a point set. We obtain a "condensed" formula for the matrix D of the final point set in terms of the numbers of (m, l)-mutations during the movement and in terms of the matrix $D(S_I)$ of the initial point set S_I.

These results are applied in the subsequent three Sections 3.9, 3.10 and 3.11. In Section 3.9 we extend the index range for which the above-mentioned Dehn-Sommerville relations $\overline{h}_{i,j} = \overline{h}_{d-i,j}$ are valid (where $\overline{h}_{i,j}$ is an "extension" of $h_{i,j}$). In addition, using solely the results from Section 3.9 we obtain the generalized Dehn-Sommerville relations "modulo a constant" (for the full index range), i.e. $\overline{h}_{i,j} = \overline{h}_{d-i,j} + C_{n,d,i,j}$, where $C_{n,d,i,j}$ depends only on n, d, i, j. This gives hope to show the generalized Dehn-Sommerville relations completely independently of the results in [Mul93]. In Section 3.10 we answer the question how the numbers of k-sets of S (i.e. the first row of the matrix D) can be computed from the numbers of j-facets of S (i.e. the last row of D) and vice versa, where S is a point set in $\mathbb{R}^3$. This result relies upon a Corollary from Section 3.7. In Section 3.11 we show that in dimensions greater 3 the first row of D does not determine the last row of D and vice versa. More precisely, we show that for each dimension there are pairs of point sets such that the first rows of their matrices D are equal, but the last rows are different (and vice versa). In addition, this construction implies that there are many pairs of point sets with this property.

In Chapter 4 we develop two algorithms for enumeration of k-sets of a point set in general position. We begin with a comparison of the known algorithms for this problem. In Section 4.1 we outline the framework of the reverse search technique which is applied in both algorithms. In Section 4.2 we study the notion of polytopal neighbors in a k-set polytope $Q_k(S)$ and show how to efficiently determine all such polytopal neighbors of a vertex of $Q_k(S)$. We also discuss one possible way to optimize a linear function on $Q_k(S)$. These results are used in the next section, where we obtain an algorithm based on the notion of polytopal neighbors for enumeration of k-sets of a given point set in $\mathbb{R}^d$. In Section 4.4 we examine the notion of combinatorial neighbors in $Q_k(S)$ and apply it to design the second algorithm for the enumeration of k-sets. The last Section of Chapter 4 treats affine k-sets. The most important result here is the fact that the 1-skeleton of a k-set polytope restricted to the vertices corresponding to the affine k-sets is not always connected.

Chapter 5 contains three smaller pieces. First, we give an alternative proof of the upper bound on the number of $n/2$-sets in $\mathbb{R}^3$ (Section 5.1). In the subsequent section we derive a formula for the exact number of j-facets of the vertex set of a simplicial, neighborly polytope. Section 5.3 is devoted to the proof of a lower bound on the maximum number of median intersections in an arrangement of pseudolines due to KLAWE, PATERSON and PIPPENGER [KPP82] (this proof has been never before written up in detail).

In Chapter 6 we overview the known bounds on the numbers of k-sets and related geometric objects. We list older and current bounds for k-sets, $(\leq k)$-sets, j-facets and $(\leq j)$-facets for the plane (Section 6.1) and for higher dimensions (Section 6.2). In the last section of Chapter 6 we give an overview on the results for some variations of k-sets and results for k-sets of special point sets.

1.4 Preliminaries

The following definitions, symbols and conventions are used throughout the whole dissertation. We denote by $S = \{p_1, \ldots p_n\}$ a set of n points in $\mathbb{R}^d$. If not otherwise stated, we assume that the points in S are in *general position*, i.e. no $i + 1$ points are on a common $(i - 1)$-flat, for $i = 1, \ldots, d$. If h is an oriented hyperplane in $\mathbb{R}^d$, then h^+ denotes the open halfspace on the positive side of h and h^- the open halfspace on the negative side of h; hence, $\mathbb{R}^d = h^+ \dot\cup h \dot\cup h^-$. An oriented, non-vertical hyperplane in $\mathbb{R}^d$ is said to be *upward-oriented* if it contains $(0, \ldots, 0, +\infty)$ on its positive side, otherwise it is said to be *downward-oriented*. For two sets A, B of points and a hyperplane h we say that h *strictly separates* A and B if $A \cap h = B \cap h = \emptyset$ and A is on the opposite side of h as B. For a point set A, its affine hull is denoted by aff(A). For a set H of hyperplanes in $\mathbb{R}^d$ we denote by $\mathcal{A}(H)$ their arrangement, see [Ede87]. The origin of $\mathbb{R}^d$ is denoted as $\mathbf{o}$.

The *moment curve* in $\mathbb{R}^d$ is the set $M_d = \{(t, t^2, \ldots, t^d) | t \in \mathbb{R}\}$ and the convex hull of n points in M_d is called a *cyclic polytope* $P_{n,d}$. A polytope P in $\mathbb{R}^d$ is called *neighborly*, if every set of at most $\lfloor d/2 \rfloor$ of vertices of P is a vertex set of a face of P. A polytope P is *simplicial* if every facet has a minimal number of d vertices, which applies especially if the vertices of P are in general position. All cyclic polytopes are simplicial and neighborly, see [Zie94]. For two vertices v, w of a polytope P their *distance in* P is the number of vertices of P in the shortest path from v to w minus 1 (so neighbors have distance 1). The *diameter* of a polytope P is the largest distance between two vertices of P. We call an i-dimensional face of a polytope P its i*-face.*

We assume that the binomial coefficient $\binom{a}{b}$ is 0, if $a < b$, or $a < 0$, or $b < 0$. For two integers i, j let $[i = j]$ be 1 if $i = j$ and 0 otherwise. We use the sum convention

$$\sum_{j=i}^{k} a_i = 0$$

if $i > k$ (for any a_i). For two vectors $\mathbf{v}$ and $\mathbf{w}$, let $\langle \mathbf{v}, \mathbf{w} \rangle$ denote their scalar product. For a matrix $\tilde{A}$ we denote by $\mathrm{rank}(\tilde{A})$ the rank of $\tilde{A}$ (the cardinality of a largest subset of linearly independent rows or columns of $\tilde{A}$). By lp(m, n) we denote the time necessary for solving a linear program with m variables and n inequalities. For algorithms for solving a linear program see [GJ97].

We use a following geometrical duality transformation $\mathcal{D}$ which maps points to hyperplanes and vice versa. The most important property is the incidence and order preservation explained below. For a more detailed discussion

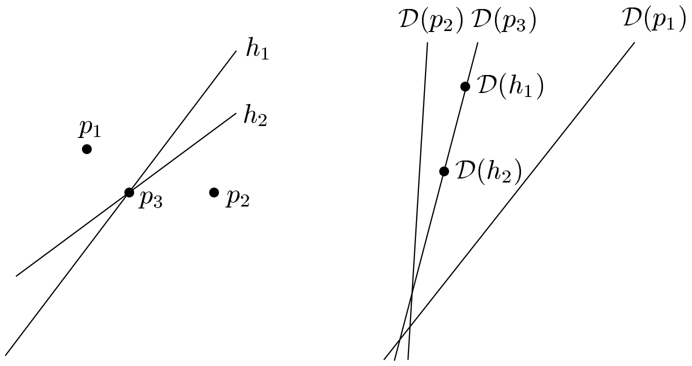

Figure 1.1: *An illustration of the duality transformation $\mathcal{D}$*

see Sections 1.4 and 3.1 of [Ede87]. It is also described in [Mul94]. The *paraboloidal transformation* $\mathcal{D}$ maps a point $p = (p_1, \ldots, p_d)$ in $\mathbb{R}^d$ to the hyperplane $\mathcal{D}(p)$ with equation

$$x_d = 2p_1x_1 + 2p_2x_2 + \ldots 2p_{d-1}x_{d-1} - p_d.$$

Vice versa, a non-vertical hyperplane h with equation

$$x_d = h_1x_1 + h_2x_2 + \ldots + h_{d-1}x_{d-1} + h_d$$

is mapped by $\mathcal{D}$ to the point

$$\mathcal{D}(h) = (h_1/2, h_2/2, \ldots, h_{d-1}/2, -h_d).$$

For a set S of points in $\mathbb{R}^d$ we denote by $\mathcal{D}(S)$ the set of hyperplanes obtained by applying $\mathcal{D}$ to the elements of S. Conversely, for a set H of non-vertical hyperplanes in $\mathbb{R}^d$ let $\mathcal{D}(H)$ denote the set of points obtained by applying $\mathcal{D}$ to the elements of H. The set of points is said to be in the *primary space*, while the set of hyperplanes is in the *dual space*. For a point p and a (non-vertical) hyperplane h in $\mathbb{R}^d$ we have the following properties:

(D1) *Incidence preservation*: p lies on h if and only if point $\mathcal{D}(h)$ lies on hyperplane $\mathcal{D}(p)$.

(D2) *Order preservation*: p lies above h if and only if point $\mathcal{D}(h)$ lies above the hyperplane $\mathcal{D}(p)$ (the same property holds with "above" interchanged by "below").

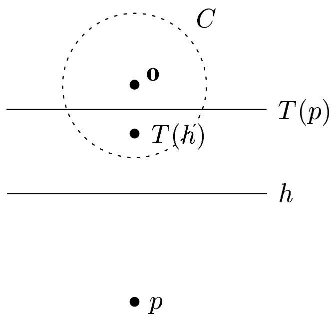

Figure 1.2: *An illustration of the polarity transformation T*

In Figure 1.1, the point p_3 is on the intersection of h_1 and h_2 and so by (D1) $\mathcal{D}(p_3)$ contains the points $\mathcal{D}(h_1)$ and $\mathcal{D}(h_2)$; also, p_1 is above h_1 and h_2 and so by (D2) the points $\mathcal{D}(h_1)$ and $\mathcal{D}(h_2)$ are above the line $\mathcal{D}(p_1)$.

Another geometric duality transformation used in this thesis is the polarity T [Mul94]. The *polarity transformation* maps a point $p = (p_1, \ldots, p_d)$ in $\mathbb{R}^d$ different from the origin **o** to the hyperplane $T(p)$ with equation

$$\langle p, x \rangle = p_1 x_1 + \ldots + p_d x_d = 1,$$

and vice versa. Note that if we consider the above equation as an equation of an oriented hyperplane, then the origin **o** is always in $T(p)^-$.

An important property of T is the fact that it maps a point p at a distance d from **o** to the hyperplane $T(p)$ at distance $1/d$ from **o** which is perpendicular to the unique line containing both **o** and p, see Figure 1.2 (C is a sphere of radius 1).

The transformation T is also incidence and order preserving. Let us assume till the end of this section that all hyperplanes which we consider are oriented in such a way that the origin **o** is always on the negative side. We have [Mul94] (see also Figure 1.2):

(T1) A point p lies on a hyperplane h if and only if the point $T(h)$ lies on the hyperplane $T(p)$.

(T2) A point p lies in a halfspace h^+ (resp. h^-) if and only if $T(h)$ lies in $T(p)^+$ (resp. $T(p)^-$).

We have also that if the origin **o** is placed in $(0, \ldots, 0, +\infty)$, then the property (D2) of the transformation $\mathcal{D}$ also applies to T. The proof is left to the reader.

Chapter 2

k-sets and related geometric objects

The main theme of this chapter are geometrical objects closely related to k-sets. We start with an overview of the most important variations of k-sets which can be found in literature: halving hyperplanes, $(\leq k)$-sets, j-facets and $(\leq j)$-facets (Section 2.1). Section 2.2 examines briefly the dual notion of j-facets, namely vertices of level j in arrangements of hyperplanes. In the same section we investigate k-levels, a notion of great importance in computational geometry. We show some relations between k-levels, vertices of level j and k-sets, and conclude with some complexity bounds on k-levels in terms of numbers of k-sets.

2.1 k-sets and related basic geometric objects

In this section we define (in addition to k-sets) the most important geometric objects, which are related to k-sets: halving hyperplanes, $(\leq k)$-sets, j-facets, halving facets and $(\leq j)$-facets. We do not show any theorems here, but we point to several basic properties of the above objects.

For $k \in \{1, \ldots, n\}$ a *k-set of S* is a subset $P \subseteq S$, $|P| = k$ that can be strictly separated from $S \backslash P$ by a hyperplane. We denote by $e_k(S)$ the number of k-sets of S and call $\mathbf{e} = \mathbf{e}(S) = (e_k(S))_{k=1}^{n-1}$ the *e-vector* of S. We define

$$e_k(d, n) = \max_{|S|=n, S \subseteq \mathbb{R}^d} e_k(S).$$

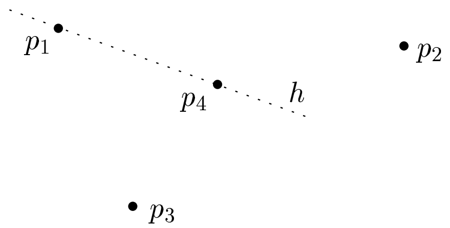

Figure 2.1: *The point set S_1 has three 1-sets*

Obviously we have $e_k(S) = e_{n-k}(S)$ and therefore $e_k(d,n) = e_{n-k}(d,n)$. It is easy to see that $e_1(d,n) = n$ for any d and n.

A *halving hyperplane* of S is a hyperplane h containing n mod 2 points of S such that there are $\lfloor n/2 \rfloor$ points of S on each side of h. In $\mathbb{R}^2$ we speak about *halving lines* and in $\mathbb{R}^3$ about *halving planes*. As the number of halving hyperplanes of S we understand the number of unordered pairs $\{S \cap h^-, S \cap h^+\}$ induced by (oriented) halving hyperplanes h of S. This number will be denoted as $e_{1/2}(S)$. Again, we put

$$e_{1/2}(d,n) = \max_{|S|=n, S \subseteq \mathbb{R}^d} e_{1/2}(S).$$

The special importance of the halving hyperplanes comes from the fact that lower bounds on the number of halving hyperplanes imply lower bounds for the number of k-sets for any k, see [Ede87].

An important variation of the notion of a k-set are $(\leq k)$-sets (pronounce: "at most k-sets"). For any $k = 1, \ldots, n-1$, an $(\leq k)$-*set* is any i-set for $i = 1, \ldots, k$. Thus, every 1-set and every 2-set is an (≤ 2)-set. We write $E_k(S)$ for the number of the $(\leq k)$-sets of S and $E_k(d,n)$ for the maximum number of $(\leq k)$-sets over all sets S of n points in $\mathbb{R}^d$. Note that $E_k(S) = \sum_{i=1}^{k} e_i(S)$, but $E_k(d,n) = \sum_{i=1}^{k} e_i(d,n)$ does not hold in general, since a point set S' such that $E_k(S') = E_k(d,n)$ is in general different from the the point set S'' with $e_k(S'') = e_k(d,n)$, for integers d, k and n. For example, in the plane we have $E_k(S') = E_k(2,n) = nk$ if S' is a set of points in a convex position, but on the other hand we have $e_k(S') = n$ (the proof is left to the reader) and $e_k(2,n) = O(nk^{1/3})$ (see Section 6.1) for any $k < n$.

We may assume that S is in general position as the number $e_k(d,n)$ is realized by a set S with such property [EW85]. To see this, assume that S_0 is a set of n points in $\mathbb{R}^d$ with $e_k(d,n) = e_k(S_0)$. Each k-set (and its corresponding $(n-k)$-set) is determined by some hyperplane. The arrangement of these hyperplanes partitions $\mathbb{R}^d$ into open regions. Within each region we may move the points of S_0 without changing the numbers of k-sets. Therefore we may perturb these points in such a way that we obtain a set S in general position with $e_k(S) = e_k(S_0)$.

A hyperplane which determines a k-set is by definition disjoint from S. By changing this assumption we obtain the objects described below, which are more useful in proofs of upper bounds on the numbers of k-sets. For $j \in \{0, \ldots, n-d\}$, a *j-facet of S* is an oriented $(d-1)$-simplex Δ spanned by d points in S that has exactly j points of S on the positive side of its affine hull aff(Δ) (the affine hull is oriented in the same way as Δ). In $\mathbb{R}^2$ sometimes we call a j-facet a *j-segment* and in $\mathbb{R}^3$ a *j-triangle*. We write $g_j(S)$ for the number of j-facets of S and call the vector $\mathbf{g} = \mathbf{g}(S) = (g_j(S))_{j=0}^{n-d}$ the *g-vector* of S. We write $g_j(d, n)$ for the maximum number of j-facets of a set of n points in $\mathbb{R}^d$. Clearly we have $g_k(S) = g_{n-k-d}(S)$ and so $g_k(d, n) = g_{n-k-d}(d, n)$. Notice that if S is in general position, then the 0-facets of S are exactly the facets (the $(d-1)$-faces) of the convex hull of S.

For even $n - d$, an $(n - d)/2$-facet has the same number of points on both sides of its affine hull. We call such facet a *halving facet*. In $\mathbb{R}^2$ sometimes we call halving facets as *halving segments* and in $\mathbb{R}^3$ we call them *halving triangles*. The number of halving facets of S is denoted as $g_{1/2}(S)$. Similarly as before, let $g_{1/2}(d, n)$ denote the maximum number of halving facets for any configuration of n points in $\mathbb{R}^d$.

For k-sets we have introduced "at most k-sets". We have an analogous definition for the j-facets. For any $j \in \{0, \ldots, n-d\}$, a *$(\leq j)$-facet* (pronounce: "at most j-facet") is any i-facet for $i \in \{0, \ldots, j\}$. We denote by $G_j(S)$ the number of $(\leq j)$-facets of S. The meaning of $G_k(d, n)$ should be clear.

2.2 k-levels in hyperplane arrangements

The notion of a k-level is fundamental for many geometric structures like hyperplane arrangements or Voronoi diagrams. k-levels are linked to k-sets by the fact that the combinatorial complexity of a k-level relates to the number of k-sets in the dual setting [AW97], and so upper bounds on the number of k-sets also bound the complexity of k-levels (this argument works in the other direction, too).

First we introduce levels of i-faces in hyperplane arrangements and state a bijection between vertices of level j, vertices of level $n-d-j$ of a hyperplane arrangement and the j-facets of the dual arrangement (Lemma 2.2.1). As next

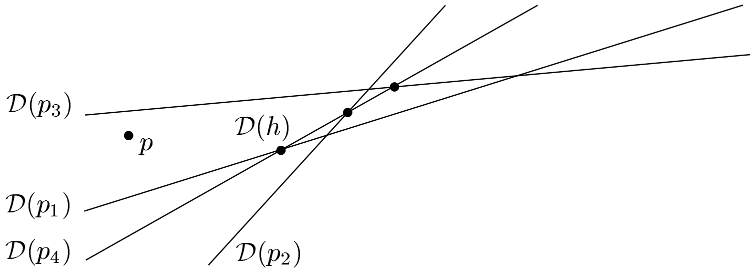

Figure 2.2: *The line arrangement $\mathcal{A}(\mathcal{D}(S_1))$ dual to the point set S_1; vertices of level 1 are marked*

we define k-levels and describe in Proposition 2.2.3 the relations between k-levels and the vertices of level j. Corollary 2.2.5 (taken from [Ede87]) states that the total number of k-sets of a point set for $k = 1, \ldots, n-1$ is

$$2\left(-1 + \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{n}{2i + d \bmod 2}\right).$$

Finally, in Proposition 2.2.6 and Proposition 2.2.7 we restate the upper bounds on the complexity of k-levels in terms of the numbers of k-sets and k-facets (shown in [Ede87]). These propositions give the result known from [CS89] that the maximum complexity of a $(\leq k)$-level in an arrangement of n hyperplanes is $\Theta(n^{\lfloor d/2 \rfloor}(k+1)^{\lceil d/2 \rceil})$.

Our definitions follow MULMULEY [Mul94] and EDELSBRUNNER [Ede87]. Let H be a set of n non-vertical hyperplanes in a general position in $\mathbb{R}^d$ and $\mathcal{A}(H)$ its arrangement. We choose a *base point* $\mathbf{b} \in \mathbb{R}^d$ (the base point is not necessarily the origin of $\mathbb{R}^d$). The *level of a point* $p \in \mathbb{R}^d$ is the number of hyperplanes which strictly separate p and $\mathbf{b}$. The *level of an i-face* (where i is the dimension of the face) of $\mathcal{A}(H)$ is the level of any point in the interior of the face. We denote as $V_j(H)$ or as V_j the set of vertices of $\mathcal{A}(H)$ of level $j \in \{0, \ldots, n-d\}$. Note that the level of a vertex of an arrangement is at most $n-d$.

Consider the configuration S_1 in Figure 2.1 and the dual arrangement $\mathcal{A}(\mathcal{D}(S_1))$ shown in Figure 2.2. Assume that the base point in Figure 2.2 is located at $(0, +\infty)$. Then each vertex v of level 1 in $\mathcal{A}(\mathcal{D}(S_1))$ is mapped by $\mathcal{D}$ to a line $\mathcal{D}(v)$ containing two points of S_1 such that there is exactly one point of S_1 above $\mathcal{D}(v)$. (This is due to the properties D1 and D2 of $\mathcal{D}$). Obviously $\mathcal{D}(v)$ is the affine hull of a 1-facet. Note that only a part of 1-facets

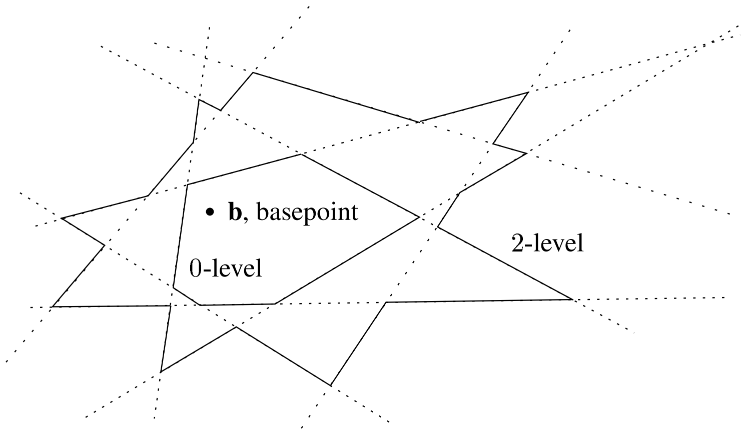

Figure 2.3: *Some k-levels in a line arrangement*

of S_1 arise in this way. The 1-facets l of S_1 whose affine hulls are downward-oriented (i.e., have the one point below l) are mapped by $\mathcal{D}$ to a vertex $\mathcal{D}(l)$ of level $n - 3 = 1$ (here $n = |S_1| = 4$). The following lemma generalizes this observation. The proof is straightforward.

Lemma 2.2.1 *Let S be a set of points in general position in $\mathbb{R}^d$ such that no affine hull of d points is vertical, and let $H = \mathcal{D}(S)$ be an arrangement of n hyperplanes with the base point **b** located in $(0, \ldots, 0, +\infty)$. Then for $j \in \{0, \ldots, n-d\}$ we have:*

(a) *There is a bijection between the vertices of level j in $\mathcal{A}(H)$ and j-facets of S whose affine hulls are upward-oriented.*

(b) *There is a bijection between the vertices of level $n - d - j$ and j-facets of S whose affine hulls are downward-oriented.*

(c) *It holds that $g_j(S) = |V_j| + |V_{n-d-j}|$ for every $j \in \{0, \ldots, n-d\}$.*

Remark 2.2.2

(a) *If the base point **b** is located in $(0, \ldots, 0, -\infty)$, then the above lemma holds with "upward-oriented" interchanged by "downward-oriented".*

(b) *The above lemma does not hold if the base point is located neither in $(0, \dots, 0, +\infty)$ nor in $(0, \dots, 0, -\infty)$. If p in Figure 2.2 is the base point, then $\mathcal{D}(h) \in V_0$, but h is the affine hull of a 1-facet, see Figure 2.1.*

(c) *Instead of the duality transformation $\mathcal{D}$ we could have used the polarity transformation T and the last lemma would hold as well. The disadvantage of this approach is the fact that then the origin must be located in $(0, \dots, +\infty)$, while the origin can be set freely when using $\mathcal{D}$.*

(d) *See also Proposition 3.1.1, Section 3.1 for an another bijection between certain j-facets and (certain) vertices of level j.*

For $k \in \{0, \dots, n-1\}$, a *k-level* $\ell_k(H)$ or ℓ_k in the arrangement $\mathcal{A}(H)$ is the boundary of the collection of the faces in $\mathcal{A}(H)$ with level less or equal to k. Figure 2.3 shows some k-levels in a line arrangement in $\mathbb{R}^2$. An important special case of k-levels occurs when the base point **b** is located in $(0, ..., 0, +\infty)$ or in $(0, ..., 0, -\infty)$. In the first case we call a k-level an *upper k-level*. In the latter case a k-level is called an *lower k-level*. (These definitions are not common in the literature and we use them in this section for the purpose of clarity). Note that an upper k-level is the same as a lower $(n-k-1)$-level. There are also alternative characterizations of upper k-levels [Ede87]. For a point $p \in \mathbb{R}^d$, let $a(p)$, $o(p)$ and $b(p)$ denote the number of hyperplanes which lie above p, contain p, or lie below p, respectively. It is not hard to see that for $k \in \{0, \dots, n-1\}$, an upper k-level is the set of all points p with

$$a(p) \leq k \qquad \text{and} \qquad b(p) \leq n-k-1.$$

Yet in other words, an upper k-level is the set of all points that lie below or on at least $k+1$ hyperplanes and above or on at least $n-k$ hyperplanes.

Consider a vertex $v \in V_j$, $j \in \{0, \dots, n-d\}$ as in Figure 2.4. A ray r from **b** which goes sufficiently close to v intersects points of levels $\ell_j, \ell_{j+1}, \dots, \ell_{j+d-1}$ (in this order) which belong to the hyperplanes defining v. Generalizing this observation, we obtain the following proposition:

Proposition 2.2.3 *Let $\mathcal{A}(H)$ be an arrangement of n hyperplanes in $\mathbb{R}^d$ and **b** a base point of $\mathbb{R}^d$.*

(a) *A point p is contained in i different k-levels of $\mathcal{A}(H)$ if and only if it is contained in i hyperplanes of H.*

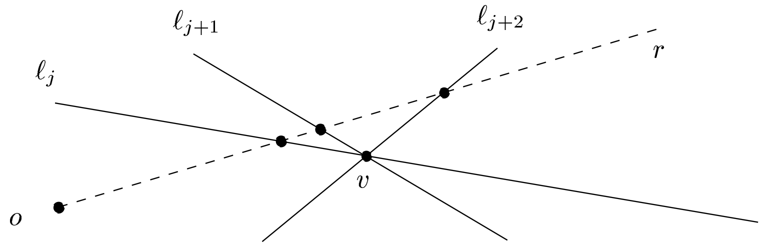

Figure 2.4: *The vertex $v \in V_j$ is contained in levels $\ell_j, \dots, \ell_{j+d-1}$*

(b) *For any $j \in \{0, \dots, n-d\}$, a vertex in $V_j(H)$ is contained exactly in the levels $\ell_j, \dots, \ell_{j+d-1}$.*

(c) *For every $k \in \{0, \dots, n-1\}$, a k-level ℓ_k contains vertices of V_j for $j \in \{\max\{0, k-d+1\}, \dots, \min\{n-d, k\}\}$.*

In Lemma 2.2.1 we have obtained a correspondence between vertices of level j in $\mathcal{A}(H)$ and j-facets of $\mathcal{D}(H)$. Due to the properties of the mapping $\mathcal{D}$ we have a similar relationship between cells of $\mathcal{A}(H)$ and the k-sets of $\mathcal{D}(H)$. We will derive this relationship in the following.

Temporarily let us define both the 0-set and the n-set of a point set S: the 0-set of S is the empty set, and the n-set of S is S itself. Let p be a point in the interior of a cell of $\mathcal{A}(H)$ such that there are k hyperplanes above p or k hyperplanes below p, for $k \in \{0, \dots, n\}$. (If $k \in \{1, \dots, n-1\}$, then the cell containing p lies directly below an upper $(k-1)$-level or directly below an upper $(n-k-1)$-level of $\mathcal{A}(H)$). Then, by the properties D1 and D2 of $\mathcal{D}$, the hyperplane $\mathcal{D}(p)$ defines a k-set in $\mathcal{D}(H)$, i.e. $\mathcal{D}(p)$ has k points on one side and contains no points of $\mathcal{D}(H)$. But this gives us no bijection yet, as for every unbounded cell there is another unbounded cell corresponding to the same k-set. To obtain a bijection, let us define a projective cell of $\mathcal{A}(H)$ as follows. Any bounded cell or unbounded cell which fits between parallel hyperplanes is a projective cell. For any other type of cell c let c' be the cell of $\mathcal{A}(H)$ such that for any hyperplane $h \in H$, c and c' are on the opposite sides of h. Then the union of c and c' is a projective cell. For a $k \in \{0, \dots, n\}$, a k-partition is a set $\{A, \mathcal{D}(H) \backslash A\}$, where A is a k-set of $\mathcal{D}(H)$ (and obviously $\mathcal{D}(H) \backslash A$ is an $(n-k)$-set).

Lemma 2.2.4 *For every $k = 0, \dots, n$ there is a bijection between the k-partitions of $\mathcal{D}(H)$ and the projective cells of $\mathcal{A}(H)$.*

PROOF. Note that if a cell c is bounded or it fits between two parallel hyperplanes, then it is the only one which defines a k-partition of $\mathcal{D}(H)$ for a certain k. It is not hard to see that for any cell c of another kind there is exactly one cell c' which defines the same k-partition as c. Furthermore, c and c' lie on the opposite side of every hyperplane $h \in H$. ∎

Corollary 2.2.5 *[Ede87] We have*

$$\sum_{i=1}^{n-1} e_i(S) = 2\left(-1 + \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{n}{2i + d \textit{ mod } 2}\right)$$

where

$$d \textit{ mod } 2 = \begin{cases} 0 & \textit{if } d \textit{ is even}, \\ 1 & \textit{otherwise}. \end{cases}$$

PROOF. As shown in Section 3 of [Ede87], the maximum number of projective cells of $\mathcal{A}(H)$ is

$$\sum_{i=0}^{d} (-1)^i \sum_{j=0}^{d-i} \binom{n}{j},$$

and this number is achieved if and only if $\mathcal{A}(H)$ is simple. In connection with Lemma 2.2.4 we obtain the desired result. ∎

The *complexity* $|\ell_k(H)|$ or $|\ell_k|$ of a k-level $\ell_k(H)$ of an arrangement $\mathcal{A}(H)$ is the number of faces of the part of the arrangement included in $\ell_k(H)$. This number is related to the number of k-facets and plays an important role in the analysis of the algorithms which use k-levels.

Proposition 2.2.6 *[Ede87] Let H be a set of n hyperplanes and **b** the base point located at* $(0, \ldots, 0, +\infty)$.

(a) *The maximum complexity of an upper k-level of $\mathcal{A}(H)$ is*

$$\Omega(\min\{g_j(d,n) | \max\{0, k-d+1\} \leq j \leq \min\{k, n-d\}\}).$$

(b) *The maximum complexity of an upper k-level is* $\Omega(e_{k+1}(d,n))$.

PROOF. **(a)**: By Proposition 2.2.3(b) and (c) every vertex in

$$\bigcup_{j=\max\{0,k-d+1\}}^{\min\{k,n-d\}} V_j$$

is contained in a k-level $\ell_k(H)$. Therefore

$$|\ell_k(H)| \geq d \min\{|V_j(H)| : \max\{0, k-d+1\} \leq j \leq \min\{k, n-d\}\}.$$

By Proposition 2.2.1(c) we have $|V_j(H)| + |V_{n-d-j}(H)| = g_j(H)$, and so the statement follows.

(b): (See Section 3.2 of [Ede87]). Let c be a cell of $\mathcal{A}(H)$ which determines (by the transformation $\mathcal{D}$) a set $\{A, \mathcal{D}(H)\backslash A\}$, where A is a $(k+1)$-set of $\mathcal{D}(H)$ and $k \in \{0, \ldots, n-2\}$. Then c contributes at least one facet to the k-level or to the $(n-k)$-level of $\mathcal{A}(H)$. This completes the proof. ∎

The following proposition provides upper bounds on the complexity of a k-level.

Proposition 2.2.7 *[Ede87] Let H be a set of n hyperplanes and **b** a base point located at* $(0, \ldots, 0, +\infty)$.

(a) *The number of vertices of $\ell_k(H)$ is at most*

$$\sum_{j=\max\{0,k-d+1\}}^{\min\{k,n-d\}} |V_j|.$$

(b) *The complexity of an upper k-level of $\mathcal{A}(H)$ is in*

$$O(\max\{g_j(d,n) | \max\{0, k-d+1\} \leq j \leq \min\{k, n-d\}\}).$$

(c) *The complexity of an upper k-level is in*

$$O(\max\{e_j(d,n) | k-d+2 \leq j \leq k+d-1\}).$$

PROOF. **(a)**: It follows by Proposition 2.2.3(c).

(b): By (a) and Proposition 2.2.1(c) the number of vertices of $\ell_k(H)$ is at most

$$\sum_{j=\max\{0,k-d+1\}}^{\min\{k,n-d\}} |V_j| \leq$$
$$d \max\{g_j(d,n) | \max\{0, k-d+1\} \leq j \leq \min\{k, n-d\}\}.$$

The claim follows, since every face of $\ell_k(H)$ is incident upon at least one vertex of $\ell_k(H)$ and every vertex of $\ell_k(H)$ is incident to at most constant number of faces of $\mathcal{A}(H)$.

(c): The statement is shown in Section 3.2 of [Ede87] as Theorem 3.3. ■

To obtain explicit upper and lower bounds on the maximum complexity of a k-level the reader should substitute in Propositions 2.2.6 and 2.2.7 the (actual best) upper bounds on the number of k-sets or on the number of j-facets, see for example [AW97]. As for so-called $(\leq k)$-levels, asymptotically tight complexity bounds can be given. For $k \in \{0, \ldots, n-1\}$, a $(\leq k)$-*level* (say: at-most-k level) $\mathcal{L}_k(H)$ is the union of i-levels for $0 \leq i \leq k$. The complexity an $(\leq k)$-level $\mathcal{L}_k(H)$ is defined analogously as for k-levels. Due to a result in [CS89], the maximum number of $(\leq k)$-facets of a set of n points in $\mathbb{R}^d$ is $\Theta(n^{\lfloor d/2 \rfloor}(k+1)^{\lceil d/2 \rceil})$. Combining this result with Propositions 2.2.6 and 2.2.7, we obtain the following theorem:

Theorem 2.2.8 *[CS89] The maximum complexity of a $(\leq k)$-level in an arrangement of n hyperplanes in $\mathbb{R}^d$ is $\Theta(n^{\lfloor d/2 \rfloor}(k+1)^{\lceil d/2 \rceil})$.*

Chapter 3

(i, j)-Partitions

In this chapter we investigate (i, j)-partitions, which generalize both k-sets and j-facets. They have been (implicitly) introduced by MULMULEY [Mul93] in the dual setting. In our investigations we focus on the numbers of (i, j)-partitions and the relations among them. The results presented in this chapter, especially in Section 3.2 and Section 3.9, together with the algorithms showed in Chapter 4 are the main results of this thesis.

Let S be a set of $n \geq 0$ points in general position in $\mathbb{R}^d$. A pair (A, B) of subsets of S is called *(i, j)-partition (of S)* , if $|A| = i$, $|B| = j$ and there is an oriented hyperplane h with $S \cap h = A$ and $S \cap h^+ = B$; we say that h *induces* the (i, j)-partition (A, B). For $i \in \mathbb{Z}$, $j \in \mathbb{Z}$ we let $D_{i,j} = D_{i,j}(S)$ denote the number of (i, j)-partitions of S.

Obviously

$$D_{i,j} = 0 \;\text{ unless }\; i \geq 0,\, j \geq 0 \;\text{ and }\; i + j \leq n.$$

For $k \in \{1, \ldots, n-1\}$, there is a natural bijective correspondence between $(0, k)$-partitions and k-sets; hence

$$D_{0,k} = e_k \;\text{ for }\; k \in \{1, \ldots, n-1\}.$$

For $n \geq d + 1$ and $j \in \{0, \ldots, n-d\}$ there is a correspondence between (d, j)-partitions and j-facets which gives

$$D_{d,j} = g_j \;\text{ for }\; j \in \{0, \ldots, n-d\}, \;\text{ provided } n \geq d + 1.$$

We denote by $D = D(S) = (D_{i,j})$ the $(d+1) \times (n+1)$-matrix with numbers $D_{i,j}$ of (i, j)-partitions of S for $i \in \{0, \ldots, d\}$ and $j \in \{0, \ldots, n\}$. In

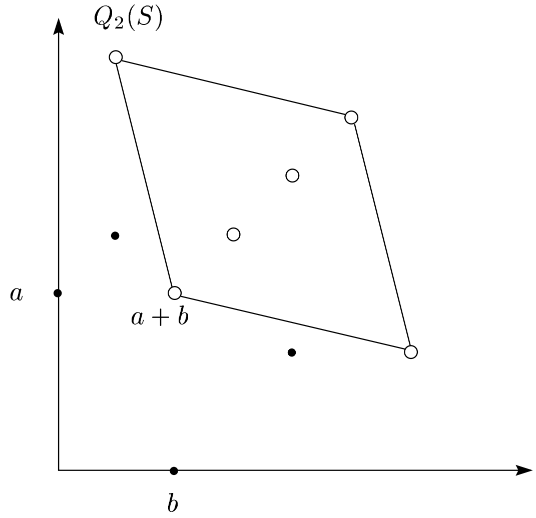

Figure 3.1: *A set S of 4 points in $\mathbb{R}^2$ (black) and the corresponding 2-set polytope*

Section 3.1 we give an example of D and we discuss the connection of (i, j)-partitions with the dual notion of faces in hyperplane arrangements.

In Section 3.2 we study the intriguing notion of k-set polytopes. For a set S of $n \geq 2$ points in general position and $k \in \{1, \ldots n-1\}$ a *k-set polytope* $Q_k(S)$ is the convex hull of the set

$$X_k(S) = \{\sum_{p \in T} p \mid T \in \binom{S}{k}\}.$$

The original motivation for the introduction of k-set polytopes is the fact that there is a bijective correspondence between the vertices of a k-set polytope and the $(0, k)$-partitions (i.e. k-sets) of the underlying point set S, see Figure 3.1. One of results of this chapter (Theorem 3.2.1) extends this relation through the following bijection between the $(i-1)$-faces of a k-set polytope (where $i \in \{2, \ldots, d\}$) and the (i, j)-partitions for $j \in \{k-i+1, \ldots, k-1\}$. For $k \in \{1, \ldots, n-1\}$ and $l \in \{0, \ldots, d-1\}$ let $f_{l,k} = f_{l,k}(S)$ be the number of l-faces of $Q_k(S)$. To simplify some formulas, we also define

$$f_{l,k} = f_{l,k}(S) = \begin{cases} 1 & \text{if } k \in \{1, \ldots, n-1\} \text{ and } l = -1, \\ 0 & \text{if } k \in \{1, \ldots, n-1\} \text{ and } l \notin \{-1, \ldots, d-1\}, \\ 0 & \text{if } k = 0 \text{ and } l \in \mathbb{Z}. \end{cases}$$

Then Theorem 3.2.1 says that for integers i, k such that $k \in \{1, \ldots, n-1\}$ and $i \in \mathbb{Z}\backslash\{0,1\}$ we have

$$\begin{aligned} f_{0,k} &= D_{0,k}, \\ f_{i-1,k} &= \sum_{j=k-i+1}^{k-1} D_{i,j}. \end{aligned}$$

While the first of the above two equations is a consequence of the well-known bijection between the vertices of $Q_k(S)$ and the k-sets of S, the second equation is an extension for higher-dimensional faces of $Q_k(S)$ shown in this thesis. A direct consequence of the above statement are formulas for the numbers of (i,j)-partitions in terms of the numbers of $(i-1)$-faces of $Q_k(S)$ (Corollary 3.2.2). Another important consequence of Theorem 3.2.1 is Theorem 3.2.3, which relates the numbers $D_{i,j}$ among each other. For example, we have for $k \in \{1, \ldots, n-1\}$ and $d \geq 2$ that

$$D_{0,k} + \sum_{i=2}^{d} \sum_{j=k-i+1}^{k-1} (-1)^{i-1} D_{i,j} = 1 + (-1)^{d-1}.$$

In addition to these results we show in Section 3.2 some other statements about k-set polytopes, which find their application in Chapter 4 for enumeration of k-sets.

In Section 3.3 we derive formulas for the numbers of (i,j)-partitions of the vertex set of a cyclic polytope in $\mathbb{R}^d$. In Section 3.4 we estimate the numbers of $(0,k)$-partitions (k-sets) of a point set S in general position in terms of the numbers of (d,j)-partitions (j-facets) of S, and vice versa. Most of these relations have been used previously in asymptotic form but they have been never worked out in detail. Section 3.5 uses duality to translate several results of MULMULEY [Mul93] obtained for faces of a hyperplane arrangement to (i,j)-partitions of the dual point set S. We introduce there the numbers $h_{i,j}$ of "refined j-facets", which play a similar role for the matrix D as the h-vector of a convex polytope plays for the f-vector of such a polytope [Zie94] (the definition of $h_{i,j}$ can be found on page 47). The results comprise among others the characterization of a part of the matrix D by the numbers $h_{i,j}$ (similarly as the f-vector of a convex polytope is determined by its h-vector). Another result is the so-called generalized Dehn-Sommerville relations, which reads as $h_{i,j} = h_{d-i,j}$ for appropriate i and j. These relations specialize for $j = 0$ and in the dual setting to the well-known Dehn-Sommerville relations $h_{d-i,0} = h_{i,0}$, $i \in \{0, \ldots, d\}$, for convex polytopes [Zie94].

In Section 3.6 we study the notion of a mutation of a point set and the notion of a mutation of an oriented matroid, and relate them to each other. A mutation of a point set occurs if a moving point crosses the affine hull of a j-facet; it causes the change of the matrix D of this point set. We classify mutations according to two integers m and l, defining so-called (m,l)-mutations. In the subsequent Section 3.7 we completely characterize the changes of the matrix D caused by an (m,l)-mutation; especially, we scrutinize the changes in the numbers of k-sets and in the numbers of j-facets. Section 3.8 is devoted to the study of the cumulative change of the matrix D while n points move continuously from an initial position S_I to a final position S_F. We obtain a "condensed" formula for $D(S_F)$ in terms of the numbers of (m,l)-mutations during the movement and in terms of the matrix $D(S_I)$.

Section 3.9 contains results which are among the most interesting in this chapter. For $i \in \{0,\dots,d\}$ and $j \geq 0$ let $\overline{h}_{i,j}$ be an integer defined by the equation

$$\overline{h}_{i,j} = \sum_{l=0}^{n}\sum_{m=0}^{d}(-1)^{d-m+i}D_{m,l}\sum_{k=0}^{d}\binom{d-1+j-l-k}{d-1}\binom{m}{k}\binom{d-m}{i-k}.$$

These integers are an "extension" of the numbers $h_{i,j}$ mentioned above for a greater index range of j. For $j=0$ and in the dual setting the vector $(\overline{h}_{d,0},\overline{h}_{d-1,0},\dots,\overline{h}_{0,0})$ is exactly the well-known h-vector of a convex polytope [Zie94]. Using a formula derived in Section 3.8 we show in Theorem 3.9.1 (independently of the results in [MUL93]) that for a point set S of $n \geq d+2$ points in general position, $i \in \{0,\dots,d\}$ and $j \in \{0,\dots,n\}$ we have

$$\overline{h}_{d-i,j} = \overline{h}_{i,j} + C_{n,d,i,j},$$

where the constant $C_{n,d,i,j}$ depends on n, d, i, and j. For pairs of integers i, j with $C_{n,d,i,j}=0$ the above equation states exactly the generalized Dehn-Sommerville relations found by MULMULEY [Mul93] in the dual setting; see the above discussion of Section 3.5. The second important result of Section 3.9 is Theorem 3.9.2. It states that the constant $C_{n,d,i,j}$ is indeed 0 for $d \geq 2$, sufficiently large n, $i \in \{0,\dots,d\}$ and $j \in \{0,\dots,k''\}$, where $k'' = \lfloor \max\{(n-3d-3)/2, n/(d+1)-2d\}\rfloor$; therefore, the generalized Dehn-Sommerville relations

$$\overline{h}_{d-i,j} = \overline{h}_{i,j}$$

are valid for these index ranges. This extends the result of MULMULEY stated in Section 3.5 by making the range of integers j such that the above equation

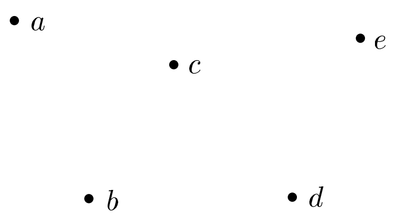

Figure 3.2: *A planar point set* S_2

is shown to be true substantially larger. Experiments give rise to the conjecture that for all $i \in \{0, \ldots, d\}$ and $j \in \{0, \ldots, n\}$ the constant $C_{n,d,i,j}$ is 0.

Section 3.10 presents a relation between the numbers of k-sets and the numbers of j-facets and vice versa in $\mathbb{R}^3$. This relation has been obtained by results from Section 3.7. Contrary to this, we show in Section 3.11 that the first row of the matrix D of a point set S does not determine the last row of $D(S)$, and vice versa. To show this we construct (using results from Section 3.7) pairs of points sets of n points which have the same first row of the matrix D but different last rows (and vice versa). Moreover, we show that this is not a singular phenomenon but occurs for many pairs of point sets.

3.1 (i, j)-partitions and duality

In this section we illustrate the most basic properties of the numbers of (i, j)-partitions giving an example of the matrix D of a planar point set. Furthermore, we discuss a bijection between the (i, j)-partitions of a point set and the $(d - i, j)$-faces of the dual hyperplane arrangement.

As stated in the introduction to this chapter, (i, j)-partitions generalize both the k-sets and the j-facets. As a consequence, the first row of D contains the e-vector, and the last row of D contains the g-vector. Analogously to the k-sets and the j-facets, the numbers of (i, j)-partitions obey the equation

$$D_{i,j} = D_{i,n-i-j}.$$

For example, the planar point set S_2 in Figure 3.2 has the following $(1, 1)$-partitions: $(\{a\}, \{b\})$, $(\{a\}, \{e\})$, $(\{b\}, \{a\})$, $(\{b\}, \{d\})$, $(\{d\}, \{b\})$, $(\{d\}, \{e\})$, $(\{e\}, \{d\})$, $(\{e\}, \{a\})$. Table 3.1 gives the complete matrix D of the point set S_2.

	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
# $(0,j)$-partitions	1	4	4	4	4	1
# $(1,j)$-partitions	4	8	12	8	4	0
# $(2,j)$-partitions	4	6	6	4	0	0

Table 3.1: *The matrix D of the point set S_2 in Figure 3.2*

It is interesting to investigate the relation between the (i,j)-partitions of a point set S and the faces of the dual hyperplane arrangement (compare also Section 2.2). For this aim we need the following definition. We say that a point $p \in \mathbb{R}^d$ (possibly $p \notin S$) is a *k-centerpoint* of S, if each oriented hyperplane h containing p fulfills the property that

$$\min\{|S \cap (h \cup h^+)|, |S \cap (h \cup h^-)|\} \geq k.$$

For each point set S there exists a so-called *centerpoint* of S which is an $\lceil n/(d+1) \rceil$-centerpoint of S [Ede87, Chapter 4]. For some special point sets there exist even $\lfloor (n-d-3)/2 \rfloor$-centerpoints, see Proposition 3.9.5. Using the notion of a k-centerpoint, we can obtain a bijection between the $(d-i)$-faces of level j of a hyperplane arrangement and the (i,j)-partitions of the dual point set. Recall the definition of the polar transformation T in Section 1.4.

Proposition 3.1.1 *Let H be a set of hyperplanes in $\mathbb{R}^d$ (disjoint from the origin $\boldsymbol{o}$) such that $S = T(H)$ is in general position. Assume that the origin of $\mathbb{R}^d$ is a k-centerpoint for some $k \geq d$. Then for $i \in \{0, \ldots, d\}$ we have:*

(a) *For $j \in \{0, \ldots, n-i\}$ there is a bijection between the $(d-i)$-faces in $\mathcal{A}(H)$ of level j and the (i,j)-partitions of S which can be induced by hyperplanes which have the origin on their negative side. This bijection maps a $(d-i)$-face F in $\mathcal{A}(H)$ of level j to the (i,j)-partition (A,B), where $T(A)$ is exactly the set of hyperplanes in H which contain F and $T(B)$ is exactly the set of hyperplanes in H which strictly separate $\boldsymbol{o}$ and F.*

(b) *For $j \leq k-d-i$, there is a bijection between the $(d-i)$-faces in $\mathcal{A}(H)$ of level j and the (i,j)-partitions of S (with the same properties as in (a)).*

(c) *If $j < k-d+i$, then the image of an (i,j)-partition under the bijection described in (a) is a bounded $(d-i)$-face of level j in $\mathcal{A}(H)$.*

PROOF. **(a)** Follows immediately from the properties of T, see Section 1.4.

(b) If $j + i \le k - d$, then by the properties of a k-centerpoint every (i,j)-partition can be induced by a hyperplane which has the origin on its negative side.

(c) Let F be an $(d-i)$-face of level j in $\mathcal{A}(H)$ and p a point in F. Then $T(p)$ is a hyperplane which induces an (i,j)-partition (A,B) of S. If F is unbounded, then p can be arbitrarily far away from **o** and so $T(p)$ can be arbitrarily close to **o**. Then there is a hyperplane $\bar{h}$ which contains the origin and has at least $(d-i) + |A \cup B|$ points in $\bar{h} \cup \bar{h}^+$. It follows that $|A \cup B| \ge k - d + i$. ■

3.2 k-set polytopes and correspondences to (i,j)-partitions

The notion of a k-set polytope has been introduced in [EVW97]. In this section we take a closer look at the properties of k-set polytopes and at their relations to the (i,j)-partitions. The main result of this section is Theorem 3.2.1, which relates the $(i-1)$-faces of $Q_k(S)$ with (i,j)-partitions of S, where $j \in \{k-i+1, \ldots, k-1\}$. This theorem immediately implies formulas for the numbers of (i,j)-partitions in terms of the numbers of $(i-1)$-faces of $Q_k(S)$ (Corollary 3.2.2). As another consequence of Theorem 3.2.1 we obtain some interesting relations among the numbers $D_{i,j}$ (stated in Theorem 3.2.3). (A special case of these relations is a well-known relation $e_k(S) = g_{k-1}(S)$ in $\mathbb{R}^2$). In addition to these results we show a few further properties of k-set polytopes, for example a characterization of the i-faces of a k-set polytope in terms of their vertices (Theorem 3.2.8(b)) and an upper bound of $\binom{n}{2}$ on the diameter of such a polytope (Proposition 3.2.14).

In this section we use the definitions stated in the beginning of this chapter.

Theorem 3.2.1 *Let i, k be integers with $k \in \{1, \ldots, n-1\}$. Then*

$$\begin{aligned} f_{-1,k} &= 1, \\ f_{0,k} &= D_{0,k}, \\ f_{i-1,k} &= \sum_{j=k-i+1}^{k-1} D_{i,j} \qquad \textit{for } i \in \mathbb{Z}\backslash\{0,1\}. \end{aligned} \tag{3.1}$$

Equation (3.1) is implied by a bijection between the $(i-1)$-faces of a k-set polytope (where $i \in \{2, \ldots, d\}$) and the (i,j)-partitions for $j \in$

$\{k-i+1,\ldots,k-1\}$. Some additional properties of this bijection are stated in Theorem 3.2.8.

Theorem 3.2.1 implies the following formulas via simple linear algebra. They express the integers $D_{i,j}$ in terms of the integers $f_{i',j'}$ and are consequences of (3.1) and essentially the fact that for integer pairs i,k such that $i \notin \{0,\ldots,d\}$ or $k \notin \{0,\ldots,n\}$ both $f_{i-1,k}$ and $D_{i,k}$ vanish.

Corollary 3.2.2

(a) *For $k \in \{0,\ldots,n-2\}$ and $i \in \mathbb{Z}\setminus\{0,1\}$ we have*

$$D_{i,k} - D_{i,k-i+1} = f_{i-1,k+1} - f_{i-1,k}$$

and

$$D_{i,k} = \sum_{m=0}^{\lfloor k/(i-1)\rfloor} \left(f_{i-1,k+1-(i-1)m} - f_{i-1,k-(i-1)m}\right).$$

(b) *For $k \in \{0,\ldots,n-2\}$ we have*

$$D_{2,k} = f_{1,k+1}.$$

(c) *For $i \in \mathbb{Z}$ we have*

$$D_{i,0} = f_{i-1,1}.$$

Theorem 3.2.1 and Corollary 3.2.10 imply the following relations on the numbers $D_{i,j}$.

Theorem 3.2.3 *Let S be a set of n points in general position in $\mathbb{R}^d$. Assume that $k \in \{1,\ldots,n-1\}$. For $d \geq 2$ we have*

$$D_{0,k} + \sum_{i=2}^{d}\sum_{j=k-i+1}^{k-1}(-1)^{i-1}D_{i,j} = 1 + (-1)^{d-1}.$$

As special cases we obtain for $d = 2$

$$D_{0,k} = D_{2,k-1}, \qquad (3.2)$$

and for $d = 3$

$$D_{0,k} - D_{2,k-1} + D_{3,k-2} + D_{3,k-1} = 2. \qquad (3.3)$$

Furthermore, for $d = 3$ we have

$$3D_{0,k} = D_{2,k-1} + 6 \tag{3.4}$$

and

$$D_{0,k} = \frac{1}{2}\left(D_{3,k-2} + D_{3,k-1}\right) + 2. \tag{3.5}$$

Remark 3.2.4 *Equation (3.2) is the known "simple" relation between the number of k-sets and the number of j-facets in $\mathbb{R}^2$, see [AW97], while Equation (3.5) is the relation between the number of k-sets and the number of j-facets in $\mathbb{R}^3$ shown in [AAH$^+$98]; see also Section 3.10.*

The proofs of Theorem 3.2.1, Corollary 3.2.2 and Theorem 3.2.3 are postponed to the end of this section. In the following we first show some lemmata and theorems which are necessary for proving the above theorems and other properties of k-set polytopes. Assume that $k \in \{1, \ldots, n-1\}$. For $T \in \binom{S}{k}$ we denote by $v(T)$ the sum $\sum_{p \in T} p$ (an element of $X_k(S)$) and for $v \in X_k(S)$ we denote by $T(v)$ a set $T \in \binom{S}{k}$ such that $\sum_{p \in T} p = v$. Note that $T(v)$ is not unique: in Figure 3.3 both $v(\{a, b\})$ and $v(\{c, d\})$ are equal to v. For $T^* \in \binom{S}{k}$ and a vector $\mathbf{c}$ of $\mathbb{R}^d$ we say that T^* *maximizes* $\mathbf{c}$ if

$$\langle \mathbf{c}, v(T^*) \rangle = \max_{T \in \binom{S}{|T^*|}} \langle \mathbf{c}, v(T) \rangle.$$

For a fixed S, $k \in \{1, \ldots, n-1\}$ and $i \in \{1, \ldots, d\}$, let F be an $(i-1)$-face of $Q_k(S)$ and let (A, B) be an $(i, k-l)$-partition of S with $l \in \{1, \ldots, i-1\}$. We say that F is *induced by* (A, B) if there is a vector $\mathbf{c}$ in $\mathbb{R}^d$ such that

1. there is a hyperplane h_1 with normal vector $\mathbf{c}$ and $h_1 \cap S = A$, $h_1^+ \cap S = B$, and
2. there is a hyperplane h_2 with normal vector $\mathbf{c}$ and $h_2 \cap Q_k(S) = F$, $h_2^+ \cap Q_k(S) = \emptyset$.

Lemma 3.2.5 *Assume that $k \in \{1, \ldots, n-1\}$, $T^* \in \binom{S}{k}$, $\mathbf{c}$ is a vector of $\mathbb{R}^d$ and h is a hyperplane with equation $\langle \mathbf{c}, x \rangle = \alpha$, where $\alpha = \min_{p \in T^*} \langle \mathbf{c}, p \rangle$. Then the following statements are equivalent:*

(a) *$h^- \cap T^* = \emptyset$ and $h^+ \cap (S \backslash T^*) = \emptyset$,*

(b) *T^* maximizes $\mathbf{c}$,*

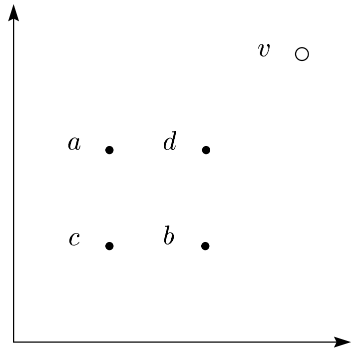

Figure 3.3: *Both $v(\{a,b\})$ and $v(\{c,d\})$ are equal to v, and so $T(v)$ is not unique*

(c) *the sets*

$$(T^* \cap h^+) \cup R \quad \textit{for all} \quad R \in \binom{S \cap h}{|T^* \cap h|}$$

are exactly the sets in $\binom{S}{k}$ which maximize $\mathbf{c}$.

PROOF. Assuming that (a) holds, we show first (b). We have $\langle \mathbf{c}, p\rangle \geq \alpha$ for all $p \in T^*$ and $\langle \mathbf{c}, p\rangle \leq \alpha$ for all $p \in S \backslash T^*$. For any $T \in \binom{S}{k}$ we have by the linearity of $\langle \cdot, \cdot \rangle$ that

$$\begin{aligned} \langle \mathbf{c}, v(T^*)\rangle - \langle \mathbf{c}, v(T)\rangle &= \sum_{p \in T^* \backslash T} \langle \mathbf{c}, p\rangle - \sum_{p \in T \backslash T^*} \langle \mathbf{c}, p\rangle \\ &\geq |T^* \backslash T|\,\alpha - |T \backslash T^*|\,\alpha \\ &= 0 \end{aligned}$$

and so $\langle \mathbf{c}, v(T^*)\rangle \geq \langle \mathbf{c}, v(T)\rangle$.

Now we assume that (b) holds; we show (a). Let $p' \in T^*$ be a point such that $\langle \mathbf{c}, p'\rangle = \alpha$. Assume that for $q \in S \backslash T^*$ we have $\langle \mathbf{c}, q\rangle > \alpha$. Then $\langle \mathbf{c}, v((T^* \cup \{q\}) \backslash \{p'\})\rangle > \langle \mathbf{c}, v(T^*)\rangle$, which is a contradiction. Therefore $\langle \mathbf{c}, q\rangle \leq \alpha$ for all $q \in S \backslash T^*$, $\langle \mathbf{c}, p\rangle \geq \alpha$ for all $p \in T^*$ (by the definition of α) and so (a) holds.

As next we assume (a) and show (c). Put $A = S \cap h$ and $B = S \cap h^+$. We have $\langle \mathbf{c}, p\rangle = \alpha$ for all $p \in A$, $\langle \mathbf{c}, p\rangle > \alpha$ for all $p \in B$ and $\langle \mathbf{c}, p\rangle <$ α for all $p \in S \backslash (A \cup B)$. Then $\langle \mathbf{c}, v(B \cup R)\rangle = \langle \mathbf{c}, v(T^*)\rangle$ for all $R \in$

$\binom{A}{|T^*\cap h|}$. We know by (b) that T^* maximizes $\mathbf{c}$, and so all the above sets do as well. Now we show that they are the only ones. Assume that a set $T' \in \binom{S}{k}$ maximizes $\mathbf{c}$. If $B \not\subseteq T'$, then there is obviously $p \in T'$ and $q \in B\backslash T'$ with $\langle \mathbf{c}, v((T' \cup \{q\})\backslash\{p\})\rangle > \langle \mathbf{c}, v(T')\rangle$; a contradiction. If $|T' \cap h| \neq |T^* \cap h|$, then $\langle \mathbf{c}, v(T')\rangle \neq \sum_{p\in B}\langle \mathbf{c}, p\rangle + \alpha|h \cap T^*| = \langle \mathbf{c}, v(T^*)\rangle$, which is again a contradiction. Thus, (c) holds.

Finally, if (c) holds, then especially T^* maximizes $\mathbf{c}$ and so (b) holds, too. ■

Lemma 3.2.6 *Assume that* $k \in \{1, \ldots, n-1\}$ *and* $T^* \in \binom{S}{k}$.

(a) T^* *is a* k*-set if and only if there is a vector* $\mathbf{c}$ *of* $\mathbb{R}^d$ *such that* T^* *maximizes* $\mathbf{c}$.

(b) $v(T^*)$ *is a vertex of* $Q_k(S)$ *if and only if there is a vector* $\mathbf{c}$ *of* $\mathbb{R}^d$ *such that* T^* *maximizes* $\mathbf{c}$.

(c) $v(T^*)$ *is a vertex of* $Q_k(S)$ *and there is a hyperplane* h *with normal vector* $\mathbf{c}$ *such that* $v(T^*) \in h$, $h^+ \cap Q_k(S) = \emptyset$ *if and only if* $\mathbf{c}$ *is a vector of* $\mathbb{R}^d$ *such that* T^* *maximizes* $\mathbf{c}$.

PROOF. **(a)**: If for some vector $\mathbf{c}$ of $\mathbb{R}^d$ the set T^* maximizes $\mathbf{c}$, then by Lemma 3.2.5 the hyperplane h with equation $\langle \mathbf{c}, x\rangle = \min_{p\in T^*}\langle \mathbf{c}, p\rangle$ has the property that $h^- \cap T^* = \emptyset$ and $h^+ \cap (S\backslash T^*) = \emptyset$. Since S is in general position, we might perturb h and obtain a hyperplane h_1 such that $h_1^+ \cap S = T^*$ and $h_1 \cap S = \emptyset$.

Conversely, if T^* is a k-set, then obviously there is a vector $\mathbf{c}$ of $\mathbb{R}^d$ such that the hyperplane h with equation $\langle \mathbf{c}, x\rangle = \min_{p\in T^*}\langle \mathbf{c}, p\rangle$ fulfills $h^- \cap T^* = \emptyset$ and $h^+ \cap (S\backslash T^*) = \emptyset$. Therefore, we are done by Lemma 3.2.5.

(b) and **(c)**: For a vector $\mathbf{c}$ of $\mathbb{R}^d$, let h be a hyperplane with equation $\langle \mathbf{c}, x\rangle = \beta$, where $\beta = \langle \mathbf{c}, v(T^*)\rangle$. If T^* maximizes $\mathbf{c}$, then $v(T) \notin h^+$ for all $T \in \binom{S}{k}$ and $v(T^*) \in h$. Therefore $v(T^*)$ is a vertex of $Q_k(S)$; by perturbing h we obtain the required hyperplane for (c).

Conversely, if $v(T^*)$ is a vertex of $Q_k(S)$, then obviously there exists a hyperplane h with normal vector $\mathbf{c}$ such that $v(T^*) \in h$ and $h^+ \cap Q_k(S) = \emptyset$. Then there is no $T \in \binom{S}{k}$ such that $\langle \mathbf{c}, v(T)\rangle > \langle \mathbf{c}, v(T^*)\rangle$ and so T^* maximizes $\mathbf{c}$. ■

The following lemma provides the argument that the dimension of a face of $Q_k(S)$ induced by an (i,j)-partition is $i-1$.

Lemma 3.2.7 *Let $A = \{p_1, \ldots, p_a\}$ be a set of $a > 1$ points (not necessarily in general position) and $b \in \{1, \ldots, a-1\}$. Then the dimensions of the affine space* $\operatorname{aff}(A)$ *and of the affine space* $\operatorname{aff}(X_b(A))$ *are equal.*

PROOF. Let $\mathbf{A}$ be a $(d+1) \times a$ -matrix such that its i-th column is a vector whose entries $1, \ldots, d$ are the coordinates of p_i and the $(d+1)$-st entry is 1. Let $\mathbf{B}$ be an analogous $(d+1) \times \binom{a}{b}$-matrix for $X_b(A)$ but with the integer b in the $(d+1)$-st entry of each column. It is well-known that the rank of $\mathbf{A}$ is exactly the dimension of $\operatorname{aff}(A)$ plus 1 and the rank of $\mathbf{B}$ is the dimension of $\operatorname{aff}(X_b(A))$ plus 1, see [Zie94]. Since each column of $\mathbf{B}$ is a linear combination of the columns of $\mathbf{A}$, we have $\operatorname{rank}(\mathbf{A}) \geq \operatorname{rank}(\mathbf{B})$. To show the converse, notice first that $\binom{a}{b} \geq a$. To complete the proof we show that the columns of $\mathbf{A}$ are linear combinations of the columns of $\mathbf{B}$. Let $\mathbf{A}'$ be the matrix consisting of the first d rows of $\mathbf{A}$, and let $\mathbf{B}'$ be the analogous matrix for $\mathbf{B}$. It is not hard to see that the sum u' of all columns of $\mathbf{B}'$ is exactly the vector

$$u' = \binom{a-1}{b-1}(\mathbf{A}'_1 + \ldots + \mathbf{A}'_a)$$

where $\mathbf{A}'_i$ is the i-th column of $\mathbf{A}'$. For $j \in \{1, \ldots a\}$, let v'_j be the sum of those columns of $\mathbf{B}'$ which correspond to the points in $X_b(A \backslash \{p_j\})$. It is not difficult to show that

$$v'_j = \binom{a-2}{b-1} \sum_{\substack{i \in \{1, \ldots, a\} \\ i \neq j}} \mathbf{A}'_i.$$

Therefore

$$\mathbf{A}'_j = \frac{1}{\binom{a-1}{b-1}} u' - \frac{1}{\binom{a-2}{b-1}} v'_j. \tag{3.6}$$

To obtain the j-th column of $\mathbf{A}$ as a linear combination of the columns of $\mathbf{B}$ we create a linear combination w_j of the columns of $\mathbf{B}$ corresponding to (3.6) (i.e. each column of $\mathbf{B}'$ in (3.6) is substituted by a corresponding column of $\mathbf{B}$). In this way the first d entries of the j-th column of $\mathbf{A}$ agree with the first d entries of w_j. It remains to take care about the last entry of w_j. To obtain u' we have added all $\binom{a}{b}$ columns of $\mathbf{B}'$ and v'_j has been obtained by adding $\binom{a}{b-1}$ columns of $\mathbf{B}'$. Therefore, the last entry of w_j is

$$\frac{\binom{a}{b}}{\binom{a-1}{b-1}} b - \frac{\binom{a}{b-1}}{\binom{a-2}{b-1}} b = 1$$

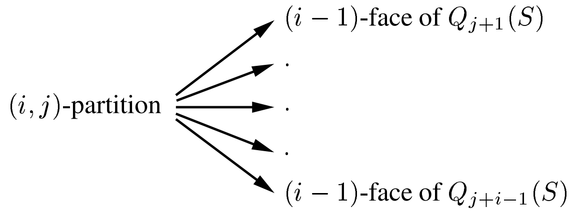

Figure 3.4: *A visualization of Theorem 3.2.8(b)*

and we are done. ■

The following theorem is the basis for the most interesting result of this section, namely Theorem 3.2.1. It is remarkable that the dimension of a face F of $Q_k(S)$ depends on the index i of an (i, j)-partition which induces F, and not on the number of vertices of F (see part (b)).

Theorem 3.2.8 **(a)** *For $k \in \{1, \ldots, n-1\}$, a set $T^* \in \binom{S}{k}$ is a k-set of S if and only if $v(T^*)$ is a vertex of $Q_k(S)$.*

(b) *For $i \in \{2, \ldots, d\}$, $j \in \{0, \ldots n-i\}$, an (i, j)-partition (A, B) of S and $k \in \{j+1, \ldots, j+i-1\}$ there is a $(i-1)$-face F of $Q_k(S)$ induced by (A, B) in the following sense: F has exactly the vertices*

$$v(B \cup R) \text{ for all } R \in \binom{A}{k-j}.$$

(c) *For $i \in \{2, \ldots, d\}$, $k \in \{1, \ldots, n-1\}$ and an $(i-1)$-face F of $Q_k(S)$ there is exactly one (i, j)-partition with $j \in \{k-i+1, \ldots, k-1\}$ which induces F (here j is determined by F), see Figure 3.5.*

(d) *For $j \in \{0, \ldots, n-2\}$ and an $(1, j)$-partition (A, B) of S there is exactly one vertex of $Q_{j+1}(S)$ induced by (A, B). Notice that there might exist several $(1, j)$-partitions of S which all induce the same vertex of $Q_{j+1}(S)$.*

(e) *For $i \in \{2, \ldots, d\}$ and $k \in \{1, \ldots, n-1\}$ there is a bijection between the set*

$$M_i = \{(A, B) \mid (A, B) \text{ is an } (i, j)\text{-partition of } S \\ \text{for } j \in \{\max\{0, k-i+1\}, \ldots, k-1\}\}$$

and the $(i-1)$-faces of $Q_k(S)$ such that the image of an element (A, B) in M_i is a face of $Q_k(S)$ induced by (A, B).

Remark 3.2.9 *The definition of k-set polytopes is also valid if S is not in general position. Also parts (a) and (e) of Theorem 3.2.8 hold in this case. We omit the argument, since it is given in [AAH+98].*

PROOF. **(a)**: It follows immediately by Lemma 3.2.6(a) and (b).

(b): Let h_1 be a hyperplane such that $h_1 \cap S = A$ and $h_1^+ \cap S = B$, and let $\mathbf{c}$ be the normal vector of h_1. Let T^* be a set such that $T^* = B \cup A'$ with $A' \in \binom{A}{k-j}$; obviously h_1 has the equation $\langle \mathbf{c}, x\rangle = \min_{p \in T^*} \langle \mathbf{c}, p\rangle$. We also have $h_1^- \cap T^* = \emptyset$, $h^* \cap (S \backslash T^*) = \emptyset$ and so by Lemma 3.2.5 the sets $B \cup R$ for all $R \in \binom{A}{k-j}$ are exactly the sets which maximize $\mathbf{c}$. By Lemma 3.2.6(c) the points $v(B \cup R)$ for all $R \in \binom{A}{k-j}$ are exactly all vertices of $Q_k(S)$ which lie in a hyperplane h with normal vector $\mathbf{c}$ such that $h^+ \cap Q_k(S) = \emptyset$, i.e. they are all vertices of a face F of $Q_k(S)$. It remains to show that the dimension of F is $i-1$. This is done by applying Lemma 3.2.7 with $i = a > 1$ and $b = k - j$.

(c): Assume that $k \in \{1, \ldots, n-1\}$, $i \in \{2, \ldots, d\}$ and F is a face of $Q_k(S)$ of dimension $i-1$. Then there is a hyperplane h_1 such that $Q_k(S) \cap h = F$, $h_1^+ \cap Q_k(S) = \emptyset$; let $\mathbf{c}$ be the normal vector of h_1. Let $v(T^*)$ be a vertex of $Q_k(S)$, where $T^* \in \binom{S}{k}$. By Lemma 3.2.6(c) T^* maximizes $\mathbf{c}$. By Lemma 3.2.5 the hyperplane h_2 with equation $\langle \mathbf{c}, x\rangle = \min_{p \in T^*} \langle \mathbf{c}, p\rangle$ has the property that $h_2^- \cap T^* = \emptyset$ and $h_2^+ \cap (S \backslash T^*) = \emptyset$. Put $A = h_2 \cap S$ and $B = h_2^+ \cap S = h_2^+ \cap T^*$. The same lemma shows that the sets $B \cup R$ for all $R \in \binom{A}{|T^* \cap h_2|}$ are exactly the sets which maximize $\mathbf{c}$. By Lemma 3.2.6(c) they are exactly the vertices of F. Now notice that $|A| > 1$ and $0 < |T^* \cap h_2| < |A|$, otherwise F would have at most one vertex and so dimension 0. Therefore, Lemma 3.2.7 is applicable to A and to $\{v(R) \mid R \in \binom{A}{|T^* \cap h_2|}\} = X_{|T^* \cap h_2|}(A)$. It follows that the dimension of $\operatorname{aff}(A)$ is $i-1$. Because S is in general position, we have $|A| = i$. Since $|B| = k - |T^* \cap h_2|$ and $0 < |T^* \cap h_2| < i$, we conclude that (A, B) is an (i,j)-partition such that $j \in \{k - i + 1, \ldots, k-1\}$ which induces F.

To show the uniqueness notice that by the above considerations the vertices of F are exactly the points $v(B' \cup R)$ for all $R \in \binom{A'}{l}$ for some $B' \subseteq S$, $A' \subseteq S$ and $l \in \{2, \ldots, |A'|\}$. Furthermore, A' and B' are unique, since $B' = \bigcap_{v \in F} T(v)$ and $A' = \left(\bigcup_{v \in F} T(v)\right) \backslash B'$ (here $v \in F$ means that we iterate over all vertices of F). On the other hand, let (A'', B'') be an (i,j)-partition of S which induces F. Then by (b) of this Theorem we have $B'' = \bigcap_{v \in F} T(v)$ and $A'' = \left(\bigcup_{v \in F} T(v)\right) \backslash B''$, and so (A'', B'') is unique.

(d): By Lemma 3.2.5 and Lemma 3.2.6(c).

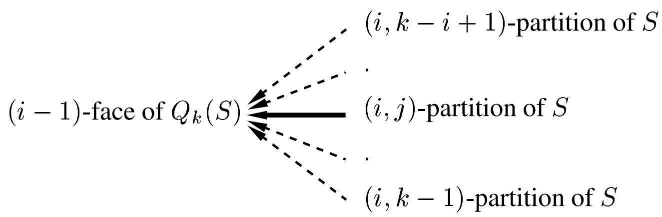

Figure 3.5: *A visualization of Theorem 3.2.8(c)*

(e): Follows from (b) and (c).

■

Corollary 3.2.10 *For $d = 3$ and $k \in \{1, \dots, n-1\}$ the polytope $Q_k(S)$ is simplicial. For $d \geq 4$ and $k \in \{2, \dots, n-1\}$ the polytope $Q_k(S)$ is not simplicial.*

PROOF. By Theorem 3.2.8(b) and (c) for $k \in \{1, \dots, n-1\}$ and a point set S of n points in general position in $\mathbb{R}^d$ the facets of the k-set polytope $Q_k(S)$ have 3 vertices if $d = 3$. By Theorem 3.2.8(b) for $d \geq 4$ and each $k \in \{2, \dots, n-1\}$ there is a $(d, k-2)$-partition of S which induces a face of $Q_k(S)$ with $\binom{d}{2} > d$ vertices. ■

PROOF OF THEOREM 3.2.1. Assume that $k \in \{1, \dots, n-1\}$. Equation

$$f_{0,k} = D_{0,k}$$

follows from Theorem 3.2.8(a). Next we show (3.1). For $i < 0$ or $i > d$ we have $f_{i-1,k} = 0$ and

$$\sum_{j=k-i+1}^{k-1} D_{i,j} = 0,$$

since $D_{i,j} = 0$ in this index range of i. For $i \in \{2, \dots, d\}$ the statement follows from the bijection stated in Theorem 3.2.8(e). ■

PROOF OF COROLLARY 3.2.2.

(a): First we show the equation

$$D_{i,k} - D_{i,k-i+1} = f_{i-1,k+1} - f_{i-1,k}. \tag{3.7}$$

If $k \in \{1, \ldots, n-2\}$ and $i \in \mathbb{Z} \setminus \{0, 1\}$, then by Theorem 3.2.1 we have

$$f_{i-1,k+1} - f_{i-1,k} = \sum_{j=k-i+2}^{k} D_{i,j} - \sum_{j=k-i+1}^{k-1} D_{i,j} = D_{i,k} - D_{i,k-i+1}.$$

For $k = 0$ and $i \in \{2, \ldots, d\}$ we have $f_{i-1,k} = 0$ and $D_{i,k-i+1} = 0$, and so (3.7) reduces to $f_{i-1,1} = D_{i,0}$. The last equation is valid, since $Q_1(S)$ is the convex hull of S and $D_{i,0}$ counts the number of its $(i-1)$-faces. For $k = 0$ and $i \notin \{0, \ldots, d\}$ (3.7) holds by definition. The second equation of part (a) follows by a recursive application of (3.7) and the fact that $D_{i,k-(i-1)m} = 0$ for $m > k/(i-1)$.

(b): It follows from Equation (3.1) of Theorem 3.2.1.

(c): By (a) we have (with $k = 0$) for all $i \notin \{0, 1\}$ that

$$f_{i-1,1} = D_{i,0} - D_{i,-i+1}.$$

For $i \geq 2$ we are done, since $-i + 1 < 0$. For $i \leq 0$ the equation to be shown follows from the definitions. Finally, for $i = 1$ observe that $Q_1(S)$ is the convex hull of S and $f_{0,1}(S)$ is the number of its vertices; this number obviously equals $D_{1,0}$.

■

PROOF OF THEOREM 3.2.3. By Corollary 3.2.10 the polytope $Q_k(S)$ is simplicial if $d = 3$, and we can apply the Dehn-Sommerville relation $f_{1,k} = 3f_{0,k} - 6$ for convex polytopes [Zie94]; in connection with Theorem 3.2.1 we obtain then (3.4). Equations (3.2) and (3.3) follow from the Euler-Poincaré formula (which holds for every polytope) and from Theorem 3.2.1. Finally, (3.5) can be obtained by substituting (3.4) into (3.3). ■

In the remainder of this section we show some properties of k-set polytopes published in [AF99]. Some of them are useful for enumerating k-sets of a given point set S, see Chapter 4.

We call two vertices of $Q_k(S)$ *polytopal neighbors* if they are connected by an edge (a 1-face) in $Q_k(S)$. The following corollary is a consequence of Theorem 3.2.8.

Corollary 3.2.11 *Assume that* $k \in \{1, \ldots, n-1\}$ *and* $T_1, T_2 \in \binom{S}{k}$, $T_1 \neq T_2$.

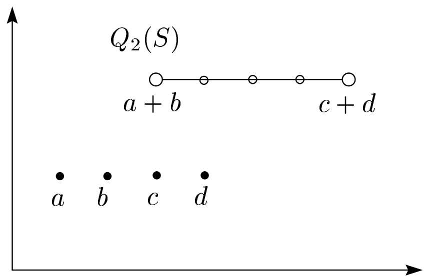

Figure 3.6: *In this example $v(\{a,b\})$ and $v(\{c,d\})$ are polytopal neighbors in $Q_2(S)$*

(a) *The points $v(T_1)$ and $v(T_2)$ are polytopal neighbors if and only if there is a hyperplane h with $|h \cap S| = 2$, $h^+ \cap S = T_1 \cap T_2$ and $|T_1 \cap T_2| = k-1$.*

(b) *If the points $v(T_1)$ and $v(T_2)$ are polytopal neighbors then $T_1 \cup T_2$ is a $(k+1)$-set.*

(c) *A vertex v of $Q_k(S)$ has at most $k(n-k)$ polytopal neighbors.*

PROOF. **(a)**: If $v(T_1)$ and $v(T_2)$ are connected by an edge F in $Q_k(S)$, then by Theorem 3.2.8(c) there is a $(2, j)$-partition of S which induces F for some integer j. By (b) of the same theorem the vertices of the edge are exactly the points $v(T_3 \cup \{p_1\})$, $v(T_3 \cup \{p_2\})$, where $T_3 = T_1 \cap T_2$, $T_1 \backslash T_3 = \{p_1\}$ and $T_2 \backslash T_3 = \{p_2\}$. Conversely, if there is a hyperplane h with $|h \cap S| = 2$, $|h^+ \cap S| = k-1$ and $h^+ \cap S = T_1 \cap T_2$ then $(h \cap S, T_1 \cap T_2)$ is a $(2, k-1)$-partition. Then by Theorem 3.2.8(b) and (c) $v(T_1)$ and $v(T_2)$ are polytopal neighbors.

(b): Follows from (a).

(c): To obtain a polytopal neighbor v' of a vertex v, we have to remove a point from $T(v)$ and add a point from $S \backslash T(v)$, since $|T(v) \cap T(v')| = k-1$ by (a). ■

Remark 3.2.12 *The statements (a) and (b) of Corollary 3.2.11 might not hold if S is not in general position, see Figure 3.6. Here $S = \{a,b,c,d\}$, $T_1 = \{a,b\}$ and $T_2 = \{c,d\}$. Since $v(T_1)$ and $v(T_2)$ are polytopal neighbors in $Q_2(S)$, statements (a) and (b) of Corollary 3.2.11 do not hold in this case.*

Lemma 3.2.13 *For a polytope Q and two vertices v_1, v_2, there exists a path from v_1 to v_2 which uses each edge direction at most once.*

PROOF. Let $\mathbf{c}_1, \mathbf{c}_2 \in \mathbb{R}^d$, $\mathbf{c}_1 \neq \mathbf{c}_2$ be the normal vectors of supporting hyperplanes at v_1, v_2, respectively. Consider the parametric function

$$h(\theta, x) = (\theta \mathbf{c}_2 + (1-\theta)\mathbf{c}_1)^T x.$$

It is not hard to see that there is a hyperplane with normal vector $(\theta \mathbf{c}_2 + (1-\theta)\mathbf{c}_1)$ supporting a vertex or an edge of $Q_k(S)$, and that this hyperplane "follows" a path from v_1 to v_2, while θ goes from 0 to 1. Simultaneously the expression in parenthesis takes no two equal values (since it is a convex hull of points represented by $\mathbf{c}_1$ and $\mathbf{c}_2$). Therefore each edge on the path from v_1 to v_2 has a different direction. ∎

Proposition 3.2.14 *The diameter of $Q_k(S)$ is at most $\binom{n}{2}$.*

PROOF. If two vertices $v(T_1)$, $v(T_2)$ are polytopal neighbors in $Q_k(S)$, then by Corollary 3.2.11(a) we have $v(T_1) - v(T_2) = p_1 - p_2$ for some $p_1, p_2 \in S$. It follows that the k-set polytope $Q_k(S)$ has at most $\binom{n}{2}$ edge directions, and we are done by Lemma 3.2.13. ∎

3.3 Numbers of (i,j)-partitions for cyclic polytopes

In this section we derive a formula for the numbers of (i,j)-partitions of a cyclic polytope with n vertices in $\mathbb{R}^d$ (Proposition 3.3.1). It was known that it is not a difficult task to obtain such a formula by purely combinatorial methods, yet the explicit form has been never stated, except for special cases $i = 0$ in [GH97] and $i = d$ [AW97]. Such a formula provides an example of a point set for which the matrix D is completely known, and thus might be helpful to show the Conjecture 3.9.6.

Let us define

$$\binom{a}{b}_{-1} = \begin{cases} 1 & \text{if } a = b = -1 \\ \binom{a}{b} & \text{else.} \end{cases}$$

Proposition 3.3.1 *For $n \geq d+1$ we have*

$$D_{i,j}(S_{n,d}) = \sum_{s=0}^{d} B(n,j,i,s)$$

where $B(n, j, i, s)$ is for $s = 2q$

$$\sum_{t_1=0}^{i}\sum_{t_2=0}^{i-t_1}\left(\binom{q+1}{t_1}\binom{j-1}{q-t_1}_{-1}\binom{q}{t_2}\cdot\binom{n-i-j-1}{q-t_2-1}_{-1}\binom{2q-t_1-t_2}{2q-i}\right)$$
$$+\sum_{t_1=0}^{i}\sum_{t_2=0}^{i-t_1}\left(\binom{q}{t_1}\binom{j-1}{q-t_1-1}_{-1}\binom{q+1}{t_2}\cdot\binom{n-i-j-1}{q-t_2}_{-1}\binom{2q-t_1-t_2}{2q-i}\right)$$

and for $s = 2q-1$ it is

$$2\sum_{t_1=0}^{i}\sum_{t_2=0}^{i-t_1}\left(\binom{q}{t_1}\binom{j-1}{q-t_1-1}_{-1}\binom{q}{t_2}\cdot\binom{n-i-j-1}{q-t_2-1}_{-1}\binom{2q-1-t_1-t_2}{2q-1-i}\right).$$

The formula (3.8) in the corollary below can be found also in [GH97].

Corollary 3.3.2 *For $n \geq d+1$ we have*

$$e_j(S_{n,d}) = D_{0,j}(S_{n,d}) = \sum_{s=0}^{d} B(n, j, s) \tag{3.8}$$

where $B(n, j, s)$ is for $s = 2q-1$

$$2\binom{j-1}{q-1}\binom{n-j-1}{q-1}$$

and for $s = 2q$ it is

$$\binom{j-1}{q}\binom{n-j-1}{q-1}+\binom{j-1}{q-1}\binom{n-j-1}{q}.$$

Furthermore, for $d = 2q-1$ we have

$$g_j(S_{n,d}) = D_{d,j}(S_{n,d}) = 2\binom{j+q-1}{q-1}\binom{n-j-q}{q-1}$$

and for $d = 2q$ *we have*

$$g_j(S_{n,d}) = D_{d,j}(S_{n,d}) = \binom{j+q-1}{q-1}\binom{n-j-q}{q} + \binom{j+q}{q}\binom{n-j-q-1}{q-1}.$$

The proofs of Theorem 3.3.1 and Corollary 3.3.2 are postponed to the end of this section. For $n \geq d+1$ let $S_{n,d} = \{p_1, \dots, p_n\}$ be a set of n points on the moment curve M_d (i.e. $S_{n,d}$ is the set of vertices of a cyclic polytope $P_{n,d}$) such that $p_i = (t_i, t_i^2, \dots, t_i^d)$ and $t_1 < t_2 < \dots < t_d$. An *ordered partition of a set* S is a t-tuple $(S_1, \dots, S_t)$ of (possibly empty) sets such that $S = S_1 \dot\cup S_2 \dot\cup \dots \dot\cup S_t$ and for S_a, S_b with $a < b$ it holds that the indices of the points in S_a are smaller than the indices of the points in S_b. The sets $S_1, \dots, S_t$ are called *blocks*.

Let h be the hyperplane $h_0 + h_1 x_1 + \dots + h_d x_d = 0$ in $\mathbb{R}^d$. The points of intersection of h with the moment curve M_d correspond to the roots of the polynomial

$$f(t) = h_0 + h_1 t + \dots + h_d t^d.$$

The graph of $f(t)$ is divided up by at most d intersections with the axis t into segments above, on and below the axis t (with zero or more points in $S_{n,d}$ in each segment). This gives rise to a following definition. A *PZN-partition of* $S_{n,d}$ *(induced by* h*)* is an ordered partition of $S_{n,d}$ such that each block of the partition corresponds to a segment of the graph of $f(t)$ created by h (i.e. the block contains zero or more points in $S_{n,d}$ in the corresponding segment). In addition, the blocks are "colored" by P, Z and N depending whether the segment of the block is above, on, or below the axis t, respectively. The order of blocks in the partition is determined by the order how the segments of the graph of $f(t)$ are traversed when t goes from $-\infty$ to $+\infty$.

It is not hard to see that the PZN-partitions of $S_{n,d}$ are exactly the ordered partitions of $S_{n,d}$ with blocks colored P, Z and N which fulfill the following conditions:

- the first block is a P-block or an N-block,
- if B is not the last block and it is a P-block (N-block), then B is directly followed by a Z-block and an N-block (a P-block) (i.e. the sequence is PZN or NZP),

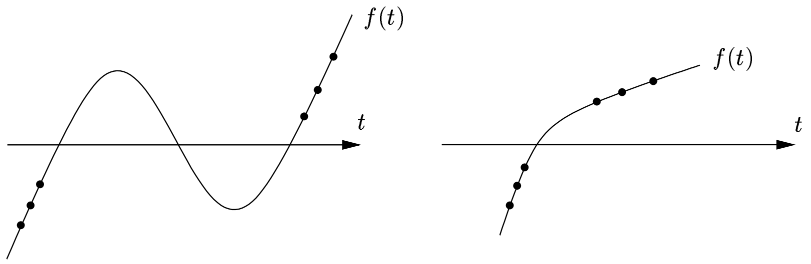

Figure 3.7: *Both PZN-partitions correspond to the same $(0, 3)$-partition*

- each Z-block has cardinality 0 or 1,
- the number of Z-blocks is at most d.

If the total cardinality of the P-blocks is j and the total cardinality of the Z-blocks is i, then such PZN-partition corresponds to an (i, j)-partition of $S_{n,d}$. Unfortunately, many different PZN-partitions can yield the same (i, j)-partition. This is illustrated in Figure 3.7. To overcome this problem, we introduce the following notion. A PZN-partition is called a *minimal PZN-partition*, if

1. all Z-blocks before the first non-empty P-block or N-block are non-empty,
2. each Z-block directly preceding an empty P-block or an empty N-block is non-empty.

Lemma 3.3.3 *Among all PZN-partitions of $S_{n,d}$ which correspond to the same (i, j)-partition of $S_{n,d}$ there is exactly one minimal PZN-partition.*

PROOF. The proof of existence is easy and left to the reader. Assume that $\mathcal{P}$ and $\mathcal{P}'$ are two minimal PZN-partitions of $S_{n,d}$ which correspond to the same (i, j)-partition of $S_{n,d}$ for some $i \in \{0, \dots, d\}$ and $j \in \{0, \dots, n-i\}$. We show first that $\mathcal{P}$ and $\mathcal{P}'$ have the same non-empty P-blocks and the same non-empty N-blocks (each non-empty block is identified by its color and by the points it contains). Assume that there is a P-block (N-block) of $\mathcal{P}$ not present in $\mathcal{P}'$. Then $\mathcal{P}'$ must contain two P-blocks (N-blocks) B_1, B_2 with $B \subseteq B_1 \cup B_2$. Since B_1 and B_2 contain consecutive points on M_d, all blocks between B_1 and B_2 must be empty. Especially, there is an empty N-block

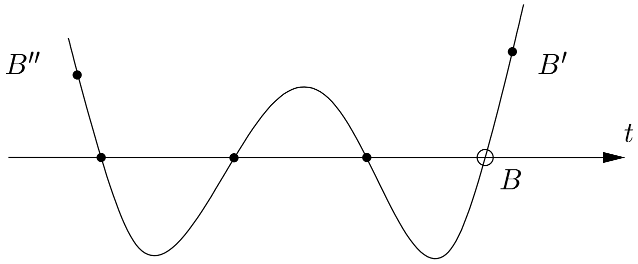

Figure 3.8: *Illustration of the argument that B must exist in $\mathcal{P}'$*

(P-block) directly preceded by an empty Z-block between B_1 and B_2, which contradicts 2. Since the non-empty Z-blocks are determined by the (i, j)-partition, $\mathcal{P}$ and $\mathcal{P}'$ have the same non-empty blocks.

Next we show that $\mathcal{P}$ and $\mathcal{P}'$ have the same empty blocks (empty blocks are identified by their color and one non-empty block directly proceeding or directly following the empty block). Assume that B is an empty Z-block of $\mathcal{P}$. Then the P-block (N-block) B' directly following B must be non-empty by 2, furthermore there is a non-empty P-block or N-block B'' before B closest to B (by 1) in $\mathcal{P}$. The number of Z-blocks between B'' and B' in $\mathcal{P}'$ is the same as in $\mathcal{P}$, since B'' has the same color in both partitions and both partitions have the same number of non-empty Z-blocks between B'' and B' (parity argument), see Figure 3.8. Since each (empty) P-block or N-block between B'' and B' is directly preceded by a non-empty Z-block in both $\mathcal{P}$ and $\mathcal{P}'$ we follow that B must exist in $\mathcal{P}'$. Now assume that B is an empty P-block (N-block) in $\mathcal{P}$. If B is the first block of $\mathcal{P}$, then clearly B also exists in $\mathcal{P}'$ by 1 and by the fact, that the first non-empty P-block or N-block has the same color in both $\mathcal{P}$ and $\mathcal{P}'$. Otherwise there is a non-empty P-block or N-block B' before B closest to B. Then B must occur in $\mathcal{P}'$ by a similar argument as before. ■

PROOF OF PROPOSITION 3.3.1. By the last lemma we have to count the minimal PZN-partitions such that the compounded cardinality of the P-blocks is j and the compounded cardinality of the Z-blocks is i. We classify the PZN-partitions by the number of their Z-blocks, and so let $B(n, j, i, s)$ be the number of such minimal PZN-partitions of $S_{n,d}$ with exactly s Z-blocks each. Among them we count the PZN-partitions which start with a P-block and have t_1 empty P-blocks and t_2 empty N-blocks. If $s = 2q - 1$, then the number of P-blocks is q and the number of N-blocks is also q. There are $\binom{q}{t_1}$ possibilities to choose the empty P-blocks among all P-blocks and $\binom{j-1}{q-t_1-1}_{-1}$ ways to partition a set of j points of $S_{n,d}$ into $q - t_1$ remaining non-empty P-blocks (if $j = 0$ and $q - t_1 = 0$, then all P-blocks are empty and we have exactly one choice). It is not hard to see that we can choose the empty N-blocks among all N-blocks independently of the choice of the empty P-blocks. There are $\binom{q}{t_2}$ choices for the empty N-blocks, and we can partition the $n - i - j$ points in $S_{n,d}$ into $q - t_2$ remaining non-empty N-blocks in $\binom{n-j-i-1}{q-t_2-1}_{-1}$ ways.

By 1 and 2 it is clear that for each empty P-block or N-block we must make a unique Z-block non-empty and that in total $t_1 + t_2$ Z-blocks become non-empty. The remaining $i - t_1 - t_2$ points in $S_{n,d}$ can be put into the remaining $s - t_1 - t_2$ Z-blocks, which is possible in $\binom{s-t_1-t_2}{s-i}$ ways. Since the same calculation also holds for the PZN-blocks which start with an N-block, the value of $B(n, j, i, s)$ is

$$2 \sum_{t_1=0}^{i} \sum_{t_2=0}^{i-t_1} \left(\binom{q}{t_1} \binom{j-1}{q-t_1-1}_{-1} \binom{q}{t_2} \binom{n-i-j-1}{q-t_2-1}_{-1} \binom{2q-1-t_1-t_2}{2q-1-i} \right).$$

The case $s = 2q$ is handled analogously. ■

PROOF OF COROLLARY 3.3.2. If $i = 0$, then the formula in Proposition 3.3.1 immediately reduces to (3.8). The formula for $g_j(S_{n,d})$ can be derived from the Proposition 3.3.1 using Vandermonde's convolution [GKP89]. Yet another proof is given in Section 5.2. ■

3.4 Estimated numbers of (i, j)-partitions

The asymptotic relations between the e-vector and the g-vector have been used frequently in proofs of the upper bounds on the number of k-sets, see e.g. [DE94]. Since we do not know any careful presentation of these relations, we treat here in more detail some estimations of the rows of the matrix D by other rows (Theorem 3.4.1). These estimations imply immediately the asymptotic relations (Corollary 3.4.2).

Theorem 3.4.1 *Let S be a set of $n > d$ points in general position in $\mathbb{R}^d$, $d \geq 2$.*

(a) *For $k \in \{1, \dots, n-1\}$ and $i \in \{2, \dots, d\}$ we have*

$$\begin{aligned} e_k(S) &\leq D_{1,k-1}(S) \quad \textit{and} \quad e_k(S) \leq D_{1,k}(S), \\ e_k(S) &\leq \frac{1}{\binom{d}{i-1}} \sum_{j=\max\{0,k-i+1\}}^{\min\{n-d,k-1\}} D_{i,j}(S) \binom{i}{k-j}, \end{aligned}$$

especially

$$e_k(S) \leq \frac{1}{d} \sum_{i=\max\{0,k-d+1\}}^{\min\{n-d,k-1\}} g_i(S) \binom{d}{k-i},$$

(b) *for $j \in \{0, \dots, n-d\}$ we have*

$$\begin{aligned} g_j(S) &\leq \sum_{k=j}^{j+d-i} D_{i,k}(S) \quad \textit{for } i \in \{1, \dots, d-1\} \quad \textit{and} \\ g_j(S) &\leq \sum_{k=j+1}^{j+d-1} e_k(S) \end{aligned}$$

(the last bound is tight for $\mathbb{R}^2$),

(c) *for $j \in \{0, \dots, n-d\}$ there exists S', $|S'| = 2|S|$ with*

$$g_j(S) \leq e_{2j+d}(S').$$

PROOF. **(a)**: Let A be a k-set of S, $k \in \{1, \dots, n-1\}$ and h a hyperplane separating A and $S \backslash A$, with $h^+ \cap S = A$, $h \cap S = \emptyset$. For $i \in \{1, \dots, d\}$ we can perturb h and obtain a hyperplane h_1 with $|h_1 \cap S| = i$ and $h_1^+ \cap S \subseteq A$. If $i = 1$, then h_1 is counted either by the number $D_{1,k-1}(S)$ (in case $h_1 \cap A \neq \emptyset$), or by the number $D_{1,k}(S)$ (in case $h_1 \cap A = \emptyset$); for every k-set, both cases are possible, which shows the first inequalities. Let a be the number of points in A which can be included one by one in a perturbed hyperplane h, and b the analogous number for $S \backslash A$. Obviously $a \geq 1$ and $b \geq 1$. To show $a + b \geq d + 1$, observe that if h is perturbed in such a way that it contains a set M of $d - 1$ points of S, then we still have a choice between at least two points of S which can be included in a perturbed h. Otherwise we could "rotate" h around M and sweep in this way the whole $\mathbb{R}^d$, which is a contradiction to $n > d$. The number of different hyperplanes h_1 such that $|h_1 \cap A| \geq 1$ and $|h_1 \cap (S \backslash A)| \geq 1$ obtained by perturbing h is obviously

$$\sum_{j=1}^{i-1} \binom{a}{j} \binom{b}{i-j}.$$

By the Vandermonde convolution formula and the fact that $\binom{a+b-1}{i} \geq \binom{a}{i} + \binom{b}{i}$ if $a \geq 1$, $b \geq 1$, $a+b-1 \geq 2$ we have

$$\sum_{j=1}^{i-1} \binom{a}{j}\binom{b}{i-j} = \binom{a+b}{i} - \binom{a}{i} - \binom{b}{i} \geq \binom{a+b-1}{i-1} \geq \binom{d}{i-1}.$$

Each such hyperplane h_1 induces an (i,j)-partition with $j \in \{k-i+1,\dots,k-1\} \cap \{0,\dots,n-d\}$, and we have shown above that at least $\binom{d}{i-1}$ such partitions yield the same k-set A. On the other hand, each (i,j)-partition with i,j as above gives rise to at most $\binom{i}{k-j}$ k-sets, and so (a) follows.

(b): Consider the arrangement $\mathcal{A}(H)$ of a set H of hyperplanes obtained from S be the duality $\mathcal{D}$ defined in Section 1.4. Since S is in general position, H is simple. We assume w.l.o.g. that no hyperplane in H is orthogonal to the x_1-axis. For $i \in \{1,\dots,d\}$, let F be a face of $\mathcal{A}(H)$ of dimension $d-i$, and v_F the unique leftmost vertex of F (i.e. a vertex of F with smallest x_1-coordinate). If the number of hyperplanes above v_F is j (and so v_F is a dual of a j-facet), then F lies below k hyperplanes in H with $k \in \{j,\dots,j+d\}$ (and so F corresponds to an (i,k)-partition of S). If $i=0$, then at least one hyperplane containing v_F is above the cell F and at least one hyperplane containing v_F is below F, and so $k \in \{j+1,\dots,j+d-1\}$. Analogously, if v_F is above exactly j hyperplanes in H, then F lies above k hyperplanes with $k \in \{j,\dots,j+d-i\}$ (or $k \in \{j+1,\dots,j+d-1\}$ if $i=0$). Since every j-facet of S dualizes to a vertex of $\mathcal{A}(H)$ with exactly j hyperplanes above or below it, we have the desired result. In $\mathbb{R}^2$ the bound for the case $i=0$ is tight as $g_j(S) = e_{j+1}(S)$ in $\mathbb{R}^2$.
(c): The set S' with a required property can be obtained as follows: we replace each point $p \in S$ by two points $p+\varepsilon\mathbf{v}$ and $p-\varepsilon\mathbf{v}$, where $\mathbf{v} \in \mathbb{R}^d$ is a vector and $\varepsilon > 0$ is a sufficiently small real number. Then every j-facet Δ of S has a unique $(2j+d)$-set in S' which is induced by a hyperplane lying between the doubled vertices of Δ. ∎

Corollary 3.4.2 *Let $d \geq 2$. For $k \in \{1,\dots,n-1\}$ and $j \in \{0,\dots,n-d\}$ we have*

$$\begin{aligned} e_k(d,n) &= O(\max_{k-d+1 \leq i \leq k-1} g_i(d,n)),\\ g_j(d,n) &= O(\max_{j+1 \leq i \leq j+d-1} e_i(d,n)), \quad \textit{and}\\ g_j(d,n) &= O(e_{2j+d}(d,2n)). \end{aligned}$$

3.5 Results on (i, j)-partitions via geometric duality

The most important goal of this section is to "translate" the main results of the paper [Mul93] by MULMULEY obtained for hyperplane arrangements into theorems concerning (i, j)-partitions. These results are presented in Theorem 3.5.2. It states in part (b) that first $k' + 1$ columns of the matrix D are determined by the matrix of the numbers $h_{i,j}$ of j-facets with exactly i incoming ridges, where $i \in \{0, \ldots, d\}$ and $j \in \{0, \ldots, k'\}$; part (c) states that this relation is invertible. The most interesting result of Theorem 3.5.2 gives the so-called generalized Dehn-Sommerville relations $h_{i,j} = h_{d-i,j}$ for each pair i, j with $i \in \{0, \ldots, d\}$ and $j \in \{0, \ldots, k'\}$. (In Section 3.9 we extend the index range for which these relations are valid.) These relations specialize for $j = 0$ (and in the dual setting) to Dehn-Sommerville relations for simplicial polytopes [Zie94]; they are fundamental in the theory of polytopes. The last part of Theorem 3.5.2 gives an upper bound on the numbers $h_{0,j} = h_{d,j}$ for $j \in \{0, \ldots, k'\}$. An extension of this result is known as the Lovász crossing lemma, see [AW97].

To obtain Theorem 3.5.2, we have to dualize the i-faces of level j of a hyperplane arrangement and the vertices of such an arrangement of level j with i so-called incoming edges. While the former objects correspond to $(d - i, j)$-partitions of the dual point set S, the latter become j-facets with i so-called incoming ridges (this is shown in Proposition 3.5.5). To ensure the conditions required in [Mul93] and to obtain a bijection between the faces of the hyperplane arrangement and the (i, j)-partitions of S it is necessary to restrict j to at most k', where k' is an integer closely related to a centerpoint of S.

Let H be a set of n hyperplanes disjoint from the origin $\mathbf{o}$ in $\mathbb{R}^d$, $\mathcal{A}(H)$ their arrangement, and S the set of points dualized by the transformation T from the hyperplanes in H, i.e. $S = \{T(h) : h \in H\}$. We assume in the following that there is no vertex $v = (v_1, \ldots, v_d)$ of $\mathcal{A}(H)$ with $v_d = 0$ and that there are no two vertices in $\mathcal{A}(H)$ with equal x_d-coordinates. These assumptions can be always fulfilled by an infinitesimally small change of the direction of the x_d-axis. Furthermore, we assume that the origin $\mathbf{o}$ is a k-centerpoint of S for some $k \geq 2d$ (see Section 3.1). We also put $k' = k - 2d$.

An essential condition for the results in [Mul93] is that the collection of faces of $\mathcal{A}(H)$ with level at most k' is bounded and simple. The latter condition translates to the fact that S is in general position, and the former condition is ensured by the choice of k' and by Proposition 3.1.1(c).

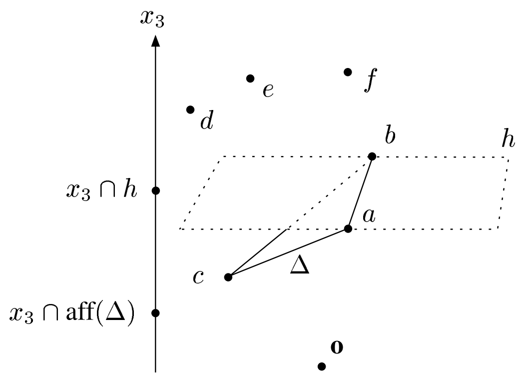

Figure 3.9: *The $(2,3)$-partition $(\{a,b\},\{d,e,f\})$ is an incoming ridge of the 3-facet Δ (x_3 denotes the x_3-axis)*

Recall from Section 2.2 that the level of a point $p \in \mathbb{R}^d$ in $\mathcal{A}(H)$ is the number of hyperplanes in H that strictly separate p and the origin **o** and the level of a face $F \in \mathcal{A}(H)$ is the level of any point in the relative interior of F. As stated in Proposition 3.1.1(b), there is a bijection between the $(d-i)$-faces of level j in $\mathcal{A}(H)$ and the (i,j)-partitions of S for each pair i,j with $i \in \{0,\dots,d\}$ and $j \in \{0,\dots,k'\}$.

The following definitions are taken from [Mul93]. For a vertex v of $\mathcal{A}(H)$ let cone(v) denote the unique cone which contains the origin **o** and which is formed by hyperplanes containing v. We orient the edges of $\mathcal{A}(H)$ in the increasing x_d-direction; this is always possible, since by assumption no two vertices of $\mathcal{A}(H)$ have the same x_d-coordinate. Then an edge of v (a 1-face of $\mathcal{A}(H)$ incident to v) in cone(v) is called an *incoming edge of v* if it is oriented towards v. We denote by $h_{i,j}$ the number of vertices in $\mathcal{A}(H)$ of level j with exactly i incoming edges, for $i \in \{0,\dots,d\}$ and $j \in \{0,\dots,k'\}$.

In following we describe the "dual" of an incoming edge. Let Δ be a j-facet of S such that $\mathbf{o} \in (\mathrm{aff}(\Delta))^-$, and let (A,B) be a (d,j)-partition of S induced by Δ. We call a $(d-1,j)$-partition (A',B') of S an *incoming ridge of* Δ if $A' \subseteq A$, $B' = B$ and no hyperplane h which induces (A',B') intersects the x_d-axis below the intersection of x_d and aff(Δ). In Figure 3.9 the 3-facet Δ in $\mathbb{R}^3$ induces the $(3,3)$-partition $(\{a,b,c\},\{d,e,f\})$. The $(2,3)$-partition $(\{a,b\},\{c,d,e\})$ induced by h is an incoming ridge of Δ, which is not hard to see.

Remark 3.5.1 *The number of incoming ridges of Δ can be also determined as follows, cf. [Wel99]. We say that a directed line l enters an oriented hyperplane h, if it intersects h in a single point, and if l is directed from the positive to the negative side of h. Now let Δ be a j-facet such that the x_d-axis enters aff(Δ). The number of incoming ridges of Δ is d, if the x_d-axis intersects Δ, otherwise it is the number of $(d-2)$-faces of Δ not visible from the intersection of aff(Δ) with the x_d-axis. If the x_d-axis does not enter aff(Δ), then it enters the affine hull of the $(n-d-j)$-facet Δ' obtained by reorienting Δ. Then the number of incoming ridges of Δ is d minus the number of incoming ridges of Δ'.*

We can now restate the results of [Mul93] for sets of points. Note that in parts (b) and (c) we use polynomials to abbreviate the presentation. The meaning of "=" in these parts is that the coefficient of $x^m y^l$ on the left-hand side of the equation is the same as the coefficient of $x^m y^l$ on the right-hand side of the equation, for every allowed integer pair m, l.

Theorem 3.5.2 *[Mul93] Let S be a set of points in general position in $\mathbb{R}^d$ such that the origin of $\mathbb{R}^d$ is a k-centerpoint of $\mathbb{R}^d$ with $k \geq 2d$. Put $k' = k - 2d$. Then*

(a) *The matrix $(h_{i,j})$, $i \in \{0, \dots, d\}$ and $j \in \{0, \dots, k'\}$, does not depend on the choice of the coordinate system (especially the x_d-axis).*

(b) *We have*

$$\sum_{i=0}^{d} \sum_{j=0}^{k'} D_{i,j} x^i y^j = \sum_{i=0}^{d} \sum_{j=0}^{k'} h_{i,j} y^j (1+x)^i (x+y)^{d-i}, \tag{3.9}$$

mod $y^{k'+1}$. Here mod $y^{k'+1}$ means that the terms that are divisible by $y^{k'+1}$ are ignored. It follows that the matrix $(D_{i,j})$ is related to the matrix $(h_{i,j})$ by a linear transformation.

(c) *The transformation (3.9) is invertible by*

$$\sum_{i=0}^{d} \sum_{j=0}^{k'} h_{i,j} x^i y^j = \left(\sum_{r \geq 0} \binom{d+r-1}{r} y^r \right) \sum_{i=0}^{d} \sum_{j=0}^{k'} D_{i,j} y^j (1-xy)^i (x-1)^{d-i} \tag{3.10}$$

mod $y^{k'+1}$.

(d) *We have the generalized Dehn-Sommerville relations*

$$h_{i,j} = h_{d-i,j} \text{ for all } i \in \{0, \dots, d\} \text{ and } j \in \{0, \dots, k'\}. \quad (3.11)$$

(e) *It holds*

$$h_{0,j} = h_{d,j} \leq \binom{j+d-1}{d-1} \text{ for } j \in \{0, \dots, k'\}.$$

Remark 3.5.3

(a) *A point set with n points has always a $\lceil n/(d+1) \rceil$-centerpoint [Ede87], so that k' can be at least $\lceil n/(d+1) \rceil - 2d$.*

(b) *Part (e) of the last Theorem is almost the same, but a slightly weaker result as the tight upper bound on the maximum number of j-facets entered by a directed line [Wel99], also known as Lovász crossing lemma [AW97]. The Lovász crossing lemma says that the inequalities of part (e) hold for* all $j \in \{0, \dots, n-d\}$.

Frequently it is useful to have an explicit form of the numbers $h_{i,j}$ in terms of $D_{i',j'}$'s. The following lemma provides such a representation.

Lemma 3.5.4 *For $i \in \{0, \dots, d\}$ and $j \in \{0, \dots, k'\}$ we have*

$$h_{i,j} = \sum_{l=0}^{n} \sum_{m=0}^{d} (-1)^{d-m+i} D_{m,l} \sum_{k=0}^{d} \binom{d-1+j-l-k}{d-1} \binom{m}{k} \binom{d-m}{i-k}.$$

PROOF. It follows by a straightforward expansion of (3.10). ∎

The following proposition gives the above-mentioned link between incoming edges and incoming ridges.

Proposition 3.5.5 *Let $v = (v_1, \dots, v_d)$ be a vertex of level $j \in \{0, \dots, k'\}$ in $\mathcal{A}(H)$ and let the j-facet Δ be the image of v under the bijection described in Proposition 3.1.1(b). Then the number of incoming edges of v equals the number of incoming ridges of Δ.*

PROOF. Let (A, B) be a (d, j)-partition of S induced by Δ. If F is an edge of cone(v) in $\mathcal{A}(H)$ then the image of F under the bijection described in Proposition 3.1.1(b) is obviously a $(d-1, j)$-partition (A', B') of S with $A' \subseteq$

A and $B' = B$ (the condition that **o** is in cone(v) translates to the fact that $B' = B$).

Let p be a point in F with coordinates $(p_1, \dots, p_d)$ such that $p_d \neq 0$. Then $T(p)$ is a hyperplane which induces (A', B') and by the definition of T the intersection z_p of $T(p)$ with the x_d-axis has the coordinates $(0, \dots, 0, 1/p_d)$. Analogously, the intersection z_v of aff(Δ) with the x_d-axis has the coordinates $(0, \dots, 1/v_d)$. If F is an incoming edge of v, then for each $p \in F$ we have $p_d < v_d$ and so the intersection z_p is above the intersection z_v on the x_d-axis. ∎

3.6 Mutations of point sets and of oriented matroids

In this and the following two sections we refine a technique of continuous motions of points in order to provide tools for proofs in Sections 3.9, 3.10 and 3.11. The fundamental notion of this technique is a mutation of a point set. It is introduced in the beginning of this section. We also refine this notion by classifying mutations into (m,l)-mutations, where $m \in \{1, \dots, d\}$ and $l \in \{0, \dots, n-d-1\}$. We regard the mutations in a more general setting of oriented matroids, and explain how the mutations of oriented matroids specialize to mutations of point sets. When the points in S are moving, then the matrix $D(S)$ of the numbers of (i,j)-partitions of S changes only when a mutation occurs. The goal of this and the next section is to characterize the change of the matrix $D(S)$ caused by an (m,l)-mutation (this goal is achieved in Section 3.7). The symbols introduced here are also used in the subsequent two sections.

We consider a continuous motion of the points in S. This paradigm has been already used in [Tve66, GH97, AAH$^+$98]. During the motion the set stays in general position, except for a finite number of discrete instances. We require for each instance that it fulfills the *mutation condition*:

1. there is a unique set $\tilde{X}$ called *mutation kernel* of $d+1$ points in a common hyperplane and $\tilde{X}$ changes the orientation, and
2. no other $i+1$ points are on a common $(i-1)$-flat, for $i = 1, \dots, d$.

If S, $|S| \geq d+2$ is the point set before the violation and S' the point set afterwards, then (S, S') is called a *mutation (of a point set)*. By the above

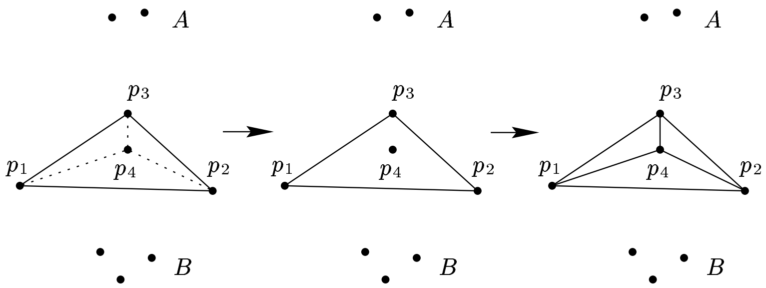

Figure 3.10: *A visualization of a* $(3,2)$*-mutation of a point set in* $\mathbb{R}^3$

condition there is a unique subset X of S (corresponding the the mutation kernel of $(S, S'))$, $|X| = d+1$, such that its convex hull is an *almost flat simplex* Δ: the affine hull of each facet of Δ partitions $S \backslash X$ into the same sets A and B. We say that a facet F of Δ is *visible (from* B*)* if the affine hull of F strictly separates B and the unique point in X which is not a vertex of F (if $B = \emptyset$, then we define such facet F of Δ as the facet whose affine hull has A and the unique point of X not in F on the same side). We show later in this section that all visible facets of Δ become invisible in the simplex Δ' of S' corresponding to Δ. We show in Section 3.7 that if m is the number of facets of Δ visible from B and $l = |A|$, then the pair (m, l) determines the change of $D(S)$. We call the corresponding mutation an (m, l)*-mutation (of a point set).* In Figure 3.10 the mutation kernel $\tilde{X}$ consists of the points $\{p_1, p_2, p_3, p_4\}$ which span the almost flat simplex Δ. In the leftmost picture of Figure 3.10 the simplex Δ has three facets visible from B, and in the rightmost picture there is only one facet of Δ visible from B. Since $|A| = 2$, we have here a $(3, 2)$-mutation.

We borrow the framework of the oriented matroid theory from [BLVS+93]. For $S = \{p_1, \ldots, p_n\}$, every hyperplane in $\mathbb{R}^d$ defines a vector $\textsc{u} \in \{+, -, 0\}^n$ by

$$(\textsc{u})_i = \begin{cases} + & \text{if } p_i \in h^+, \\ 0 & \text{if } p_i \in h, \\ - & \text{if } p_i \in h^-, \end{cases}$$

where $(\textsc{u})_i$ is the i-th entry of $\textsc{u}$. We call $\textsc{u}$ a *covector of* S. Note that a hyperplane h induces an (i, j)-partition only if it induces a covector $\textsc{u}$ where the number of 0's is i and the number of $+$'s is j. A detailed discussion of k-sets in the setting of oriented matroids is given in [Kor95].

The set $\mathcal{L}_S$ of covectors of S induced by every possible hyperplane in $\mathbb{R}^d$ determines the so-called oriented matroid of S. In general, if a subset $\mathcal{L}$ of $\{+,-,0\}^n$ fulfills certain conditions, then it determines an *oriented matroid* $\mathcal{M}(\mathcal{L})$ and the elements of $\mathcal{L}$ are called *covectors* (see [BLVS+93] for the full definition). Oriented matroids which arise from sets of points are called *realisable*. An oriented matroid is called *uniform* (and of *rank* $d+1$), if the number of 0's in every covector in $\mathcal{L}$ is at most d. For realisable oriented matroids this means that the set S inducing $\mathcal{L}_S$ is in general position. If the number of 0's of a covector of a uniform oriented matroid is d, then it is called a *minimal covector*.

We consider in the following oriented matroids which are uniform but not necessarily realisable. For $\mathcal{M}(\mathcal{L})$ let $D_{i,j}(\mathcal{L})$ denote the number of covectors in $\mathcal{L}$ where the number of 0's is i and the number of $+$'s is j, for $i = 0, \ldots, d$, $j = 0, \ldots, n$. We denote by $D(\mathcal{L}) = (D_{i,j}(\mathcal{L}))$ the matrix of these numbers. Since for a covector $\mathrm{U} \in \mathcal{L}$ also $-\mathrm{U} \in \mathcal{L}$, we have $D_{i,j}(\mathcal{L}) = D_{i,n-i-j}(\mathcal{L})$. If $\mathcal{M}(\mathcal{L})$ is realisable, then $D_{i,j}(\mathcal{L})$ is the number $D_{i,j}(S)$ of (i,j)-partitions in a point set S inducing $\mathcal{L}$.

A *mutation (of an oriented matroid)* is a pair $(\mathcal{L}, \mathcal{L}')$ of covector sets of two oriented matroids $\mathcal{M}(\mathcal{L})$, $\mathcal{M}(\mathcal{L}')$ which fulfills the following conditions.

- $\mathcal{L} \backslash \mathcal{L}'$ has as minimal covectors exactly $2(d+1)$ covectors $\mathrm{C}_1, \ldots, \mathrm{C}_{d+1}, -\mathrm{C}_1, \ldots, -\mathrm{C}_{d+1}$ of the form

  $$\mathrm{C}_i = (\overbrace{0, \ldots, 0}^{i-1} s_i \overbrace{0, \ldots, 0}^{d+1-i})\mathrm{Z}, \quad s_i \in \{+,-\}, \quad \mathrm{Z} \in \{+,-\}^{n-d-1},$$

  i.e. C_i has $s_i \in \{+,-\}$ in position i, 0's in positions $1, \ldots, i-1, i+1, \ldots, d+1$ and a sign vector $\mathrm{Z} \in \{+,-\}^{n-d-1}$ common to all $\mathrm{C}_1, \ldots, \mathrm{C}_{d+1}$ (for $i = 1, \ldots, d+1$).

- $\mathcal{L}' \backslash \mathcal{L}$ has as minimal covectors exactly $2(d+1)$ covectors $\mathrm{C}'_1, \ldots, \mathrm{C}'_{d+1}, -\mathrm{C}'_1, \ldots, -\mathrm{C}'_{d+1}$, where

  $$\mathrm{C}'_i = (\overbrace{0, \ldots, 0}^{i-1}(-s_i)\overbrace{0, \ldots, 0}^{d+1-i})\mathrm{Z},$$

  i.e. C'_i is the same as C_i but with negated sign at position i, for $i = 1, \ldots, d+1$.

We say that a mutation is an (m,l)-*mutation (of an oriented matroid)* or a *mutation of type* (m,l), if m is the number of $+$'s among $s_1, \ldots, s_{d+1}$ and l

the number of $+$'s in the sign vector z. We assume w.l.o.g. that $s_1 = \ldots = s_m = +$, $s_{m+1} = \ldots = s_{d+1} = -$ and that $m \in \{1, \ldots, n\}$.

Assume that in Figure 3.10 the points $p_1, \ldots, p_4$ have indices $1, \ldots, 4$ (resp.), that the two points in A have indices 5 and 6 and that the points in B have indices $7, 8, 9$. Furthermore let us denote as S the point set in the leftmost picture of Figure 3.10 and as S' the rightmost picture in Figure 3.10. Then the minimal covectors of $\mathcal{L}_S \backslash \mathcal{L}_{S'}$ are exactly

$$\begin{aligned} \mathrm{c}_1 &= (+000\mathrm{z}), \\ \mathrm{c}_2 &= (0+000\mathrm{z}), \\ \mathrm{c}_3 &= (00+0\mathrm{z}), \\ \mathrm{c}_4 &= (000-\mathrm{z}), \end{aligned}$$

and $-\mathrm{c}_1, \ldots, -\mathrm{c}_4$, where $\mathrm{z} = (++---)$. Consequently, the minimal covectors of $\mathcal{L}_{S'} \backslash \mathcal{L}_S$ are exactly

$$\begin{aligned} \mathrm{c}'_1 &= (-000\mathrm{z}), \\ \mathrm{c}'_2 &= (0-000\mathrm{z}), \\ \mathrm{c}'_3 &= (00-0\mathrm{z}), \\ \mathrm{c}'_4 &= (000+\mathrm{z}), \end{aligned}$$

and $-\mathrm{c}'_1, \ldots, -\mathrm{c}'_4$.

Remark 3.6.1 *If $(\mathcal{L}, \mathcal{L}')$ is an (m, l)-mutation, then $(\mathcal{L}', \mathcal{L})$ is an $(d+1-m, l)$-mutation. Furthermore, since $\mathrm{u} \in \mathcal{L}$ implies $-\mathrm{u} \in \mathcal{L}$, each (m, l)-mutation is simultaneously an $(d+1-m, n-d-1-l)$-mutation.*

The following lemma is necessary to show that if (S, S') is an (m, l)-mutation of a point set, then $(\mathcal{L}_S, \mathcal{L}_{S'})$ is an (m, l)-mutation of an oriented matroid. The meaning of Δ, Δ', X, A, B is as in the beginning of this section. Furthermore, h^0 denotes the hyperplane containing the unique set of $d+1$ coplanar points in the instance of the point set violating the general position.

Lemma 3.6.2 *Each facet of Δ visible from B in S becomes an invisible facet of Δ' in S'.*

PROOF. Fix a point p in the set X corresponding to the mutation kernel. Let h (h') be the hyperplane spanned by the points $X \backslash \{p\}$ in S (in S'), and let q (q')

be the orthogonal projection of p onto h (h'). We denote by $\mathcal{A}(h)$ the $(d-1)$-dimensional arrangement in h of the affine hulls of all subsets in $X \setminus \{p\}$ with $d-1$ elements; the meaning of $\mathcal{A}(h^0)$ and of $\mathcal{A}(h')$ is analogous.

By the mutation condition, the arrangement $\mathcal{A}(h)$ is (combinatorially) the same as $\mathcal{A}(h^0)$ and $\mathcal{A}(h')$ (otherwise we would have i affine dependent points for some $i \in \{2, \ldots, d\}$ in the instance violating the general position). By the same reason we may assume that q lies in an open region of $\mathcal{A}(h)$ and q' lies in an open region of $\mathcal{A}(h')$. Furthermore, q is in the same region of $\mathcal{A}(h)$ as q' in $\mathcal{A}(h')$ (here regions are the same, if they have the same set of vertices).

Now, it is not hard to see that the facets of Δ incident to p and visible from B are exactly the facets of Δ' incident to p, which are not visible from B in S' (sloppy speaking, in both Δ and Δ' the point p stays in an open polyhedra of dimension d whose facets are orthogonal to h^0 and essentially project to the boundary of the region containing p in h). ■

Corollary 3.6.3 *If (S, S') is an (m, l)-mutation of a point set, then $(\mathcal{L}_S, \mathcal{L}_{S'})$ is an (m, l)-mutation of the oriented matroid $\mathcal{M}(\mathcal{L}_S)$.*

PROOF. Each C_i in $\mathcal{L}_S$ becomes C'_i in $\mathcal{L}_{S'}$ as required (each C_i corresponds to a facet of Δ visible from B in S, and B is exactly the set of points p_i with $(\mathrm{Z})_i = -$). ■

Finally, we show that an $(\overline{m}, l)$-mutation for $l \in \{0, \ldots, n-d-1\}$ and $\overline{m} \in \{1, \ldots, d\}$ can be transformed into an (m, l)-mutation for every $m \in \{1, \ldots, d\}$ by some local changes of the mutation kernel.

Lemma 3.6.4 *Assume that $(\bar{S}, \bar{S}')$ is an $(\bar{m}, l)$-mutation. Then there are point sets S, S' such that (S, S') is an (m, l)-mutation for every $m \in \{1, \ldots, d\}$.*

PROOF. Consider a simplex Δ in $\mathbb{R}^d$. We orient the affine hulls of the facets of Δ in such a way that Δ is on the negative side of each of them. It is not hard to see that these hyperplanes create an arrangement $\mathcal{A}(\Delta)$ of $\mathbb{R}^d$ such that for each $m \in \{0, \ldots, d\}$ there is a cell of $\mathcal{A}(\Delta)$ which is on the positive side of exactly m hyperplanes. (If $m = 0$, then the cell is the relative interior of Δ). If Δ is an almost flat simplex and a convex hull of a set $\bar{X} \subseteq \bar{S}$ corresponding to the mutation kernel of $(\bar{S}, \bar{S}')$, then the cell of $\mathcal{A}(\Delta)$ which contains B is on the positive side of exactly $\bar{m}$ hyperplanes of $\mathcal{A}(\Delta)$. Now, to obtain S we rotate Δ until the set B is contained in a cell of $\mathcal{A}(\Delta)$ which lies on the positive side of exactly m hyperplanes for a given $m \in \{1, \ldots, d\}$ and then possibly stretch $\bar{X}$ such that it becomes an almost flat simplex in the new position. It is not hard to see that also in the case that B is empty we can obtain in this way an almost flat simplex with exactly m facets visible (from B). Finally, we obtain S' by moving the points in $\bar{X}$ in such a way that during the movement there is exactly one violation of the general condition assumption and it fulfills the mutation condition. ■

3.7 Changes in numbers of (i, j)-partitions due to a mutation

In this section we characterize the change of the matrix $D(\mathcal{L})$ due to a mutation of an oriented matroid $\mathcal{M}(\mathcal{L})$. The main result of this section is the Theorem 3.7.1, which completely characterizes such a change of the matrix $D(\mathcal{L})$. The most interesting special cases of this theorem are the changes of the e-vector and the g-vector due to a mutation. They are stated in Corollary 3.7.2.

For integers d, i, j, a, b let us define

$$T_{d,i,j}(a, b) = \binom{a}{d+1-i-j+b}\binom{d+1-a}{j-b}.$$

Theorem 3.7.1 *Let $(\mathcal{L}, \mathcal{L}')$ be an (m, l)-mutation with $l \in \{0, \dots, n-d-1\}$ and $m \in \{1, \dots, d\}$. Then for every $i \in \{0, \dots d\}$ and $j \in \{0, \dots, n\}$ the difference $D''_{i,j}(m, l) := D_{i,j}(\mathcal{L}') - D_{i,j}(\mathcal{L})$ is:*

$$\begin{aligned} D''_{i,j}(m, l) &= T_{d,i,j}(m, l) - T_{d,i,j}(d+1-m, l) \\ &\quad + T_{d,i,j}(d+1-m, n-d-1-l) - T_{d,i,j}(m, n-d-1-l). \end{aligned}$$

If $l < \lceil (n-3d)/2 \rceil$ or $l > \lceil n/2 \rceil - 1$, then the formula reduces to

$$D''_{i,j}(m, l) = T_{d,i,j}(m, l) - T_{d,i,j}(d+1-m, l).$$

Corollary 3.7.2 *Let (S, S') be an (m, l)-mutation of a point set.*

(a) *For $j \in \{0, \dots, n-d\}$ we have $g_j(S') - g_j(S) = g_{n-d-j}(S') - g_{n-d-j}(S)$, for $j \notin \{l, l+1, n-d-l-1, n-d-l\}$ we have $g_j(S') - g_j(S) = 0$, and*

$$\begin{aligned} &g_l(S') - g_l(S) \\ &\quad = (2m-d-1)(1 - [2l = n-d-1] + [2l = n-d]), \\ &g_{l+1}(S') - g_{l+1}(S) \\ &\quad = (d+1-2m)(1 - [2l = n-d-1] + [2l = n-d-2]). \end{aligned}$$

(b) *If $m = d+1-m$, then $e_k(S') - e_k(S) = 0$ for all $k \in \{1, \dots, n-1\}$. Otherwise we have for $k \in \{1, \dots, n-1\}$ that $e_k(S') - e_k(S) = e_{n-k}(S') - e_{n-k}(S)$, for $k \notin \{l+m, l+d+1-m, n-l-m, n-l-d-1+m\}$ we have $e_k(S') - e_k(S) = 0$, and*

$$\begin{aligned} & e_{l+m}(S') - e_{l+m}(S) \\ & \qquad = -1 + [2l = n-d-1] - [2l = n-2m], \\ & e_{l+d+1-m}(S') - e_{l+d+1-m}(S) \\ & \qquad = 1 - [2l = n-d-1] + [2l = n-2d-2+2m]. \end{aligned}$$

The proofs of Theorem 3.7.1 and Corollary 3.7.2 can be found at the end of this section. We proceed now with a lemma necessary to show Theorem 3.7.1. For two covectors U_1 and U_2, their *composition* is the covector $\mathrm{U}_2 \circ \mathrm{U}_1$ with

$$(\mathrm{U}_2 \circ \mathrm{U}_1)_i = \begin{cases} (\mathrm{U}_2)_i & \text{if } (\mathrm{U}_1)_i = 0, \\ (\mathrm{U}_1)_i & \text{if } (\mathrm{U}_1)_i \in \{+,-\}, \end{cases}$$

for $i = 1, \dots, n$. We also call a composition the successive composition (from right to left)

$$\mathrm{U}_t \circ \mathrm{U}_{t-1} \circ \dots \circ \mathrm{U}_1.$$

Each covector can be obtained as a composition of finitely many minimal covectors [BLVS$^+$93, p. 141].

Lemma 3.7.3 *Let $(\mathcal{L}, \mathcal{L}')$ be a mutation. Then the covectors in $\mathcal{L} \backslash \mathcal{L}'$ are exactly the compositions $\mathrm{U}_t \circ \dots \circ \mathrm{U}_1$ of covectors $\mathrm{U}_1, \dots, \mathrm{U}_t \in \mathcal{L}$ with*

$$\mathrm{U}_1 \in \{\mathrm{C}_1, \dots, \mathrm{C}_{d+1}, -\mathrm{C}_1, \dots, -\mathrm{C}_{d+1}\},$$

and the covectors in $\mathcal{L}' \backslash \mathcal{L}$ are exactly the compositions $\mathrm{U}'_t \circ \dots \circ \mathrm{U}'_1$ of covectors $\mathrm{U}'_1, \dots, \mathrm{U}'_t \in \mathcal{L}'$ with

$$\mathrm{U}'_1 \in \{\mathrm{C}'_1, \dots, \mathrm{C}'_{d+1}, -\mathrm{C}'_1, \dots, -\mathrm{C}'_{d+1}\}.$$

PROOF. Let $\mathrm{U} \in \mathcal{L}$ be a covector with a non-0 entry in position $r \in \{1, \dots, d+1\}$. Let $\mathrm{U}_1, \dots, \mathrm{U}_t$ be covectors in $\mathcal{L}$ such that $\mathrm{U} = \mathrm{U}_t \circ \dots \circ \mathrm{U}_1$. Assume that $\mathrm{U}_1 \notin \{\mathrm{C}_1, \dots, \mathrm{C}_{d+1}, -\mathrm{C}_1, \dots, -\mathrm{C}_{d+1}\}$. Then there is a smallest index $j \in \{2, \dots, t\}$ with $\mathrm{U}_j \circ \dots \circ \mathrm{U}_1 \in \mathcal{L} \backslash \mathcal{L}'$. Then $\mathrm{U}_j \in \{\mathrm{C}_1, \dots, \mathrm{C}_{d+1}, -\mathrm{C}_1, \dots, -\mathrm{C}_{d+1}\}$ and $\mathrm{U}_{j-1} \circ \dots \circ \mathrm{U}_1 \in \mathcal{L}'$. Assume that $\mathrm{U}_j = \mathrm{C}_i$ for some $i \in \{1, \dots, d+1\}$ (the case $\mathrm{U}_j = -\mathrm{C}_i$ is handled analogously). Obviously $\mathrm{U}_j \circ \dots \circ \mathrm{U}_1$ and $\mathrm{C}_r \circ \mathrm{U}_{j-1} \circ \dots \circ \mathrm{U}_1 = \mathrm{C}'_r \circ \mathrm{U}_{j-1} \circ \dots \circ \mathrm{U}_1$

are equal in positions $d+2, \dots, n$ and have no 0's in these positions. Therefore $\mathrm{U}_j \circ \dots \circ \mathrm{U}_1 = (-\mathrm{C}'_i) \circ \mathrm{C}'_r \circ \mathrm{U}_j \circ \dots \circ \mathrm{U}_1$ (since $(\mathrm{U}_j)_i = (-\mathrm{C}'_i)_i$). As $\mathrm{U}_{j-1} \circ \dots \circ \mathrm{U}_1 \in \mathcal{L}'$, $-\mathrm{C}'_i \in \mathcal{L}'$ and $\mathrm{C}'_r \in \mathcal{L}'$, also $\mathrm{U}_j \circ \dots \circ \mathrm{U}_1 \in \mathcal{L}'$, and we have a contradiction. The second part of the lemma ($\mathrm{U}' \in \mathcal{L}'$) is treated analogously. ■

PROOF OF THEOREM 3.7.1. By Lemma 3.7.3 every covector in $\mathcal{L}\backslash\mathcal{L}'$ has form UZ or $-$ (UZ) for certain $\mathrm{U} \in \{+,-,0\}^{d+1}$. We consider the contribution of the covectors of the form UZ to $D''_{i,j}(m,l)$ first. Let $a(i,j)$ be the number of such covectors in $\mathcal{L}\backslash\mathcal{L}'$ with i 0's (in positions $1, \dots, d+1$) and j $+$'s in all positions, $j \geq l$. Let $a'(i,j)$ be the analogous number for $\mathcal{L}'\backslash\mathcal{L}$. The number of covectors counted for $a(i,j)$ is the number of covectors of the form UZ, where $\mathrm{U} \in \{+,-,0\}^{d+1}$, the number of 0's of U is i and the number of $+$'s is $(j-l)$ (there are $\binom{d+1}{i}\binom{d+1-i}{j-l}$ such covectors UZ), minus covectors "forbidden" by Lemma 3.7.3, which are of the form $\mathrm{U}_1\mathrm{U}_2\mathrm{Z}$, where $\mathrm{U}_1 \in \{0,-\}^m$, $\mathrm{U}_2 \in \{+,0\}^{d+1-m}$, the number of $+$'s in U_2 is $(j-l)$ and the number of 0's in U_1 is $i-((d+1-m)-(j-l))$ (there are $\binom{m}{i-d-1+m+j-l}\binom{d+1-m}{j-l} = T_{d,i,j}(m,l)$ of them). Therefore,

$$a(i, j-l) = \binom{d+1}{i}\binom{d+1-i}{j-l} - T_{d,i,j}(m,l).$$

For $a'(i,j)$ we count the covectors of the form UZ, where $\mathrm{U} \in \{+,-,0\}^{d+1}$, the number of 0's in U is i and the number of $+$'s in U is $(j-l)$, minus covectors of the form $\mathrm{U}'_1\mathrm{U}'_2\mathrm{Z}$, where $\mathrm{U}'_1 \in \{+,0\}^m$, $\mathrm{U}'_2 \in \{0,-\}^{d+1-m}$, the number of $+$'s in U'_1 is $(j-l)$ and the number of 0's in U'_2 is $i-(m-(j-l))$. Therefore,

$$a'(i, j-l) = \binom{d+1}{i}\binom{d+1-i}{j-l} - T_{d,i,j}(d+1-m,l).$$

The contribution of the covectors of the form $-(\mathrm{UZ})$, $\mathrm{U} \in \{+,-,0\}^{d+1}$ to $D''_{i,j}(m,l)$ is the following. Each covector $-(\mathrm{UZ})$ with i 0's and j $+$'s in $\mathcal{L}\backslash\mathcal{L}'$ has a counterpart UZ with i 0's and $(n-i-j)$ $+$'s, and so the number of such covectors is $a(i, n-i-j)$. Analogously, the number of covectors in $\mathcal{L}'\backslash\mathcal{L}$ of type $-(\mathrm{UZ})$ with i 0's and j $+$'s is $a'(i, n-i-j)$. We conclude that

$$D''_{i,j}(m,l) = a'(i,j) - a(i,j) + a'(i,n-i-j) - a(i,n-i-j).$$

For the special case notice that $a'(i,j) - a(i,j) \neq 0$ only if $j \in \{l, \dots, l+d\}$ and $a'(i,n-i-j) - a(i,n-i-j) \neq 0$ only if $j \in \{n-i-l-d, \dots, n-i-l\}$.

If $l < \lceil (n-3d)/2 \rceil$ or $l > \lceil n/2 \rceil - 1$, then $\{l, \ldots, l+d\} \cap \{n-i-l-d, \ldots, n-i-l\} = \emptyset$, and so $D''_{i,j}(m,l) = D''_{i,n-i-j}(m,l) = a'(i,j) - a(i,j)$. ■

Remark 3.7.4 *We have* $D''_{i,j}(m,l) = D''_{i,n-i-j}(m,l)$*, furthermore* $D''_{i,j}(m,l) \neq 0$ *only if* $j \in \{l, \ldots, l+d\} \cup \{n-i-l-d, \ldots, n-i-l\}$*. Also notice that Remark 3.6.1 implies* $D''_{i,j}(m,l) = -D''_{i,j}(d+1-m,l)$*, as well as* $D''_{i,j}(m,l) = D''_{i,j}(d+1-m, n-d-1-l)$*. Both equations are confirmed by the explicit formula for* $D''_{i,j}(m,l)$*.*

3.8 Changes in numbers of (i,j)-partitions due to continuous motion

In this section we solve the following problem: assume that we start with a point set S_I in general position and continuously move the points until they form another (given) set S_F of points in general position. If we store the numbers of (m,l)-partitions for each pair m,l with $m \in \{1, \ldots, d\}$ and $l \in \{0, \ldots, n\}$, can we determine $D(S_F)$ if $D(S_I)$ is known? The answer to this question is "yes", and we give an explicit formula for each entry $D_{i,j}$ of the matrix $D(S_F)$ in Theorem 3.8.1. We can write this result in a more concise form using a two-variable polynomial. To this aim let us define a polynomial $D(S,x,y)$ associated with $D(S)$ (for a given S)) as

$$D(S,x,y) = \sum_{j=0}^{n} \sum_{i=0}^{d} D_{i,j}(S) x^i y^j. \tag{3.12}$$

Then in Theorem 3.8.2 we obtain a one-line formula for the for the matrix $D(S_F)$. This result is the main argument of the Lemma 3.9.4 in Section 3.9. Before we state the exact results, we need some definitions.

Assume that S_I and S_F are point sets in general position in $\mathbb{R}^d$, with $n \geq d+2$ points each. It is not hard to see that a following continuous transformation $\mathcal{T}$ from S_I (the "initial" point set) into S_F (the "final" point set) fulfills the mutation condition defined in Section 3.6. For every $i = 1, \ldots, n$ move p_i from its position in S_I to its position in S_F such that the trajectory of p_i is a line segment. During this movement, the position of every p_j, $j \in \{1, \ldots, n\} \backslash \{i\}$ remains unchanged. If the mutation condition is violated, then we slightly perturb the position of p_i in order to avoid this violation.

For $m = 1, \dots n$ and $l = 0, \dots n - d - 1$, let $c_{m,l}$ be the number of (m, l)-mutations minus the number of $(d+1-m, l)$-mutations which occurred during the transformation $\mathcal{T}$. For the sake of convenience we put $c_{m,l} = 0$ if $l \notin \{0, \dots, n-d-1\}$ or if $m \notin \{1, \dots, d\}$. Such definition of $c_{m,l}$ makes sense since by Remark 3.6.1 an $(d + 1 - m, l)$-mutation compensates the effect of an (m, l)-mutation on the matrix D. Thus $c_{m,l}$ counts the "net" number of the (m, l)-mutations. By the same remark we have

$$c_{m,l} = -c_{d+1-m,l} \quad \text{and} \tag{3.13}$$

$$c_{m,l} = c_{d+1-m,n-d-1-l}. \tag{3.14}$$

Theorem 3.8.1 *For $n \geq d + 2$, $i = 0, \dots, d$ and $j = 0, \dots, n$ we have*

$$D_{i,j}(S_F) = D_{i,j}(S_I) + \sum_{l,m} c_{m,l} T_{d,i,j}(m, l).$$

Theorem 3.8.2 *For point sets S_I, S_F such that there is a continuous transformation from S_I to S_F fulfilling the mutation condition we have*

$$D(S_F, x, y) = D(S_I, x, y) + (x + y) \sum_{l=0}^{n} \sum_{m=0}^{d} c_{m,l}\, y^l (1 + x)^m (x + y)^{d-m} \tag{3.15}$$

mod y^{n+1}, where mod y^{n+1} means that terms divisible by y^{n+1} are ignored.

Proof of Theorem 3.8.1. For integers m, l let $m' = d + 1 - m$ and $l' = n - d - 1 - l$. Let $\overline{c}_{m,l}$ be the number of (m, l)-mutations during $\mathcal{T}$ (so $c_{m,l} = \overline{c}_{m,l} - \overline{c}_{m',l}$). Since $\overline{c}_{m,l}$ and $\overline{c}_{m',l'}$ both count the (m, l)-mutations, we have by Theorem 3.7.1

$$\begin{aligned} D_{i,j}(S_F) \quad = \quad & D_{i,j}(S_I) + \frac{1}{2} \sum_{l,m} \overline{c}_{m,l} \left(T_{d,i,j}(m, l) - T_{d,i,j}(m', l) + \right. \\ & \left. T_{d,i,j}(m', l') - T_{d,i,j}(m, l')\right) \end{aligned}$$

By changing the summation variable we obtain

$$-\sum_{l,m} \overline{c}_{m,l} T_{d,i,j}(m', l) = -\sum_{l,m} \overline{c}_{m',l} T_{d,i,j}(m, l),$$

and so

$$\sum_{l,m}(\overline{c}_{m,l}-\overline{c}_{m',l})T_{d,i,j}(m,l)=\sum_{l,m}c_{m,l}T_{d,i,j}(m,l).$$

Analogously, using $\overline{c}_{m,l}=\overline{c}_{m',l'}$ we have

$$\sum_{l,m}\overline{c}_{m,l}T_{d,i,j}(m',l')=\sum_{l,m}\overline{c}_{m',l'}T_{d,i,j}(m',l')=\sum_{l,m}\overline{c}_{m,l}T_{d,i,j}(m,l),$$

and

$$-\sum_{l,m}\overline{c}_{m,l}T_{d,i,j}(m,l')=-\sum_{l,m}\overline{c}_{m',l'}T_{d,i,j}(m,l')=-\sum_{l,m}\overline{c}_{m',l}T_{d,i,j}(m,l),$$

which yields the desired result. ∎

PROOF OF THEOREM 3.8.2. We have

$$D(S_F,x,y)=D(S_I,x,y)+\sum_{l=0}^{n}\sum_{m=0}^{d}c_{m,l}\left(\sum_{j=0}^{n}y^j\sum_{i=0}^{d+1}x^i\,T_{d,i,j}(m,l)\right)$$

by Theorem 3.8.1 and the fact that by (3.13) the coefficient of $x^{d+1}y^j$ in the last expression is $\sum_{m=0}^{d}c_{m,j}=\sum_{m=0}^{d+1}c_{m,j}=0$. We introduce a new variable $k=j-l$. The expression $T_{d,i,j}(m,l)$ is non-zero only if $k\in\{0,\ldots,d+1-m\}$. Therefore, for fixed l and m it holds

$$\begin{aligned}
&\sum_{j=0}^{n}y^j\sum_{i=0}^{d+1}x^iT_{d,i,j}(m,l)\\
&\qquad=\sum_{k=0}^{d+1-m}y^{l+k}\sum_{i=0}^{d+1}x^i\binom{m}{d+1-i-k}\binom{d+1-m}{k}\bmod y^{n+1}\\
&\qquad=\sum_{k=0}^{d+1-m}y^{l+k}\binom{d+1-m}{k}\sum_{i=0}^{d+1}x^i\binom{m}{d+1-i-k}\bmod y^{n+1}.
\end{aligned}$$

We have

$$\begin{aligned}
\sum_{i=0}^{d+1}x^i\binom{m}{d+1-i-k} &= \sum_{i=d+1-m-k}^{d+1-k}x^i\binom{m}{m-d-1+i+k}\\
&= \sum_{i'=0}^{m}x^{i'+d+1-m-k}\binom{m}{i'}\\
&= (1+x)^m x^{d+1-m-k}
\end{aligned}$$

since the binomial is zero for $i < d+1-m-k$ and for $i > d+1-k$. Furthermore, we have

$$\begin{aligned}\sum_{k=0}^{d+1-m} & y^{l+k}\binom{d+1-m}{k}(1+x)^m x^{d+1-m-k} \\ &= y^l(1+x)^m \sum_{k=0}^{d+1-m} y^k x^{d+1-m-k}\binom{d+1-m}{k} \\ &= y^l(1+x)^m(x+y)^{d+1-m}\end{aligned}$$

which completes the proof. ■

Remark 3.8.3 *While we can always start with a point set of n points in general position and obtain by a sequence of mutations any other point set of n points in general position (by a transformation described above), it is not clear whether this statement holds for oriented matroids. The reason is that it is not known whether for an oriented matroid there is a mutation at all (see [BLVS+93, Section 7.3]). While by Shannon's Theorem [Sha79] every realizable uniform oriented matroid on n points (and so every set of n points in $\mathbb{R}^d$) allows at least n mutations,* RICHTER-GEBERT *constructed, for every $n \geq 2$, a uniform oriented matroid of rank 4 on $4n$ points which allows only $3n+1$ mutations [Ric92]. Due to this, we consider in this section set of points and not oriented matroids.*

Assume that S_I is a point set for which the matrix $D(S_I)$ is known (this is the case if S_I is a vertex set of a cyclic polytope). Then knowing $c_{m,l}$ for all $m = 0, \dots, d$ and $l = 0, \dots, n$ we could obtain $D(S_F, x, y)$ by (3.15). Since by (3.13) and (3.14) there is a redundancy among the numbers $c_{m,l}$, we conjecture that already few rows of the matrix $D(S_F)$ determine the remaining rows of $D(S_F)$. More precisely, experiments support the following conjecture:

Conjecture 3.8.4 *For each $d \geq 2$ and each $n \geq d+2$ there are subsets of $\lfloor d/2 \rfloor$ rows of $D(S_F)$ which determine the remaining rows of $D(S_F)$ by (3.13), (3.14) and (3.15).*

To support this conjecture, we show in Section 3.10 how to obtain in $\mathbb{R}^3$ the first row of $D(S_F)$ from the last row and vice versa.

3.9 Generalized Dehn-Sommerville relations

In this section we extend the range of the index j for which the generalized Dehn-Sommerville relations $h_{i,j} = h_{d-i,j}$ shown in Theorem 3.5.2(d) are valid: we extend the range from $j \in \{0,\ldots,\lceil n/(d+1) - 2d\rceil\}$ to $j \in \{0,\ldots,k''\}$, where $k'' = \lfloor \max\{(n-3d-3)/2, n/(d+1) - 2d\}\rfloor$ (Theorem 3.9.2). In addition to this result, we show that the generalized Dehn-Sommerville relations are valid "modulo a constant" for all $j \in \{0,\ldots,n\}$ (Theorem 3.9.1). This result gives hope to show the generalized Dehn-Sommerville relations using only the results about motions of point sets and independently from the results in [Mul93], see the discussion preceding Conjecture 3.9.6. The results in this section are an application of the results based on motions of point sets presented in Sections 3.6, 3.7 and 3.8.

Before we can state the exact results, we have to "redefine" the numbers $h_{i,j}$. The reason is that the numbers $h_{i,j}$ in Theorem 3.5.2 are defined only for $j \leq k'$, where k' is an integer such that there is a $(k'+2d)$-centerpoint of S. The following definition overcomes this weakness by defining $\overline{h}_{i,j}$ as a linear combination of $D_{i',j'}$ for $i' \leq i$ and $j' \leq j$. Unfortunately, for $j > k'$ we could not find any geometrical interpretation of the numbers $\overline{h}_{i,j}$. For each $j \leq k'$ (and each $i \in \{0,\ldots,d\}$) we have $\overline{h}_{i,j} = h_{i,j}$ and so $\overline{h}_{i,j}$ have a geometric interpretation for this range, see Section 3.5.

For a point set S of $n \geq d+2$ points in general position in $\mathbb{R}^d$, $i \in \{0,\ldots,d\}$ and $j \geq 0$, let us define the coefficients $\overline{h}_{i,j} = \overline{h}_{i,j}(S)$ by the equation

$$\sum_{i=0}^{d}\sum_{j\geq 0}\overline{h}_{i,j}\,x^i y^j = \qquad (3.16)$$

$$\left(\sum_{r\geq 0}\binom{d+r-1}{r}y^r\right)\sum_{i=0}^{d}\sum_{j\geq 0}D_{i,j}\,y^j(1-xy)^i(x-1)^{d-i}.$$

A comparison with (3.10) makes clear that $\overline{h}_{i,j} = h_{i,j}$ for $i \in \{0,\ldots,d\}$ and $j \in \{0,\ldots,k'\}$ (where k' is an integer described in Theorem 3.5.2). By a straightforward expansion of (3.16) one can show that the numbers $\overline{h}_{i,j}$ for $i \in \{0,\ldots,d\}$ and $j \geq 0$ can be equivalently defined by the equation

$$\overline{h}_{i,j} = \sum_{l=0}^{n}\sum_{m=0}^{d}(-1)^{d-m+i}D_{m,l}\sum_{k=0}^{d}\binom{d-1+j-l-k}{d-1}\binom{m}{k}\binom{d-m}{i-k}.$$

The following two theorems are the main results of this section.

Theorem 3.9.1 *Let S be a set of $n \geq d+2$ points in general position in $\mathbb{R}^d$. Then for any $i \in \{0, \dots, d\}$ and any $j \in \{0, \dots, n\}$ we have*

$$\overline{h}_{d-i,j}(S) = \overline{h}_{i,j}(S) + C_{n,d,i,j}, \tag{3.17}$$

where $C_{n,d,i,j}$ is a constant depending only of n, d, i and j.

Theorem 3.9.2 *Let S be a point set in general position with $n \geq \min\{3d+3, 2d(d+1)\}$ points in $\mathbb{R}^d$, $d \geq 2$, and $k'' = \lfloor \max\{(n-3d-3)/2, n/(d+1) - 2d\} \rfloor$. Then for all $i \in \{0, \dots, d\}$ and all $j \in \{0, \dots, k''\}$ we have the generalized Dehn-Sommerville relations*

$$\overline{h}_{i,j}(S) = \overline{h}_{d-i,j}(S). \tag{3.18}$$

The proofs of the above theorems can be found later in this section. Before we start proving lemmata and propositions necessary to show Theorem 3.9.1 and Theorem 3.9.2, let us "invert" (3.16) in order to obtain an analogous relation as (3.9) in part (a) of the following lemma. In part (b) we give the numbers of (i,j)-partitions as (positive) linear combinations of the numbers $\overline{h}_{i,j}$; notice that this is analogous to the situation for convex polytopes, where entries of the f-vector are positive linear combinations of the entries of the h-vector [Zie94, page 249].

Lemma 3.9.3

(a) *For a point set S in general position we have*

$$\sum_{i=0}^{d} \sum_{j \geq 0} D_{i,j} x^i y^j = \sum_{i=0}^{d} \sum_{j \geq 0} \overline{h}_{i,j}\, y^j (1+x)^i (x+y)^{d-i}. \tag{3.19}$$

Here the equality means that the coefficient of $x^i y^j$ in the formal series on the l.h.s. is equal to the coefficient of $x^i y^j$ in the formal series on the r.h.s of (3.19), for all $i \in \{0, \dots, d\}$, $j \geq 0$.

(b) *For a point set S in general position, $i \in \{0, \dots, d\}$ and $j \in \{0, \dots, n\}$ we have*

$$D_{i,j} = \sum_{l \geq 0} \sum_{m \geq 0} \overline{h}_{d+l+m-i-j,l} \binom{d+l+m-i-j}{m} \binom{i+j-l-m}{i-m}.$$

PROOF. **(a)**: To avoid questions of convergence, we consider each side of (3.16) as a generating function [GKP89, Section 7.2]. First, we have

$$\sum_{r\geq 0}\binom{d+r-1}{r}y^r = \frac{1}{(1-y)^d}$$

for $y \neq 1$. Thus, for $y \neq 1$ and $x \neq 1$ the right hand side of (3.16) can be written as

$$\left(\frac{x-1}{1-y}\right)^d \sum_{i=0}^{d}\sum_{j\geq 0} D_{i,j}\, y^j \left(\frac{1-xy}{x-1}\right)^i.$$

Let us substitute $\tilde{x} = \frac{1-xy}{x-1}$ in the both sides of (3.16). Since the condition $y \neq 1$ implies $\tilde{x}+y \neq 0$, the lemma is shown by the equation

$$(\tilde{x}+y)^d \sum_{i=0}^{d}\sum_{j\geq 0}\overline{h}_{i,j}\left(\frac{1+\tilde{x}}{\tilde{x}+y}\right)^i y^j = \sum_{i=0}^{d}\sum_{j\geq 0} D_{i,j}\tilde{x}^i y^j.$$

(b): This equation is obtained by a straightforward expansion of the formula given in (a). ■

The next lemma shows a surprising relation between the numbers $c_{m,l}$ defined in Section 3.8 and the numbers $\overline{h}_{i,j}$. Furthermore, for $j \geq 0$ and $i \in \{0,\ldots,d\}$ Equation (3.21) shows that the generalized Dehn-Sommerville relations $\overline{h}_{i,j}(S) = \overline{h}_{d-i,j}(S)$ hold in the full indices range for an *arbitrary* point set S in general position if $h_{i,j}(S') = h_{d-i,j}(S')$ hold in the full indices range for a *special* point set S' in general position. The same equation implies Theorem 3.9.1.

Lemma 3.9.4 *Let S_I, S_F be point sets in general position and $\mathcal{T}$ a continuous transformation between S_I and S_F as described in Section 3.8. Then for any $i \in \{0,\ldots,d\}$ and any $j \in \{0,\ldots,n\}$ we have*

$$\overline{h}_{i,j}(S_F) = \overline{h}_{i,j}(S_I) + \sum_{r=0}^{i}(c_{r,j-1} - c_{r,j}), \tag{3.20}$$

$$\overline{h}_{d-i,j}(S_F) = \overline{h}_{i,j}(S_F) - \overline{h}_{i,j}(S_I) + \overline{h}_{d-i,j}(S_I). \tag{3.21}$$

PROOF. For x, y such that $x+y \neq 0$ let us define the function

$$\tilde{D}(S_F,x,y) = D(S_I,x,y) + (x+y)^{d+1}\sum_{i=0}^{d}\sum_{j=0}^{n} c_{i,j}\, y^j \left(\frac{1+x}{x+y}\right)^i. \tag{3.22}$$

Let

$$\tilde{D}(S_F, x, y) = \sum_{i \geq 0} \sum_{j \geq 0} \tilde{D}_{i,j} x^i y^j \tag{3.23}$$

be the Taylor-expansion of $\tilde{D}(S_F\, x, y)$ in the neighborhood of $(0, 0)$. For $x+y \neq 0$ the function $\tilde{D}(S_F, x, y)$ is the same as the right hand side of (3.15). Comparing (3.15) with (3.23) and using (3.12) we conclude that $\tilde{D}_{i,j} = D_{i,j}$ for all $i \in \{0, \ldots, d+1\}$ and $j \in \{0, \ldots, n\}$ (especially, $\tilde{D}_{d+1,j} = 0$ for all $j \in \{0, \ldots, n\}$). Furthermore, $\tilde{D}_{i,j} = 0$ if $i > d+1$ or if $j > n+d+1$. Thus, the Taylor-expansion (3.23) is a polynomial. For x, y in a small neighborhood of $(0, 0)$ we define the coefficients $\tilde{h}_{i,j}$, $i \in \{0, \ldots, d\}$, $j \geq 0$ by the equation

$$\begin{aligned} &\sum_{i=0}^{d} \sum_{j \geq 0} \tilde{h}_{i,j}\, x^i y^j = \\ &\qquad \left(\sum_{r \geq 0} \binom{d+r-1}{r} y^r \right) \sum_{i=0}^{d} \sum_{j \geq 0} \tilde{D}_{i,j}\, y^j (1-xy)^i (x-1)^{d-i}. \end{aligned} \tag{3.24}$$

It is not hard to see that for $i \in \{0, \ldots, d\}$ and $j \in \{0, \ldots, n\}$ $\tilde{h}_{i,j}$ is a linear combination of $\tilde{D}_{i',j'}$ with $i' \leq i$ and $j' \leq j$. By comparison of (3.24) and (3.16) and by the fact that $\tilde{D}_{i',j'} = D_{i',j'}$ in this range, we conclude that $\overline{h}_{i,j}(S_F) = \tilde{h}_{i,j}$ for all $i \in \{0, \ldots, d\}$ and all $j \in \{0, \ldots, n\}$.

Substituting $\tilde{x} = \frac{1+x}{x+y}$ in both (3.22) and in (3.23) and multiplying by $\left(\frac{\tilde{x}-1}{1-y}\right)^d$, we have in the small neighborhood of $(0, 0)$ such that $x + y \neq 0$, $\tilde{x} - 1 \neq 0$, $1 - y \neq 0$ that

$$\frac{1}{(1-y)^d} \sum_{i=0}^{d} \sum_{j \geq 0} \tilde{D}_{i,j}\, y^j (1 - \tilde{x}y)^i (\tilde{x} - 1)^{d-i}$$

equals

$$\begin{aligned} &\frac{1}{(1-y)^d} \sum_{i=0}^{d} \sum_{j=0}^{n} D_{i,j}(S_I)\, y^j (1 - \tilde{x}y)^i (\tilde{x} - 1)^{d-i} \\ &\qquad + \left(\frac{1-y}{\tilde{x}-1} \right) \sum_{i=0}^{d} \sum_{j=0}^{n} c_{m,l} \tilde{x}^i y^j. \end{aligned} \tag{3.25}$$

By the fact that the Taylor-expansion of $\frac{1}{(1-y)^d}$ in the neighborhood of 0 is

$$\sum_{r\geq 0}\binom{d+r-1}{r}y^r$$

and by (3.24) the expression (3.25) must coincide with the polynomial $\sum_{i=0}^{d}\sum_{j\geq 0}\tilde{h}_{i,j}\tilde{x}^i y^j$.

For $i\in\{0,\dots,d\}$ and $j\in\{0,\dots n\}$ the coefficient of $\tilde{x}^i y^j$ in the Taylor-expansion in the neighborhood of $(0,0)$ of the first summand of (3.25) is exactly $\overline{h}_{i,j}(S_I)$ by (3.16). As next we calculate the coefficient of $\tilde{x}^i y^j$ of the second summand of (3.25) in its Taylor-expansion. The Taylor-expansion of $\frac{1}{1-\tilde{x}}$ is $\sum_{r\geq 0}\tilde{x}^r$, and so

$$\begin{aligned}\left(\frac{y-1}{1-\tilde{x}}\right)\sum_{i=0}^{d}\sum_{j=0}^{n}c_{i,j}\tilde{x}^i y^j &= \left(\sum_{r\geq 0}\tilde{x}^r\right)\sum_{i=0}^{d}\sum_{j=0}^{n+1}(c_{i,j-1}-c_{i,j})\tilde{x}^i y^j\\ &= \sum_{i=0}^{d}\sum_{j=0}^{n+1}\tilde{x}^i y^j\left(\sum_{r=0}^{i}(c_{r,j-1}-c_{r,j})\right)\end{aligned}$$

since $c_{i,j}=0$ if $i>d$ or $j<0$ or $j>n$ and since $\sum_{r=0}^{d}c_{r,j}=0$ for any $j\geq 0$ by (3.13). This shows (3.20), because $\overline{h}_{i,j}(S_F)=\tilde{h}_{i,j}$ for all $i\in\{0,\dots,d\}$ and all $j\in\{0,\dots,n\}$.

To show (3.21) observe that for $i\leq\lfloor d/2\rfloor$ and $j\in\{0,\dots,n\}$ we have

$$\sum_{r=0}^{d-i}(c_{r,j-1}-c_{r,j})=\sum_{r=0}^{i}(c_{r,j-1}-c_{r,j})$$

as

$$\sum_{r=i+1}^{\lfloor(d+1)/2\rfloor}(c_{r,j-1}-c_{r,j})=\sum_{r=\lfloor(d+1)/2\rfloor+1}^{d-i}(c_{r,j-1}-c_{r,j})$$

by (3.13).

■

Proof of Theorem 3.9.1. Equation (3.21) in Lemma 3.9.4 says (with $S_F=S$) that the value of $\overline{h}_{d-i,j}(S)-\overline{h}_{i,j}(S)=\overline{h}_{d-i,j}(S_I)-\overline{h}_{i,j}(S_I)$

is the same for every point set S of n points in general position in $\mathbb{R}^d$ and for every $i \in \{0, \ldots, d\}$, $j \in \{0, \ldots, n\}$. Since this value is determined by S, it does not depend on S_I and can only depend on n, d, i and j. ■

The next proposition was communicated by WELZL. It allows (in conjunction with (3.21)) the extension of the index range for which the generalized Dehn-Sommerville relations hold, i.e. it is necessary for showing Theorem 3.9.2.

Proposition 3.9.5 *For any $n \geq d + 3$ there are sets of n points in general position with an $\lfloor (n - d - 3)/2 \rfloor$-centerpoint.*

PROOF. Let $d' = n - d - 1$, and let P' be a neighborly, simplicial polytope with n vertices in $\mathbb{R}^{d'}$. Recall from Section 1.4 that a polytope P in $\mathbb{R}^d$ is called *neighborly*, if every set of at most $\lfloor d/2 \rfloor$ of vertices of P is a vertex set of a face of P. An example of such polytope is a cyclic polytope. Let S_A be the set of vertices of P' and A an $n \times d'$-matrix of real numbers such that the *i*-th row of A gives the coordinates of the *i*-th vertex of P'. Let B be an $n \times d$-matrix of real numbers such that $(A|\vec{1})^\top B = 0^{(d'+1)\times d}$, where $(A|\vec{1})$ denotes the $n \times (d' + 1)$-matrix which consists of A and one extra column full of 1's. We show that the matrix B exists and explain how to compute it (see also [Zie94, Theorem 6.14]). Obviously rank $(A|\vec{1})$ is $d' + 1$, and so by elementary row operations we can put it into a normal form $(I_{d'+1}|M)$, where I_r denotes an $r \times r$ identity matrix, for a positive integer r. Then we can choose B to be

$$\begin{pmatrix} M \\ -I_d \end{pmatrix}.$$

We can interpret the rows of B as points in $\mathbb{R}^d$. Let S_B be the set of these points. Since P' is simplicial, we may assume that S_A is in general position, and so by the choice of B also S_B. If p is the vertex of P' represented by the *i*-th row of A, then we denote by p^* the point represented by the *i*-th row of B. By an argument given in [Wel99] we have that if F is a set of vertices of F which forms a face of P', then the convex hull of the set $\{p^* | p \notin F\}$ contains the origin $\mathbf{o}$ of $\mathbb{R}^d$. Thus, by the choice of P' the convex hull of every subset S' of S_B with $n - \lfloor d'/2 \rfloor$ points contains $\mathbf{o}$. But then every hyperplane h through $\mathbf{o}$ has at most $n - \lfloor d'/2 \rfloor - 1$ points in S_B on one side, and so at least $\lfloor d'/2 \rfloor - 1 = \lfloor (n - d - 3)/2 \rfloor$ points on the other side. ■

PROOF OF THEOREM 3.9.2. If $n/(d + 1) - 2d \geq (n - 3d - 3)/2$, then the statement is shown by Theorem 3.5.2 and the fact that each point set S has a $\lceil n/(d+1)\rceil$-centerpoint. Otherwise we apply (3.21) of Lemma 3.9.4 with $S = S_F$ and S_I as a point set guaranteed in Proposition (3.9.5). By Theorem 3.5.2 we have $\overline{h}_{i,j}(S_I) = \overline{h}_{d-i,j}(S_I)$ for all $i \in \{0,\dots,d\}$, $j \in \{0,\dots,\lfloor (n-3d-3)/2\rfloor\}$, and we are done. ∎

We give in Table 3.2 some examples of the generalized Dehn-Sommerville relations (3.18) for small d's and small n's. The relations (3.18) might be shown without using Theorem 3.5.2, if the relations $\overline{h}_{i,j}(S') = \overline{h}_{d-i,j}(S')$ could be obtained directly for some special point set S' in general position. Such proof is probable e.g. if the entries of the matrix D for S' are given by a formula. The author has attempted to show this for S' being the vertex set of a cyclic polytope, for which a formula for the entries of D has been derived in Section 3.3 (it is also known that $S_{n,d}$ is in general position for $d \geq 1$, see [GH97]). Yet, the formulas involved are too complex. On the other hand, numerical experiments suggest that the relations $\overline{h}_{i,j}(S') = \overline{h}_{d-i,j}(S')$ are valid for every $j \in \{0,\dots,n\}$ and every $i \in \{0,\dots,d\}$.

Conjecture 3.9.6 *For a point set S in general position, the relations (3.18) are valid for all $j \in \{0,\dots,n\}$ and all $i \in \{0,\dots,d\}$.*

The next proposition is not used in any results of this thesis, but it shows yet another interesting symmetry related to the coefficients $\overline{h}_{i,j}$ defined by (3.16).

Proposition 3.9.7 *For a point set S in general position, $i \in \{0,\dots,d\}$ and $j \in \{0,\dots,n\}$ let $A_{i,j}$ be the coefficient of $x^i y^j$ in the polynomial*

$$\sum_{m=0}^{d}\sum_{l=0}^{n} D_{m,l}\, y^l (1-xy)^m (x-1)^{d-m}.$$

Then $A_{i,j} = (-1)^d A_{d-i,n-j}$ for all $i \in \{0,\dots,d\}$ and all $j \in \{0,\dots,n\}$.

PROOF. The above polynomial expands to

$$\sum_{m=0}^{d}\sum_{l=0}^{n}\sum_{r=0}^{d}\sum_{s=0}^{d} D_{m,l}\, y^{l+r} x^{s+r} (-1)^{d-m-s+r} \binom{m}{r}\binom{d-m}{s}.$$

We put $i = s + r$ and $l = j - r$. Then

$$A_{i,j} = \sum_{m=0}^{d}\sum_{r=0}^{m} D_{m,j-r} (-1)^{d-m+i} \binom{m}{r}\binom{d-m}{i-r}.$$

d	j	i	$\overline{h}_{i,j} - \overline{h}_{d-i,j} = 0$
2	0	0	$-D_{2,0} + D_{1,0} = 0$
	1	0	$-D_{2,1} + D_{1,1} - 2\,D_{2,0} + D_{1,0} = 0$
	2	0	$-D_{2,2} + D_{1,2} - 2\,D_{2,1} + D_{1,1} - 2\,D_{2,0} + D_{1,0} = 0$
	3	0	$-D_{2,3} + D_{1,3} - 2\,D_{2,2} + D_{1,2} - 2\,D_{2,1} + D_{1,1} - 2\,D_{2,0} + D_{1,0} = 0$
3	0	0	$-D_{3,0} + D_{2,0} - D_{1,0} + 2\,D_{0,0} = 0$
		1	$-D_{2,0} + 3\,D_{1,0} - 6\,D_{0,0} = 0$
	1	0	$-D_{3,1} + D_{2,1} - D_{1,1} + 2\,D_{0,1} - 3\,D_{3,0} + 3\,D_{2,0} - 4\,D_{1,0} + 6\,D_{0,0} = 0$
		1	$-D_{2,1} + 3\,D_{1,1} - 6\,D_{0,1} + 3\,D_{3,0} - 7\,D_{2,0} + 12\,D_{1,0} - 18\,D_{0,0} = 0$
	2	0	$-D_{3,2} + D_{2,2} - D_{1,2} + 2\,D_{0,2} - 3\,D_{3,1} + 3\,D_{2,1} - 4\,D_{1,1} + 6\,D_{0,1} - 6\,D_{3,0} + 7\,D_{2,0} - 9\,D_{1,0} + 12\,D_{0,0} = 0$
		1	$-D_{2,2} + 3\,D_{1,2} - 6\,D_{0,2} + 3\,D_{3,1} - 7\,D_{2,1} + 12\,D_{1,1} - 18\,D_{0,1} + 12\,D_{3,0} - 19\,D_{2,0} + 27\,D_{1,0} - 36\,D_{0,0} = 0$
4	0	0	$-D_{4,0} + D_{3,0} - D_{2,0} + D_{1,0} = 0$
		1	$-D_{3,0} + 2\,D_{2,0} - 2\,D_{1,0} = 0$
	1	0	$-D_{4,1} + D_{3,1} - D_{2,1} + D_{1,1} - 4\,D_{4,0} + 4\,D_{3,0} - 4\,D_{2,0} + 3\,D_{1,0} = 0$
		1	$-D_{3,1} + 2\,D_{2,1} - 2\,D_{1,1} + 4\,D_{4,0} - 7\,D_{3,0} + 8\,D_{2,0} - 6\,D_{1,0} = 0$
5	0	0	$-D_{5,0} + D_{4,0} - D_{3,0} + D_{2,0} - D_{1,0} + 2\,D_{0,0} = 0$
		1	$-D_{4,0} + 2\,D_{3,0} - 3\,D_{2,0} + 5\,D_{1,0} - 10\,D_{0,0} = 0$
		2	$-D_{3,0} + 4\,D_{2,0} - 10\,D_{1,0} + 20\,D_{0,0} = 0$

Table 3.2: *Examples of the relations (3.18)*

Since $D_{m,n-j-r} = D_{m,j-(m-r)}$, we have

$$\begin{aligned} A_{i,n-j} &= \sum_{m=0}^{d}\sum_{r=0}^{m} D_{m,j-(m-r)}(-1)^{d-m+i}\binom{m}{m-r}\binom{d-m}{d-i-(m-r)} \\ &= (-1)^d \sum_{m=0}^{d}\sum_{r=0}^{m} D_{m,j-r}(-1)^{d-m+d-i}\binom{m}{r}\binom{d-m}{d-i-r} \\ &= (-1)^d A_{d-i,j}. \end{aligned}$$

■

3.10 Relations between $\mathbf{e}(S)$ and $\mathbf{g}(S)$ in $\mathbb{R}^3$

In this section we show for a point set S in general position in $\mathbb{R}^3$ how to obtain the first row of $D(S)$, i.e. the e-vector from its last row, i.e. the g-vector, and vice versa (Theorem 3.10.1). The relations between these two vectors are surprisingly simple in $\mathbb{R}^3$: we have for example $e_j(S) = \frac{1}{2}(g_{j-1}(S) + g_{j-2}(S)) + 2$ for $j \in \{2, \dots, n-2\}$. There are two proofs for these relations: the one stated here uses the relations in Theorem 3.8.1 and (3.13), and thus it supports Conjecture 3.8.4. The second proof found by WELZL [AAH+98] uses the notion of k-set polytopes, see Theorem 3.2.3 in Section 3.2.

Theorem 3.10.1 *Let S be a set of $n \geq 5$ points in $\mathbb{R}^3$ in general position. Then $\mathbf{e}(S)$ can be determined if $\mathbf{g}(S)$ is given and vice versa. In particular we have*

$$\begin{aligned} e_1(S) &= \frac{1}{2}g_0(S) + 2, \qquad (3.26) \\ e_j(S) &= \frac{1}{2}(g_{j-1}(S) + g_{j-2}(S)) + 2 \quad \textit{for} \quad j \in \{2, \dots, n-2\}, \\ e_{n-1}(S) &= \frac{1}{2}g_{n-3}(S) + 2, \end{aligned}$$

and

$$\begin{aligned} &g_0(S) = 2e_1(S) - 4, \\ &g_j(S) = 2\left(\sum_{i=0}^{j}(-1)^{i+j}e_{i+1}(S) + (-1)^{j+1} - 1\right) \\ &\qquad\qquad\qquad \textit{for} \;\; j \in \{1, \dots, n-3\}. \end{aligned}$$

The remainder of this section is devoted to proving Theorem 3.10.1. Assume that $d = 3$ and that S_I, S_F and $c_{m,l}$ are defined as in Section 3.8. We also assume that S_I is the set of all n vertices of a cyclic polytope in $\mathbb{R}^3$. By simplifying the formula stated in Proposition 3.3.1 we obtain the following Lemma (also shown in [CS89, AW97]).

Lemma 3.10.2 *Let S_I be a set of all n vertices of a cyclic polytope in $\mathbb{R}^3$. We have*

$$\begin{array}{rcll} e_k(S_I) & = & n + 2(n-k-1)(k-1) & \textit{for} \quad 1 \leq k \leq n-1, \\ g_j(S_I) & = & 2(n-j-2)(j+1) & \textit{for} \quad 0 \leq j \leq n-3. \end{array} \tag{3.27}$$

Theorem 3.8.1 implies that if we know the numbers $c_{m,l}$ for all $m \in \{1, 2, 3\}$ and all $l \in \{0, \ldots, n-4\}$ then we can calculate $\mathbf{g}(S_F)$ (under the assumption that the matrix $D(S_I)$ is known). In an analogous way we can calculate $\mathbf{e}(S_F)$. Yet some of the integers $c_{m,l}$ are redundant for the calculation of $\mathbf{e}(S_F)$ and of $\mathbf{g}(S_F)$. First, it follows from Corollary 3.7.2 that a $(2, l)$-mutation for any $l \in \{0, \ldots, n-4\}$ changes neither the g-vector nor the e-vector. Furthermore, we have $c_{3,l} = -c_{1,l}$ for any $l \in \{0, \ldots, n-4\}$ by (3.13). Let us write $\mathbf{c} = (c_{3,0}, \ldots, c_{3,n-4})^T$. By the above considerations, the vector $\mathbf{c}$ completely determines $\mathbf{g}(S_F)$ and $\mathbf{e}(S_F)$. In more detail, we have the following result:

Lemma 3.10.3 *For every $l = 0, \ldots, n-4$ let $\mathbf{w}_l = (w_{0,l}, \ldots, w_{n-3,l})^T$ be a vector of integers with*

$$w_{j,l} = \begin{cases} 2(1 - [2l = n-4] + [2l = n-3]) & \textit{if } j \in \{l, n-3-l\}, \\ -2(1 - [2l = n-4] + [2l = n-5]) & \textit{if } j \in \{l+1,\ n-4-l), \\ 0 & \textit{otherwise}. \end{cases}$$

For every $l \in \{0, \ldots, n-4\}$, let $\mathbf{v}_l = (v_{1,l}, \ldots, v_{n-2,l})^T$ be a vector of integers with

$$v_{j,l} = \begin{cases} 1 - [2l = n-4] + [2l = n-2] & \textit{if } j \in \{l+1, n-1-l\}, \\ -1 + [2l = n-4] - [2l = n-6] & \textit{if } j \in \{l+3,\ n-3-l\}, \\ 0 & \textit{otherwise}. \end{cases}$$

Then

$$\mathbf{g}(S_F) = \mathbf{g}(S_I) + \mathbf{W}\mathbf{c} \tag{3.28}$$

where $\mathbf{W}$ is the matrix with columns $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_{n-4}$, and

$$\mathbf{e}(S_F) = \mathbf{e}(S_I) + \mathbf{V}\mathbf{c} \tag{3.29}$$

where $\mathbf{V}$ is the matrix with columns $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{n-4}$.

PROOF. Using Corollary 3.7.2 it is routine to check that for $l \in \{0,\dots,n-4\}$ and a $(3,l)$-mutation (S,S') of a point set we obtain $\mathbf{g}(S')$ by adding the vector $\mathbf{w}_l$ to $\mathbf{g}(S)$. Analogously, we have $\mathbf{g}(S') = \mathbf{g}(S) - \mathbf{w}_l$ if a $(1,l)$-mutation occurred. The equation for $\mathbf{e}(S_F)$ follows by an analogous argument. ■

We are prepared now to show the main theorem of this section.

PROOF OF THEOREM 3.10.1. We put $S = S_F$. The proof uses mostly linear algebra. Given $\mathbf{g}(S_F)$, we could compute $\mathbf{c}$ (and thus $\mathbf{e}(S_F)$ by (3.29)) using Lemma 3.10.2 and (3.28), provided that W is invertible. Unfortunately, this is not the case, but we can modify (3.28) and (3.29) and restrict both $\mathbf{W}$ and $\mathbf{V}$ suitable to our needs in the following way. The matrices $\mathbf{W}$ and $\mathbf{V}$ have the property that

$$\mathbf{v}_j = -\mathbf{v}_{n-4-j} \text{ and } \mathbf{w}_j = -\mathbf{w}_{n-4-j} \quad \text{for every } j = 0,\dots,n-4$$

as easily checked. (Also here a geometric interpretation is not hard to find). Therefore putting $t = \lceil n/2 \rceil - 3$ and

$$d_i = c_{3,i} - c_{3,n-4-i} \quad \text{for all } i = 0,\dots,t$$

we have by Lemma 3.10.3

$$\mathbf{g}(S_F) = \mathbf{g}(S_I) + (\mathbf{w}_0\, d_0 + \ldots + \mathbf{w}_t\, d_t) \tag{3.30}$$

and

$$\mathbf{e}(S_F) = \mathbf{e}(S_I) + (\mathbf{v}_0\, d_0 + \ldots + \mathbf{v}_t\, d_t) \tag{3.31}$$

(t has the value $\lceil n/2 \rceil - 3$ because the vectors $\mathbf{w}_{\lceil n/2 \rceil -2} = -\mathbf{w}_{n-4-(\lceil n/2 \rceil -2)}$ and $\mathbf{v}_{\lceil n/2 \rceil -2} = -\mathbf{v}_{n-4-(\lceil n/2 \rceil -2)}$ are both zero-vectors).

Thus, to find $\mathbf{e}(S_F)$ we only need to calculate the vector $(d_0,\dots,d_t)$ from $\mathbf{g}(S_F)$ and $\mathbf{g}(S_I)$. To this aim we define the quadratic $(t+1)\times(t+1)$-matrix $\mathbf{W}'$ with

$$w'_{i,j} = w_{i,j} \quad \text{for} \quad i = 0,\dots,t \quad \text{and} \quad j = 0,\dots,t$$

(i.e. $\mathbf{W}'$ is $\mathbf{W}$ restricted to the entries in the first $t+1$ rows and first $t+1$ columns).

For $n = 5$ and $n = 6$ we have $\mathbf{W}' = (1)$ and for $n \geq 7$ it holds

$$\mathbf{W}' = \begin{pmatrix} 2 & & & 0 \\ -2 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & -2 & 2 \end{pmatrix},$$

as can be shown by induction on n.

We put

$$g_i' = g_i(S_F) - g_i(S_I) \quad \text{for} \quad i = 0, \dots, n-3$$

and

$$e_i' = e_i(S_F) - e_i(S_I) \quad \text{for} \quad i = 1, \dots, n-1$$

and read off from the matrix $\mathbf{W}'$ that

$$d_j = \frac{1}{2} \sum_{i=0}^{j} g_i' \quad \text{for every} \quad j = 0, \dots, t. \tag{3.32}$$

Analogously we have

$$d_0 = e_1', \quad d_1 = e_2', \quad \text{and} \quad d_j = \sum_{i=0}^{\lfloor j/2 \rfloor} e_{j-2\,i+1}' \quad \text{for every} \quad j = 2, \dots, t. \tag{3.33}$$

Using (3.31) we obtain

$$e_1' = \frac{1}{2}\, g_0',$$

and

$$e_j' = \frac{1}{2}\, (g_{j-1}' + g_{j-2}') \quad \text{for } j = 2, \dots, t+1.$$

If n is odd, then we can show by induction that $e_{t+2}' = -d_{t-1} - d_t$ and so

$$e_{t+2}' = -\sum_{i=0}^{t-1} g_i' - \frac{1}{2}\, g_t'.$$

If n is even, then $e'_{t+2} = -d_{t-1}$ and $e'_{t+3} = -2\,d_t$, and so

$$e'_{t+2} = -\frac{1}{2}\sum_{i=0}^{t-1} g'_i \quad \text{and} \quad e'_{t+3} = -\sum_{i=0}^{t} g'_i.$$

Using (3.27) and the identities

$$\sum_{i=0}^{j} g_i(S_I) = (j+1)\,(j+2)\,(3\,n - 2\,j - 6)/3,$$

$$\sum_{i=0}^{j}(-1)^{i+j} e_{i+1}(S_I) = n\,(j+1) - j\,(j+3) + (-1)^j - 1,$$

which are not hard to show, we obtain the equations

$$\begin{aligned}
e_1(S_F) &= \frac{1}{2}\,g_0(S_F) + 2,\\
e_j(S_F) &= \frac{1}{2}\,(g_{j-1}(S_F) + g_{j-2}(S_F)) + 2 \quad \text{for} \quad j = 2, \ldots, t+1,\\
e_{t+2}(S_F) &= -\sum_{i=0}^{t-1} g_i(S_F) - \frac{1}{2}\,g_t(S_F) + n\,(n-1)\,(n-2)/6 + 2\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for odd } n,\\
e_{t+2}(S_F) &= -\frac{1}{2}\sum_{i=0}^{t-1} g_i(S_F) + n\,(n-1)\,(n-2)/12 + 2 \quad \text{for even } n,\\
e_{t+3}(S_F) &= -\sum_{i=0}^{t-1} g_i(S_F) + n\,(n-1)\,(n-2)/6 + 2 \quad \text{for even } n.
\end{aligned}$$

In the following we simplify the above equations. First, it is not hard to see that

$$\sum_{i=0}^{n-3} g_i(S_F) = 2\binom{n}{3}$$

always holds.

Assume that n is even. We have $\sum_{i=0}^{t+1} g_i(S_F) = \binom{n}{3}$ and $g_{t+1}(S_F) = g_{t+2}(S_F)$. Then by a straightforward substitution we obtain

$$e_j(S_F) = \frac{1}{2}\,(g_{j-1}(S_F) + g_{j-2}(S_F)) + 2 \quad \text{for } j = t+2 \text{ and } j = t+3.$$

Now let n be odd. Then $g_{t+1}(S_F) = 2\binom{n}{3} - 2\sum_{i=0}^{t} g_i(S_F)$ and so we have

$$e_{t+2}(S_F) = \frac{1}{2}\left(g_{t+1}(S_F) + g_t(S_F)\right) + 2.$$

Finally, we use again the relations $e_i(S_F) = e_{n-i}(S_F)$ for $i = 1, \ldots, n-1$, and $g_j(S) = g_{n-d-j}(S)$ for $j = 0, \ldots, n-3$ to show that for $2 \leq j \leq \lfloor n/2 \rfloor$ we have

$$e_{n-j}(S_F) = \frac{1}{2}\left(g_{n-2-j}(S_F) + g_{n-1-j}(S_F)\right) + 2.$$

Summarizing, we have shown that

$$\begin{aligned} e_1(S_F) &= \frac{1}{2}\, g_0(S_F) + 2, \\ e_j(S_F) &= \frac{1}{2}\left(g_{j-1}(S_F) + g_{j-2}(S_F)\right) + 2 \quad \text{for} \quad j = 2, \ldots, n-2, \\ e_{n-1}(S_F) &= \frac{1}{2}\, g_{n-3}(S_F) + 2. \end{aligned}$$

Using elementary linear algebra we obtain from these relations the formulas for $g_j(S_F)$, $j = 0, \ldots n-3$. ■

Another proof based on the notion of k-set polytopes (Section 3.2) has been found by WELZL [AAH+98]: we obtain immediately the Equations (3.26) using (3.4) substituted into the Equations (3.3).

3.11 Relations between $\mathbf{e}(S)$ and $\mathbf{g}(S)$ in $\mathbb{R}^d$ for $d \geq 4$

In this section we show that for $d \geq 4$ the information given by the vector $\mathbf{g}(S)$ is in general not sufficient to determine the vector $\mathbf{e}(S)$, and vice versa. In more detail, we show that for every $d \geq 4$ and every $n \geq 4(d+1)$ there are sets S, S' in $\mathbb{R}^d$ of n points in general position with $\mathbf{g}(S) = \mathbf{g}(S')$ and $\mathbf{e}(S) \neq \mathbf{e}(S')$, and that there are many such pairs of point sets for each $d \geq 4$ and each $n \geq 4(d+1)$. An analogous statement is shown for the case when the roles of $\mathbf{e}$ and of $\mathbf{g}$ are replaced. It seems that at least $\lfloor d/2 \rfloor$ rows of the matrix $D(S)$ are necessary to compute $\mathbf{e}(S)$ or $\mathbf{g}(S)$, and according to Conjecture 3.8.4 certain sets of $\lfloor d/2 \rfloor$ rows of the matrix $D(S)$ would be sufficient to determine the whole matrix. The results of this section are a consequence of the results about mutations of point sets (Corollary 3.7.2).

The problem of obtaining $\mathbf{e}(S)$ or $\mathbf{g}(S)$ from other rows of $D(S)$ is the most interesting, yet a special case. By similar arguments as in the next theorem it should be possible to show that each row of $D(S)$ cannot be determined by only one other row of $D(S)$.

Theorem 3.11.1 **(a)** *For any $d \geq 4$ and $n \geq 4(d+1)$ there are sets S, S' in $\mathbb{R}^d$ of n points in general position with $\mathbf{g}(S) = \mathbf{g}(S')$ and $\mathbf{e}(S) \neq \mathbf{e}(S')$. In more detail, for each $d \geq 4$, $k \geq 1$, $l \geq 0$ and $n \geq 4k(l+d+1)$ there are sets $S_{k,l}$, $S'_{k,l}$ in $\mathbb{R}^d$ of n points in general position such that $\mathbf{g}(S_{k,l}) = \mathbf{g}(S'_{k,l})$ and*

$$e_{l+m}(S_{k,l}) - e_{l+m}(S'_{k,l}) = k,$$

where

$$m = \begin{cases} d/2 & \text{if } d \text{ is even}, \\ (d+5)/2 & \text{otherwise}. \end{cases}$$

(b) *For any $d \geq 4$ and $n \geq \frac{3}{2}(d+1)^2$ there are sets S', S'' in $\mathbb{R}^d$ of n points in general position with $\mathbf{e}(S') = \mathbf{e}(S'')$ and $\mathbf{g}(S') \neq \mathbf{g}(S'')$. In more detail, for each $d \geq 4$, $l \geq \lfloor d/2 \rfloor - 1$ and $n \geq (l+d+1)(d+1)$ there are sets S'_l, S''_l in $\mathbb{R}^d$ of n points in general position such that $\mathbf{e}(S'_l) = \mathbf{e}(S''_l)$ and*

$$g_{l-m}(S'_l) - g_{l-m}(S''_l) = 1,$$

where

$$m = \begin{cases} d/2 - 1 & \text{if } d \text{ is even}, \\ (d-3)/2 & \text{otherwise}. \end{cases}$$

PROOF. **(a)**: Assume that $d \geq 4$. To illustrate the idea of the proof, we start with a special case that d is even, $k = 1$ and $l = 0$. Assume that S is a set of $4(d+1)$ points in general position such that there are four pairwise disjunct $(d+1)$-sets of S denoted as $T_1, \ldots, T_4$, see Figure 3.11. We deform T_1 in such a way that it allows a $(d/2+2, 0)$-mutation, and we deform T_2, T_3 and T_4 in such a way that each of these sets allows a $(d/2, 0)$-mutation. Let S' be a set obtained by "executing" a corresponding mutation for each of the sets $T_1, \ldots, T_4$. By Corollary 3.7.2 we have $\mathbf{g}(S) = \mathbf{g}(S')$, but on the other hand we have by the same corollary

$$\begin{aligned} e_{d/2+2}(S') &= e_{d/2+2}(S) - 1, \\ e_{d/2-1}(S') &= e_{d/2-1}(S) + 1, \\ e_{d/2}(S') &= e_{d/2}(S) - 3, \\ e_{d/2+1}(S') &= e_{d/2+1}(S) + 3, \end{aligned}$$

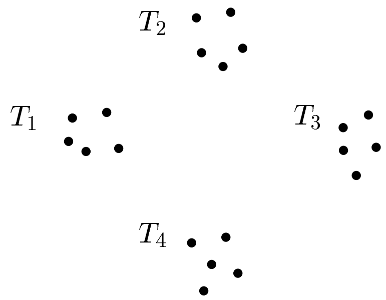

Figure 3.11: *A projection of a set S in $\mathbb{R}^4$ with four $(d+1)$-sets*

and so $\mathbf{e}(S) \neq \mathbf{e}(S')$.

The general case is based on the same scheme. We construct two sets S and S' of n points each in $\mathbb{R}^d$ such that S' is obtained by moving the points in S under a continuous transformation which fulfills the mutation condition, see Section 3.6. Let d, k, l, and n be fixed and fulfill the conditions described in the theorem statement.

Assume first that d is even. We put $t = 4k$, $m' = d/2 + 2$ and $m'' = d/2$. Assume that S is constructed in such a way that it has pairwise disjunct $(l + d + 1)$-sets $T_1, \ldots, T_t$; this is possible for example if $T_1, \ldots, T_t$ are close to the surface of a sufficiently large sphere S^{d-1}. The remaining $n - t(l + d + 1)$ points are placed e.g. in the middle of the sphere. Furthermore, for each $i \in \{1, \ldots, t\}$ we deform T_i in such a way that for a given integer $m_i \in \{1, \ldots, d\}$ a subset of $d + 1$ points in T_i forms a mutation kernel of a (potential) (m_i, l)-mutation M_i. It is not hard to see that such a construction is possible, see proof of Lemma 3.6.4. Now S' is obtained by "executing" the mutations $M_1, \ldots, M_t$, where $m_1 = \ldots = m_k = m$ and $m_{k+1} = \ldots = m_t = m'$. By Corollary 3.7.2 exactly three (m', l)-mutations compensate the changes of the g-vector caused by one (m, l)-mutation, and so we have immediately $\mathbf{g}(S) = \mathbf{g}(S')$. Since m, m', $d + 1 - m$ and $d + 1 - m'$ are pairwise different integers (m and $d + 1 - m$ have different parity) and since $2l < n-2d-2$ (and so the special cases of Corollary 3.7.2 like $[2l = n-d-1]$

etc. do not need to be considered), we have by the same corollary that

$$\begin{aligned} e_{l+m}(S') &= e_{l+m}(S) - k, \\ e_{l+d+1-m}(S') &= e_{l+d+1-m}(S) + k, \\ e_{l+m'}(S') &= e_{l+m'}(S) - 3\,k, \\ e_{l+d+1-m'}(S') &= e_{l+d+1-m'}(S) + 3\,k. \end{aligned}$$

If d is odd, then we put $t = 3k$, $m = (d+5)/2$ and $m' = (d-1)/2$. The remainder of the proof goes through as above. The statement of the theorem follows with $S = S_{k,l}$ and $S' = S'_{k,l}$.

(b): We show only the general case. Assume that d, l and n are given and fulfill the conditions of the theorem statement. First the treat the case that d is even. Let S, S' and S'' be sets of n points in general position in $\mathbb{R}^d$ with following properties. We construct S in such a way that it has $(l+d+1)$-sets $T_0, \ldots, T_t$, where $t = d$. Furthermore, the $(l+d+1)$-set T_0 is deformed in such a way that it allows an $(1,l)$-mutation M_0, and for each $i \in \{1, \ldots, t\}$ we deform T_i in such a way that the set T_i allows an $(d/2, l-d/2+i)$-mutation M_i. Now S' is obtained from S by "executing" the mutation M_0, and S'' is obtained from S by executing the mutations $M_1, \ldots, M_t$. The existence of such construction is not hard to see. Next we show that $\mathbf{e}(S') = \mathbf{e}(S'')$ but $\mathbf{g}(S') \neq \mathbf{g}(S'')$. We have $2l < n-2d-2$ and so by Corollary 3.7.2 it follows that

$$\begin{aligned} e_{l+1}(S') &= e_{l+1}(S) - 1, \\ e_{n-l-1}(S') &= e_{n-l-1}(S) - 1, \\ e_{l+d}(S') &= e_{l+d}(S) + 1, \\ e_{n-l-d}(S') &= e_{n-l-d}(S) - 1. \end{aligned}$$

Furthermore, for each $i \in \{1, \ldots, t\}$, the mutation M_i changes the e-vector in the following way:

$$\begin{aligned} e_{l+i}(S) \text{ and } e_{n-l-i}(S) &\quad \text{are changed by } -1, \\ e_{l+i+1}(S) \text{ and } e_{n-l-i-1}(S) &\quad \text{are changed by } +1. \end{aligned}$$

The cumulative effect of these changes is

$$\begin{aligned} e_{l+1}(S'') &= e_{l+1}(S) - 1, \\ e_{n-l-1}(S'') &= e_{n-l-1}(S) - 1, \\ e_{l+d}(S'') &= e_{l+d}(S) + 1, \\ e_{n-l-d}(S'') &= e_{n-l-d}(S) - 1, \end{aligned}$$

and so $\mathbf{e}(S') = \mathbf{e}(S'')$. On the other hand, by Corollary 3.7.2 (and the fact that $2l < n - 2d - 2$) we have

$$\begin{aligned} g_l(S') &= g_l(S) + 1 - d, \\ g_{n-d-l}(S') &= g_{n-d-l}(S) + 1 - d, \\ g_{l+1}(S') &= g_{l+1}(S) + d - 1, \\ g_{n-d-l-1}(S') &= g_{n-d-l-1}(S) + d - 1, \end{aligned}$$

which are the only differences between $\mathbf{g}(S)$ and $\mathbf{g}(S')$. Moreover, we have for example

$$g_{l-d/2+1}(S'') = g_{l-d/2+1}(S) - 1,$$

and it follows that $\mathbf{g}(S') \neq \mathbf{g}(S'')$.

If d is odd, then S' is the same as above and S'' is obtained from S by exactly $t = (d-1)/2$ mutations $M_1, \ldots, M_t$, where for $i \in \{1, \ldots, t\}$ M_i is an $((d-1)/2, l - (d+1)/2 + 2i)$-mutation. The remainder of the proof goes through as above. The statement follows with $S_l' = S'$ and $S_l'' = S''$. ■

Chapter 4

Enumeration of k-sets

Given a set of points S in general position in $\mathbb{R}^d$ and an integer $k \in \{1, \ldots, n-1\}$, we consider the problem of enumerating all k-sets of S. In the following sections we present two output-sensitive algorithms based on the reverse search technique (by AVIS and FUKUDA [AF96]) which solve this task. Both algorithms are joint work with FUKUDA [AF99]. By using the reverse search technique we obtain algorithms which are easy to implement, memory-efficient and allow parallel computation [BMFN97]. On the other hand, upper bounds on the running time of the both algorithms are greater than the upper bounds for the previously known algorithms. In more detail, the running time of the first of the two algorithms (Section 4.3) is

$$O(|Q_k(S)|\, kn^2\, \mathrm{lp}(d,n)),$$

where $|Q_k(S)|$ is the number of k-sets of a point set S in general position in $\mathbb{R}^d$ and $\mathrm{lp}(d,n)$ is a time for solving a linear program with d variables and n inequalities. For $\mathbb{R}^2$ and $\mathbb{R}^3$ this running time decreases to

$$O(|Q_k(S)|\, n \log n).$$

The second algorithm, presented in Section 4.4 has running time

$$O(|Q_k(S)|\, k\, n\, \mathrm{lp}(d+1,n)).$$

In order to obtain these algorithms, we study the notion of polytopal neighbors of a k-set and the notion of combinatorial neighbors of a k-set. We describe in Section 4.2 how to find efficiently all polytopal neighbors of a k-set, while in Section 4.4 we show how to find quickly all combinatorial neighbors of a k-set. The first algorithm exploits the notion of k-set polytopes which are studied in Section 3.2. Also the proof of correctness of the second algorithm implicitly depends on this notion.

In addition to these algorithms, we consider in Section 4.5 the notion of affine k-sets. An affine k-set P of S is a k-set of S which can be strictly separated from the origin (and from $S \setminus P$) by a hyperplane. We show that the 1-skeleton of a k-set polytope restricted to the vertices corresponding to affine k-sets is not always connected. As a consequence, a graph search technique (and reverse search is one) cannot be applied to an output-sensitive enumeration of affine k-sets. We show also some properties of affine k-sets.

Enumeration of k-sets has among others the following two applications. The first one is the computation of order k Voronoi diagrams in $\mathbb{R}^d$, since the regions of such diagrams corresponds to the k-sets of the convex point set which is obtained by "lifting" the sites of a Voronoi diagram onto a unit paraboloid [AW97]. Another application is orthogonal L_1 line fitting [KM93, AW97]. Here we want to find a line in $\mathbb{R}^2$ such that the sum of distances of n points in the orthogonal L_1-norm to this line is minimized for all lines in $\mathbb{R}^2$. It turns out that an optimal line must be a halving facet for even n and an $(n-3)/2$-facet for odd n.

For $d = 2$ the following algorithms for enumeration of k-sets (or equivalently, for computing the k-level of a dual hyperplane arrangement) are known. EDELSBRUNNER and WELZL give in [EW86] an algorithm to obtain a k-level in the plane in time $O(n \log n + b \log^2 n)$, where b is the size of the output. The required space is $O(n + b)$. COLE, SHARIR and YAP solve in [CSY87] the (essentially) dual problem of computation of a k-hull in time $O(n \log n + b \log^2(k + 1))$. This improves the algorithm in [EW86] if k is small. AGARWAL, DE BERG, MATOUŠEK and SCHWARZKOPF obtain in [AMdS94] a randomized $O(n\sqrt{(k+1)\log n} + n \log^2 n)$ expected time algorithm for computing a k-level. Still, it is not output-sensitive. This algorithm can be adapted for higher dimensions.

For $d > 2$ the following algorithms are known. AGARWAL and MATOUŠEK give in [AM95] an algorithm with $O((n + b)^{1+\epsilon})$ running time which constructs a k-level in $\mathbb{R}^3$, where $\epsilon > 0$ is a small constant. For $\mathbb{R}^d$, $d \geq 4$ they obtain an algorithm for the computation of a k-level with a slightly weaker bound on running time, namely

$$O\left(n^{1+\epsilon} + \min\{b n^{1-2/(\lfloor d/2 \rfloor+1)+\epsilon}, b^{d/(d+1)} n^{d/(d+1)+\epsilon}\}\right)$$

for $\epsilon > 0$.

In addition to these algorithms, there are several algorithms for computing the $(\leq k)$-levels: an optimal $O(n \log n + nk)$-time algorithm for $d = 2$ by EVERETT, ROBERT and VAN KREVELD [ERv93], an optimal $O(n \log n + nk^2)$ expected running time algorithm for $d = 3$ by CHAN [Cha98] and an $O(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil})$ expected running time algorithm for $d \geq 4$ by MULMULEY [Mul91]. Of course, if k is large and we are interested in obtaining only the k-level, then these algorithms are much slower than the algorithms for a single k-level.

The disadvantage of the above algorithms is that they use sophisticated data structures and therefore are difficult to implement. In addition, some of them have high memory requirements. Since the two algorithms presented in the next sections are free of these disadvantages, they are a serious alternative to the previously known algorithms.

4.1 The framework of the reverse search technique

In this section we describe briefly the framework of the reverse search technique by AVIS and FUKUDA [AF96]. The basic idea of the reverse search is to traverse in a depth-first search manner a tree G whose nodes are exactly the k-sets of a set S of points in general position. (In the next two sections we describe the graph G in more detail). The root of G is a distinguished k-set T^*. The advantages of the reverse search technique come from the fact that we hold essentially only one node of G in memory and compute online its father and its children if necessary.

To implement a reverse search algorithm for k-sets we need two functions f and Adj. The function f is called a *local search function*; for a k-set $T \neq T^*$ it gives the successor $f(T)$ of T on the unique path from T to T^* in G. The function Adj is called an *adjacency oracle* and for an integer $i \in \{0, \ldots, \delta\}$ and a k-set T it gives the i-th neighbor of T in G. Here δ is an integer not smaller than any node in G. The function Adj fulfills the condition that when i goes from 0 to δ, then one obtains all neighbors of T in G and each of them exactly once; see [AF96] for the complete description.

These two functions are used in the following procedure taken from [AF96], which performs a depth-first search on G (this is the "main loop" of a reverse search algorithm).

```
procedure ReverseSearch(Adj,δ,T*,f);
  T := T*; j := 0; (* j: neighbor counter *)
  repeat
    while j < δ do
      j := j + 1;
      next := Adj(T, j);
      if next ≠ 0 then
        if f(next) = T then (* reverse traverse *)
          T :=next; j := 0;
        endif
      endif
    endwhile;
    if T ≠ T* then (* forward traverse *)
      R := T; T := f(T);
      j := 0; repeat j := j + 1 until Adj(T, j) = R (* restore *)
    endif
  until T = T* and j = δ.
```

Both algorithms for enumerating the k-sets of S differ by the way how f and Adj are implemented and by the graph G (which is implicitly given by Adj and f). These implementations of f and Adj are presented in Sections 4.2, 4.3 and 4.4.

4.2 k-cells, polytopal neighbors and optimization on $Q_k(S)$

In this section we study the notion a polytopal neighbor of a k-set and one possible way to optimize a linear function on a k-set polytope (see Section 3.2). We show some basic properties of polytopal neighbors in Lemma 4.2.1 and proceed with a description of a routine for efficient finding of all polytopal neighbors of a k-set. Finally, we apply these results to describe a possible way to optimize a linear function on a k-set polytope. This optimization will be used in Section 4.3 to obtain an algorithm for enumeration of k-sets which uses polytopal neighbors.

Let us identify $U = \mathbb{R}^{\mathrm{d}}$ with a hyperplane in $\mathbb{R}^{d+1}$ determined by the equation $x_{d+1} = 1$. Let S^d be the unit sphere in $\mathbb{R}^{d+1}$ with center in the origin. Each point q in S^d or in U determines an oriented hyperplane $h(q)$ which contains the origin $\mathbf{o}$ of $\mathbb{R}^{d+1}$ and has normal vector $(q - \mathbf{o})$. We denote the sphere which is an intersection of S^d with $h(q)$ by $h_{\text{sp}}(q)$. The intersection of $h(q)$ with U is denoted by $h_{\text{aff}}(q)$. Now given a (central) hyperplane h in $\mathbb{R}^{d+1}$ with normal vector $\mathbf{n}$ containing the origin (or a corresponding sphere in S^d or a corresponding hyperplane h in U), let l be an oriented line which is the linear space spanned by $\mathbf{n}$ such that its orientation agrees with $\mathbf{n}$. We denote the intersection of l with S^d by $p_{\text{sp}}(h)$. In this way we have implicitly introduced a modification of the geometric duality T described in Section 1.4. The difference is that here a point $p = (p_1, \ldots, p_d)$ in U is mapped to a hyperplane in U with equation

$$\langle p, x\rangle = p_1x_1 + \ldots + p_dx_d = -1,$$

and in addition we can map points in S^d to hyperplanes in $\mathbb{R}^{d+1}$ or spheres in S^d, and vice versa. It is not hard to see that our duality has the same properties of incidence and order preservation as T.

The *horizon of S^d* is the intersection of S^d with the hyperplane determined by the equation $x_{d+1} = 0$. We denote by $\mathcal{A}_{\text{sp}}$ the spherical arrangement induced by the oriented spheres $h_{\text{sp}}(p)$, $p \in S$. The cell of $\mathcal{A}_{\text{sp}}$ containing the "north pole", i.e. the point $(0, \ldots, 0, 1)$ is called the *base cell B*. It is not hard to see that B is contained on the positive side of $h_{\text{sp}}(q)$ for each $p \in S$. For a cell c of $\mathcal{A}_{\text{sp}}$, we call a ridge r (a $(d-2)$-face of $\mathcal{A}_{\text{sp}}$) of c a *horizon ridge*, if it is an intersection of spheres h_1, h_2 of $\mathcal{A}_{\text{sp}}$ such that h_1, h_2 bound c, c lies on the positive side of h_1 and simultaneously c lies on the negative side of h_2. We say that a sphere h bounding a cell c is *visible (for c)* if h has c on its negative side, otherwise h is called *invisible* (for c).

For $k \in \{0, \ldots, n-1\}$, a cell c of $\mathcal{A}_{\text{sp}}$ is called a *k-cell* if the shortest path in S^d from the relative interior of the base cell to the relative interior of c traverses exactly k spheres of $\mathcal{A}_{\text{sp}}$. It is not hard to see that the duality stated above gives us a bijective mapping between the k-sets of S and the k-cells of $\mathcal{A}_{\text{sp}}$ for all $k \in \{1, \ldots, n-1\}$. We denote by $c(T)$ the k-cell corresponding to a k-set T of S.

Recall that from Section 3.2 that two vertices of $Q_k(S)$ are called *polytopal neighbors* if they are connected by an edge in $Q_k(S)$. The corresponding k-sets and the corresponding k-cells are also called polytopal neighbors. The following lemma suggests how to determine the polytopal neighbors of a given k-cell c if the horizon ridges of c are given. Then using the bijective mapping between the k-sets and the k-cells, we can then compute all polytopal neighbors of a given vertex of $Q_k(S)$.

Lemma 4.2.1 *Assume that $k \in \{2, \dots, n-1\}$ and T_1, T_2 are k-sets.*

(a) *The vertices $v(T_1)$ and $v(T_2)$ of $Q_k(S)$ are polytopal neighbors if and only if the k-cells $c(T_1)$ and $c(T_2)$ of $\mathcal{A}_{sp}$ share a horizon ridge.*

(b) *Assume that the horizon ridge r is an intersection of spheres $h_{sp}(p_1)$, $h_{sp}(p_2)$, $p_1, p_2 \in S$ such that $c(T_1)$ is on the positive side of $h_{sp}(p_1)$ and simultaneously on the negative side of $h_{sp}(p_2)$. Then $c(T_2)$ is separated from the base cell B by the spheres in the set*

$$\{h_{sp}(p) | p \in T_1 \backslash \{p_2\}\} \cup \{h_{sp}(p_1)\}.$$

PROOF. **(a)**: In this proof, we use instead of spheres of $\mathcal{A}_{\text{sp}}$ their corresponding central hyperplanes. By Corollary 3.2.11(a) we have to show that $c(T_1)$ and $c(T_2)$ share a horizon ridge in $\mathcal{A}_{\text{sp}}$ if and only if there is a central hyperplane h of S which induces an $(2, k-1)$-partition (A, B) of S with $A = (T_1 \backslash T_2) \cup (T_2 \backslash T_1)$ and $B = T_1 \cap T_2$. Assume that the latter condition is true, and denote by p_1, p_2 the both points in $S \cap h$ with $p_1 \in T_1$, $p_2 \in T_2$. Let h' be a hyperplane which strictly separates T_1 and $S \backslash T_1$. Obviously we can perturb h' from its initial position passing a position where h' equals h, into a position where h' strictly separates T_2 and $S \backslash T_2$. By duality, this movement corresponds to a movement of a point q dual to h' starting in the relative interior of $c(T_1)$, passing the intersection r of the both central hyperplanes corresponding to $h_{\text{sp}}(p_1)$ and $h_{\text{sp}}(p_2)$, and ending in the relative interior of $c(T_2)$. As q does not pass any other hyperplanes corresponding to the spheres of $\mathcal{A}_{\text{sp}}$, the cells $c(T_1)$ and $c(T_2)$ are neighbors separated by the ridge r. By the property of the duality we have also that $c(T_1)$ ($c(T_2)$) is on the negative side (positive side) of $h_{\text{sp}}(p_1)$ and that $c(T_1)$ ($c(T_2)$) is on the positive side (negative side) of $h_{\text{sp}}(h_2)$, and so r is a horizon ridge. The converse can be shown analogously.

(b): Trivial. ■

For a $k \in \{2, \dots, n-1\}$, an arrangement $\mathcal{A}_{\text{sp}}$ and its k-cell c, let $R(c)$ be the set of horizon ridges of c, and $G(R(c))$ the incidence graph of $R(c)$ (i.e. the vertices of $G(R(c))$ are horizon ridges, and there is an edge between them if both horizon ridges meet). First note that for $d > 2$ the graph $G(R(c))$ is connected, since $R(c)$ corresponds to the set of facets of some $(d-1)$-dimensional polytope. Thus, if a ridge $r \in R(c)$ is given, we can enumerate $R(c)$ (starting at r) by tracing the graph $G(R(c))$.

Since it is algorithmically easier to work with hyperplanes than with spheres, we consider in the following the arrangement of the central hyperplanes corresponding to the spheres of $\mathcal{A}_{\text{sp}}$. Thus we interchange in the following presentation "sphere" by "hyperplane" if necessary.

We trace the graph $G(R(c))$ by finding all neighbors of a ridge r in $R(c)$ and repeating this process until all ridges in $R(c)$ are found. (To this aim the reverse search could be used but it would make the description even more complicated). Finding all neighbors of a ridge r in $R(c)$ can be done by finding all central hyperplanes bounding r (we consider r itself as an $(d-1)$-dimensional polytope). As r lies in the intersection of two central hyperplanes, we can find all central hyperplanes bounding r by finding the non-redundant hyperplanes in (3) of

$$\begin{array}{llcll} (1) & a_1 x & = & b_1 & \text{(the hyperplane visible for } c \text{ containing } r) \\ (2) & a_2 x & = & b_2 & \text{(the hyperplane invisible for } c \text{ containing } r) \\ (3) & \tilde{A}x & \leq & \tilde{b} & \text{(the remaining hyperplanes bounding } c). \end{array}$$

The latter problem is the redundancy removal problem (for r), which can be solved in time $O(n\,\mathrm{lp}(d,m))$, where m is number of non-redundant hyperplanes in (3) [OSS95].

Assume that $\tilde{H}$ is the set of all hyperplanes bounding a horizon ridge r of a cell c. Then a horizon ridge r' incident to r in $G(R(c))$ can be obtained by interchanging the hyperplane visible for c containing r (1) with a hyperplane in $\tilde{H}$ visible for c, or by interchanging the hyperplane invisible for c containing r (2) with an hyperplane in $\tilde{H}$ invisible for c. In this way we can traverse $G(R(c))$ and find all polytopal neighbors of c. It is not hard to see that the time required to find all horizon ridges of c and thus all neighbors of c is $O(n_c\, n\,\mathrm{lp}(d,n))$, where n_c is the number of horizon ridges of a cell c (it is the time for executing a depth-first search or a breadth-first search on $G(R(c))$, but here the number of edges is in $O(n_c\, n\,\mathrm{lp}(d,n)))$. Recall that $n_c \leq k(n-k)$ by Corollary 3.2.11(c).

In $\mathbb{R}^2$, the graph $G(R(c))$ is disconnected, since the horizon ridges are points. We can find all horizon ridges of a cell c by computing all lines bounding c. This operation takes time $O(n \log n)$, since it is equivalent to computing the convex hull of the points dual to the lines (spheres) of $\mathcal{A}_{\text{sp}}$. Then we traverse the sequence of these lines. A horizon ridge is detected if among two consecutive lines one is visible and the other not.

In $\mathbb{R}^3$, a horizon ridge of a cell c is incident to exactly another horizon ridge at each endpoint. This allows a more efficient computation of $R(c)$. First, given the combinatorial information of c (its covector), we compute all vertices of

c in time $O(n \log n)$. For each vertex v we store all three planes containing v. In addition, for every two planes h_1, h_2 bounding c we store both vertices of c in $h_1 \cap h_2$, if they exist. Now given a vertex v contained in a horizon ridge of c, we check all three pairs of planes containing v and determine the two pairs forming the horizon ridges. For each of these two horizon ridges we look up in our data structure the vertex other than v. If both are already visited, we have determined all horizon ridges of c. Otherwise we choose a vertex not yet visited as the next vertex v. In this way we need constant time for each horizon ridge (once the vertices of c are known). Since the number n_c of horizon ridges of c is $O(n)$ (c has $O(n)$ edges), the computation of $R(c)$ needs time $O(n \log n + n_c) = O(n \log n)$.

Let $k \in \{1, \dots, n\}$ be fixed. Assume that the points in S are numbered according to their lexicographical sorting. For two sets $T_1, T_2 \in \binom{S}{k}$ we say that T_1 is *lexicographically smaller* than T_2 if and only if the set of indices of the points in T_1 is lexicographically smaller than the set of indices of the points in T_2. We write $T_1 <_{lex} T_2$. An analogous definition applies to the vertices $v(T_1)$, $v(T_2)$ of $Q_k(S)$ if T_1, T_2 are k-sets. Obviously $\{p_1, \dots, p_k\}$ is the lexicographically smallest k-set. We denote the corresponding vertex of $Q_k(S)$ by w_0.

In order to find a unique path from a vertex w of $Q_k(S)$ to w_0, we determine first all polytopal neighbors of w as explained above. Then the successor of w on the path to w_0 is the lexicographically smallest vertex of $Q_k(S)$ among these neighbors. We repeat this step until we have reached w_0.

Note that each path found in this way terminates in w_0 since the edge graph of $Q_k(S)$ is connected and since finding a lexicographically smallest set among the polytopal neighbors of w is equivalent to minimizing a (special) linear function. The latter statement is shown as follows. If $\mathbf{c}$ is a row vector of length d such that its i-th entry equals 2^{d-i} for $i = 1, \dots, d$, then obviously $\langle \mathbf{c}, v(T_1) \rangle < \langle \mathbf{c}, v(T_2) \rangle$ if and only if $T_1 <_{lex} T_2$ for $T_1, T_2 \in \binom{S}{k}$. The lexicographically smallest polytopal neighbor of $T(w)$ is the neighbor with the smallest value $\langle \mathbf{c}, v(T) \rangle$, so indeed the above method optimizes $\langle \mathbf{c}, v(T) \rangle$ over $Q_k(S)$.

4.3 Enumerating k-sets using polytopal neighbors

In this section we describe an algorithm for enumeration of k-sets of a point set S in general position in $\mathbb{R}^d$. The algorithm is based on the technique of reverse search and the notion of polytopal neighbors of a k-set studied in the previous section. At the end of this section we derive upper bounds on the running time of this algorithm.

Assume that k, $Q_k(S)$ and w_0 are the same as in previous section, that W is the set of vertices of $Q_k(S)$ and that $|Q_k(S)|$ denotes the number of vertices of $Q_k(S)$. Let us describe first the local search on $Q_k(S)$, cf. [AF96]. Let $f : W\backslash\{w_0\} \longrightarrow W$ be the function which maps a vertex $w \in W\backslash\{w_0\}$ to its successor on the unique path to w_0 in the way described in the previous section. Then we define a graph G with vertex set W whose edges are $\{w, f(w)\}$ for each $w \in W\backslash\{w_0\}$. By considerations in the previous section G is connected. The local search as described in [AF96] is then given by the triple $(G, \{w_0\}, f)$.

The adjacency oracle (A-oracle) Adj [AF96] for the graph G can be implemented in the following way (here (A1), (A2), (A3) correspond to the analogous conditions in [AF96]).

(A1) The vertices of G can be represented by the sets $T(w)$ for $w \in W$ or by integer encoding of k-tuples of indices of the elements in $T(w)$.

(A2) The integer δ is at most $k(n-k)$ by Corollary 3.2.11(c).

(A3) The adjacency list oracle Adj(w, m) is implemented as follows. We compute all polytopal neighbors of the vertex w and sort them lexicographically by the indices of the points in the corresponding k-sets. Then Adj(w, m) returns the m-th vertex in this sequence.

The *trace* of a graph G is the directed subgraph $T = (W, E(f))$, where $E(f) = \{(w, f(w)) : w \in W\backslash\{w_0\}\}$. The *height* of T is the length of the longest path in T. (The height of the trace in G might exceed the diameter of $Q_k(S)$, see Section 3.2). We can reduce the total time for executing the adjacency list oracle if we store the information about adjacent neighbors of a vertex w encountered for the first time during the reverse search in a data structure which allows access in time $O(1)$. In the same step we compute and store the value of $f(w)$, which reduces the total time for computing f. This information is kept until all vertices of G in the subtree below w are visited. In total, the number of vertices for which the neighbor information is stored simultaneously is bounded by the height of the trace.

Recall that the time for finding all horizon ridges of a k-cell is $O(n_c\, n\, \mathrm{lp}(d, n))$. Since each horizon ridge is computed for at most two different k-cells, the total time for computing the neighbor information of all vertices is

$$(\text{total number of horizon ridges of all } k\text{-cells}) O(n \,\mathrm{lp}(d,n)).$$

Now, each call of Adj(w,m) takes time $t(\text{Adj}) = O(1)$ and each call of f takes time $t(f) = O(1)$. By the general complexity bound $O(|Q_k(S)|\, \delta(t(\text{Adj}) + t(f)))$ for the reverse search given in [AF96][Corollary 2.3] we have

$$O((\text{total number of horizon ridges of all } k\text{-cells})\, n \,\mathrm{lp}(d,n))$$

as the total time complexity for reverse search using polytopal neighbors. Using the fact that the total number of horizon ridges of all k-cells is at most $k\,(n-k)\,|Q_k(S)|$ and that $|Q_k(S)| = o(n^d)$ [ABFK92] we obtain

$$O(|Q_k(S)|\, kn^2 \,\mathrm{lp}(d,n)) = \mathrm{o}(kn^{d+2} \,\mathrm{lp}(d,n))$$

as an upper bound on the above-mentioned time complexity.

In $\mathbb{R}^2$ the time for computation of the neighbors for all vertices of $Q_k(S)$ is $O(|Q_k(S)|\, n \log n)$. Since $\delta \leq 2$ in this case, the total time complexity of reverse search using polytopal neighbors is

$$O(|Q_k(S)|\, n \log n).$$

By the upper bound $|Q_k(S)| = O(nk^{1/3})$ on the number of k-sets in the plane we have $O(|Q_k(S)|\, n \log n) = O(n^2 k^{1/3} \log n)$ for $\mathbb{R}^2$.

In $\mathbb{R}^3$ we need time $O(n \log n)$ for computing and sorting of all polytopal neighbors of a vertex. Since $\delta = O(n)$ in this case, the total time complexity of reverse search is

$$O(|Q_k(S)|\, n \log n).$$

By the bound $|Q_k(S)| = O(nk^{5/3})$ on the number of k-sets in $\mathbb{R}^3$ this time complexity is $O(n^2 k^{5/3} \log n)$.

4.4 Enumerating k-sets using combinatorial neighbors

We begin this section with a study of the notion of combinatorial neighbors of a k-set, which is an refinement of the notion of polytopal neighbors. The most important result concerning combinatorial neighbors is a method for efficient finding all combinatorial neighbors of a k-set. Based on this result we obtain the second algorithm for enumeration of k-sets. This algorithm is subsequently sped up by the idea of storing adjacency. An analysis of the running time of both variations of this algorithm finishes this section.

In the following let $k \in \{2, \ldots, n-1\}$ be fixed, and assume that the points in S are numbered as described in Section 4.2. We say that two k-sets T_1, T_2 in S are *combinatorial neighbors*, if $|T_1 \cap T_2| = k - 1$. The corresponding definitions apply to vertices of $Q_k(S)$ and cells in $\mathcal{A}_{\text{sp}}$. Obviously a set $T_1 \in \binom{S}{k}$ has at most $k(n-k)$ combinatorial neighbors. We also say that two distinct sets T_1 and T_2 in $\binom{S}{k}$ differ by an (i,j)*-flip* if $T_2 = (T_1 \backslash \{p_i\}) \cup \{p_j\}$ for some integers $i, j \in \{1, \ldots, n\}$, $i \neq j$. If two sets T_1, T_2 in $\binom{S}{k}$ are combinatorial neighbors then they differ by an (i,j)-flip. If two k-sets are polytopal neighbors, then by Corollary 3.2.11(b) they are also combinatorial neighbors.

Remark 4.4.1 *If S is not in general position, then by Remark 3.2.12 the Corollary 3.2.11(b) might not hold, and indeed the example in Figure 3.6 shows that there are polytopal neighbors which are not combinatorial neighbors.*

Remark 4.4.2 *Even if S is in general position, then the fact that two k-sets are combinatorial neighbors does not imply that both k-sets are also polytopal neighbors. This shows the example in Figure 4.1: we can reach the 4-cell c_2 starting from c_1 by passing the hyperplane h_1 visible for c_1 and the hyperplane h_2 invisible for c_1. (This corresponds to a $(1,2)$-flip). Obviously c_1 and c_2 are combinatorial neighbors, but since they do not share a horizon ridge, they are not polytopal neighbors (Lemma 4.2.1(a)).*

We can test as follows whether $T' \in \binom{S}{k}$ is a k-set. The following linear inequality system is feasible if and only if T' is a k-set. The system tests whether there is a hyperplane $\langle \mathbf{c}', \mathbf{x} \rangle = b$ which strictly separates T' and $S \backslash T'$: find $\mathbf{c}' \in \mathbb{R}^d$ and $b \in \mathbb{R}$, such that

$$\begin{array}{lll} \langle \mathbf{c}', p \rangle < b & \text{for all} & p \in T', \\ \langle \mathbf{c}', p \rangle > b & \text{for all} & p \in S \backslash T'. \end{array}$$

The time for solving the above system is at most $\text{lp}(d+1, n)$.

Using the definitions of the Section 4.2, we show first how to obtain a unique path from any vertex of $Q_k(S)$ to the unique optimal vertex w_0 of $Q_k(S)$. Let

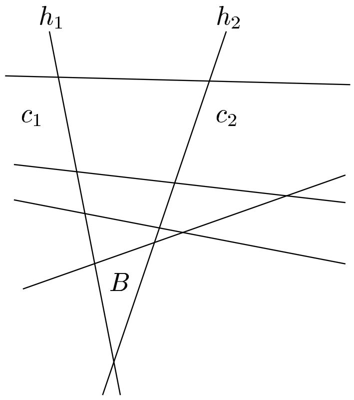

Figure 4.1: *The* 4*-cells* c_1*,* c_2 *are combinatorial neighbors but not polytopal neighbors*

W be the set of vertices of $Q_k(S)$. Let f be a function $f : W \backslash \{w_0\} \longrightarrow W$ which maps a vertex w to its successor w' in the unique path to w_0. The vertex $f(w)$ is computed in the following way. For each pair $(i, j), i, j \in \{1, \ldots , n\}, i \neq j$, let $T_{i,j}$ be the set of $\binom{S}{k}$ which differs from $T(w)$ by the (i, j)-flip, if $T_{i,j}$ exists. We denote by $\mathcal{Z}$ the family of these sets. Among the sets in $\mathcal{Z}$ we determine the lexicographically smallest one. The corresponding set $T_{i',j'}$ is then tested for being a k-set as described above. If this is the case, $v(T_{i',j'})$ is the value of $f(w)$. Otherwise we discard $T_{i',j'}$ and search for the lexicographically smallest set among the remaining sets in $\mathcal{Z}$. This procedure is repeated until success. The time $t(f)$ for computing $f(w)$ is bounded by $O(kn\,\mathrm{lp}(d + 1, n))$, since $|\mathcal{Z}| \leq k(n - k)$.

Since each polytopal neighbor is also a combinatorial neighbor, there exists a path from a vertex w to w_0. Furthermore, the above procedure yields a (unique) path ending in w_0, since it can be regarded as the optimization on the polytope $Q_k(S)$ by analogous arguments as in Section 4.2. The finite local search is then the triple $(G, \{w_0\}, f)$, where the graph G has W as a set of vertices and $\{\{w, f(w)\} | w \in W \backslash \{w_o\}\}$ as the set of edges.

The time $t(\mathrm{Adj})$ for evaluating the following adjacency oracle for a vertex w and an integer $m \leq \delta$ is clearly dominated by the time to test whether T' is a k-set, and so $t(\mathrm{Adj}) = \mathrm{lp}(d + 1, n)$. The adjacency oracle is implemented as follows.

(A1) The vertices of G can be represented by the sets $T(w)$ for $w \in W$ or by integer encoding of k-tuples of indices of the elements in $T(w)$.

(A2) The integer δ is at most $k(n-k)$, as each vertex $w \in W$ has at most that many combinatorial neighbors.

(A3) The adjacency oracle Adj(w, m) is implemented as follows. First, we determine the lexicographically m-th pair (i, j) among the pairs $\{(i,j) | i, j \in \{1, \ldots, n\}, i \neq j\}$. If the set $T' \in \binom{S}{k}$ which differs from $T(w)$ by the (i, j)-flip exists, then it is tested for being a k-set. If this is the case, T' (or its integer encoding) is returned; otherwise the oracle returns 0.

The running time of the associated Reverse Search $(Adj, \delta, \{w_0\}, f)$ is then

$$O(|Q_k(S)|\, \delta(t(Adj) + t(f))) = O(|Q_k(S)|\, k^2 n^2\, \mathrm{lp}(d+1, n))$$

by [AF96][Corollary 2.3].

Similarly as described in Section 4.3, we can compute and store the values of Adj and of f for each newly encountered vertex w until all vertices in the subtree of G below w are visited. We compute the set of combinatorial neighbors of w in the following way. For each pair $(i, j), i, j \in \{1, \ldots, n\}, i \neq j$ we test whether the set $T_{i,j} \in \binom{S}{k}$ which differs from $T(w)$ by a (i, j)-flip exists and is a k-set. The time for this computation is at most $k(n-k)\mathrm{lp}(d+1, n)$. Simultaneously, the value of $f(w)$ is computed. This information is then stored in a data structure which gives the value of Adj(w, m) in time $O(1)$. In the whole reverse search the total time for computing the neighbor information for each vertex is then $O(|Q_k(S)| kn\, \mathrm{lp}(d+1, n))$. Each call of Adj$(w, k)$ takes time $O(1)$, and each call of f takes time $O(1)$. The total time for reverse search is then

$$O(|Q_k(S)|\, k\, n\, \mathrm{lp}(d+1, n)).$$

Remark 4.4.3 *It is not hard to see that both the local search function f and the adjacency oracle described in this section also work for points sets which are not in general position. This suggests the possibility that the reverse search which uses combinatorial neighbors might work for degenerate point sets. Unfortunately, by Remark 4.4.1 the graph G to be traversed by reverse search might not be connected if S is not in general position; an example of such situation is given in Figure 3.6. Therefore, any graph search method (and reverse search is one) fails in this case.*

4.5 Affine k-sets and their enumeration

In this section we introduce and study the notion of affine k-sets of a point set in general position, which are k-sets that can be separated from the origin by a hyperplane. We describe both dual and primal representation of affine k-sets and subsequently show that the 1-skeleton of a k-set polytope restricted to the vertices corresponding to affine k-sets is not always connected in $\mathbb{R}^2$. This implies that any graph search method (and a reverse search is one) cannot be applied for an output-sensitive enumeration of k-sets in $\mathbb{R}^2$. This possibly applies to higher dimensions, too.

We conclude that the enumeration of affine k-sets in an efficient and output-sensitive way remains an open problem. A possible yet not output-sensitive way to enumerate the affine k-sets of S is to enumerate all k-sets of S and to check each output for being an affine k-set by one of the tests described above. The enumeration of affine k-sets can be used to solve the problem of finding the maximum feasible subsystem of linear relations MAX FLS. It includes many interesting special cases such as the minimum feedback arc set [AK95].

We use the notation of Section 4.2. Given a set of points S and the corresponding spherical arrangement $\mathcal{A}_{\text{sp}}$, let us call as $\mathcal{A}_{\text{aff}}$ the arrangement of hyperplanes $h_{\text{aff}}(p)$, $p \in S$. We can think about $\mathcal{A}_{\text{aff}}$ as about the central projection of the part of $\mathcal{A}_{\text{sp}}$ lying in the "northern" hemisphere of S^d onto U. Let us call a cell of $\mathcal{A}_{\text{sp}}$ which is not completely below the horizon an *affine k-cell*. Obviously there is a bijective mapping between the cells of $\mathcal{A}_{\text{aff}}$ and the affine k-cells of $\mathcal{A}_{\text{sp}}$. We call a k-set of S an *affine k-set* if it corresponds to an affine k-cell of $\mathcal{A}_{\text{sp}}$. Under the assumption that the 0-cell of $\mathcal{A}_{\text{sp}}$ contains the point $(0, \dots, 0, 1)$ of S^d, we can characterize the affine k-sets in the following way.

Lemma 4.5.1 *Let $k \in \{1, \dots, n\}$. Assume that the 0-cell of $\mathcal{A}_{sp}$ contains the point $(0, \dots, 0, 1)$. Then a k-set $P \subseteq S$ is an affine k-set if and only if there is a hyperplane which strictly separates P from $S \backslash P$ and from the origin $\boldsymbol{o}$ of U.*

PROOF. First note that a central hyperplane h has the point $(0, \dots, 0, 1)$ on its positive side if and only if the point $p_{\text{sp}}(h)$ is above the horizon of S^d. Assume that c is an affine k-cell of $\mathcal{A}_{\text{sp}}$. Let q be a point of c above the horizon, and h_q the central hyperplane such that $p_{\text{sp}}(h_q) = q$. Then h_q has the origin of U (which coincides with the point $(0, \dots, 0, 1)$ of $\mathbb{R}^{d+1}$) on its positive side. By the property of the duality, the k points of S corresponding to the hyperplanes separating the 0-cell and the cell c are on the negative side of h_q. The converse can be shown analogously. ■

If we allow a translation of the hyperplanes forming $\mathcal{A}_{\text{aff}}$ (or, equivalently, a translation of points in S), then the condition of Lemma 4.5.1 can be always satisfied: simply translate the hyperplanes forming $\mathcal{A}_{\text{aff}}$ in such a way that the origin of U is located in the 0-cell of $\mathcal{A}_{\text{aff}}$. Thus we can assume in the following, that the point $(0, \dots, 0, 1)$ is contained in the 0-cell of $\mathcal{A}_{\text{sp}}$.

We can test whether a set $T' \in \binom{S}{k}$ is an affine k-set in the analogous way as described in Section 4.4: T' is an affine k-set if we can find $\mathbf{c}' \in \mathbb{R}^d$ and $b \in \mathbb{R}$, such that

$$\begin{array}{lll} \langle \mathbf{c}', p \rangle < b & \text{for all} & p \in T' \\ \langle \mathbf{c}', p \rangle > b & \text{for all} & p \in S \backslash T' \cup \{\mathbf{o}\}, \end{array}$$

i.e. if the corresponding inequality system is feasible. The running time for this test is $\text{lp}(d+1, n+1)$.

The *graph of affine k-sets* of S has affine k-sets as nodes, and two nodes share an edge if the corresponding affine k-sets are polytopal neighbors. The reverse search algorithm can be applied to enumerating affine k-sets only under the condition that the graph of affine k-sets is connected, i.e. if one can reach each affine k-set from any other affine k-set by "moving" over affine k-sets on the polytope $Q_k(S)$.

Unfortunately, this condition is not always fulfilled in $\mathbb{R}^2$ and it is not known to be fulfilled in higher dimensions. In Figure 4.2 the solid polygon represents the 4-set polytope $Q_4(S_1)$ of the points with coordinates $(-3.9, -6.8)$, $(-9.1, -4.8)$, $(3.1, -0.8)$, $(-11.1, -6.8)$, $(-1, -5)$, $(11, 9)$, $(10, 14)$. Except for the both (polytopal) neighbors of v_1 all vertices of $Q_4(S_1)$ are affine. Therefore v_1 cannot be reached from any other affine vertex of $Q_k(S_1)$ and so the graph of affine 4-sets of S_1 is disconnected.

We explain in the following how to obtain such counterexamples in $\mathbb{R}^2$ (and possibly also in higher dimensions). For a $k \in \{2, \dots, n\}$, consider the polytope $Q_k(S \cup \{\mathbf{o}\})$ and the set $X_k(S \cup \{\mathbf{o}\})$. We have

$$X_k(S \cup \{\mathbf{o}\}) = X_k(S) \cup X_{k-1}(S),$$

since for a $T \in \binom{S \cup \{\mathbf{o}\}}{k}$, T contributes to $X_k(S)$ if $\mathbf{o} \notin T$, and T contributes to $X_{k-1}(S)$ if $\mathbf{o} \in T$. Therefore, $Q_k(S \cup \{\mathbf{o}\})$ is a convex hull of the polytopes $Q_k(S)$ and $Q_{k-1}(S)$.

Lemma 4.5.2 *A vertex v of $Q_k(S)$ is also a vertex of $Q_k(S \cup \{\mathbf{o}\})$ if and only if v corresponds to an affine k-set.*

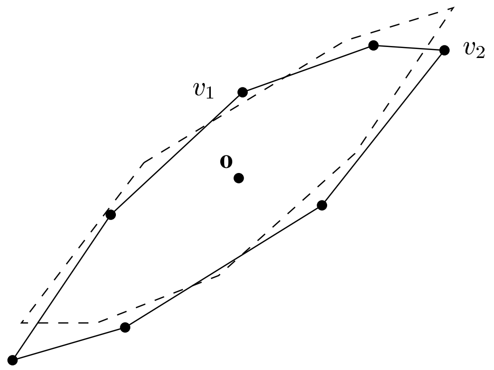

Figure 4.2: *The graph of the affine* 4*-sets of* S_1 *is not connected*

PROOF. If v is affine, then the set $T(v)$ is a k-set in $S \cup \{\mathbf{o}\}$, too. If v is not affine, then by definition the set $T(v)$ is not a k-set in $S \cup \{\mathbf{o}\}$. ∎

Now we show that $Q_k(S)$ and $Q_{k-1}(S)$ can be translated relatively to each other by an arbitrary vector, thus allowing to "hide" some vertices of $Q_k(S)$ by moving them into the relative interior of $Q_{k-1}(S)$. We translate the points in S by a vector $\mathbf{v} \in \mathbb{R}^d$ and obtain $S' = \{p + \mathbf{v} | p \in S\}$. Each point $x_1 \in X_k(S)$ becomes then $x_1 + k\mathbf{v}$, and each point $x_2 \in X_{k-1}(S)$ becomes $x_2 + (k-1)\mathbf{v}$. Therefore, $Q_k(S')$ is the polytope $Q_k(S)$ translated by $k\mathbf{v}$ and $Q_{k-1}(S')$ is the polytope $Q_{k-1}(S')$ translated by $(k-1)\mathbf{v}$. Relative to $Q_k(S)$, the polytope $Q_{k-1}(S)$ is translated by $-\mathbf{v}$.

By experimenting with some point sets and translation vectors $\mathbf{v}$, we could easily find the configuration shown in Figure 4.2. There, both neighbors of v_1 are inside the dashed polytope $Q_3(S_1)$, and we conclude that they are the only non-affine vertices of $Q_4(S_1)$. It is also easily possible to enlarge S_1 and still keep its property of being a counterexample.

Chapter 5

Some selected proofs

In this chapter we present three smaller pieces. They comprise an improved proof of an upper bound on halving triangles in $\mathbb{R}^3$ in Section 5.1, a formula for the number of j-facets of a vertex set of a neighborly, simplicial polytope (Section 5.2), and an elaboration of a known proof concerning the maximum number of median intersections in a pseudoline arrangement in Section 5.3 (the proof of this result has been never written up carefully).

5.1 Maximum number of halving triangles in $\mathbb{R}^3$

DEY and EDELSBRUNNER prove in [DE94] an upper bound $O(n^{8/3})$ on the number of halving triangles in $\mathbb{R}^3$ by exploring the combinatorics of crossings between edges and triangles in $\mathbb{R}^3$ (this result has been improved in [AACS98] to $O(nj^{5/3})$ for arbitrary j-facets in $\mathbb{R}^3$). Here we state the proof in [DE94] but instead of an inductive proof of Theorem 5.1.1 we present a probabilistic one. The used probabilistic technique has been introduced by SZÉKELY to obtain a variety of incidence bounds, for example an upper bound $O(|E|^3/|V|^2)$ on the number of crossing edge pairs of a drawing of a graph $G = (V, E)$ (communicated by SHARIR).

Let S be a set of n points in $\mathbb{R}^3$ and T a set of t (non-oriented) triangles with vertices in S. We say that two triangles have a *nontrivial intersection* if their intersection is neither empty nor a vertex or edge of both. A special case occurs when both triangles are vertex disjunct. We say then that they *cross.* An edge *crosses a triangle* when it intersects a triangle without sharing a vertex with it. We write $x(S,T)$ for the number of crossing triangle pairs in T and we set

$$x(n,t) = \min_{|S|=n,\ |T|=t} x(S,T).$$

Furthermore, let $y(S,T)$ be the maximum number of triangles in T crossed by a single edge of a triangle in T and let

$$y(n,t) = \min_{|S|=n,\ |T|=t} y(S,T).$$

Our first observation is that a lower bound $b(n,t)$ for $x(n,t)$ implies a lower bound $2b(n,t)/(3t)$ for $y(n,t)$. First, considering the two possible geometrical cases when two triangles cross, we see that two of the six participating edges cross a triangle. We give one unit of credit to each such edge and so with $x(n,t) \geq b(n,t)$ we see that at least $2b(n,t)$ units of credits has been given. Because there are $3t$ edges, it follows by the pigeonhole principle that one of the edges crosses at least $2b(n,t)/(3t)$ triangles.

Theorem 5.1.1 *For* $c_0 n^2 \leq t \leq \binom{n}{3}$ *we have*

$$c_1 \frac{t^4}{n^6} < x(n,t)$$

and therefore

$$\frac{2c_1}{3}\frac{t^3}{n^6} < y(n,t)$$

with $c_0 = 2.16$ *and* $c_1 = 1/41$.

Now assume that T is a set of halving triangles and $t = |T| \geq c_0 n^2$ (here we consider halving triangle as not oriented; their number is half of the number of oriented halving triangles as defined in Section 2.1). Then by Lovász crossing lemma [AW97] no line in $\mathbb{R}^3$ crosses more than $n^2/8$ of these triangles. On the other hand by Theorem 5.1.1 there is a line which crosses at least $(2c_1/3)t^3/n^6$ halving triangles. Thus, we obtain

$$t < \left(\frac{3}{16c_1}\right)^{1/3} n^{8/3} \leq 2n^{8/3}$$

as an upper bound on the number of halving triangles in $\mathbb{R}^3$.

To show the lower bound for $x(n,t)$ we need part (b) of the following lemma.

Lemma 5.1.2 *[DE94]*

(a) *If no two triangles in T have a nontrivial intersection, then $|T| < n^2$.*

(b) *If no two triangles in T cross, then $|T| < \frac{3}{2}n^2$.*

PROOF. **(a)**: For any point $p_i \in S$ let δ_i be the number of edges of triangles in T with endpoint p_i, and so $\delta_i \leq n-1$. We intersect the triangles incident to p_i with a sphere around p_i which is small enough so that each incident triangle intersect the sphere in a connected great circle arc. If no two triangles have a nontrivial intersection, then these arcs create on the surface of the sphere a planar graph with δ_i vertices and thus at most $3\delta_i$ arcs. We conclude that p_i is incident to at most $3\delta_i$ triangles. The total number of triangles is then

$$\frac{1}{3}\sum_{p_i \in S} 3\delta_i < n^2$$

as each triangle is counted three times.

(b): Assume that no two triangles in T cross. Then an edge e crossing a triangle σ must belong to a triangle that share a vertex with σ. For fixed e and σ there are at most three such triangles. Remove e together with at most three triangles. Let k is the number of such edges needed to be removed until no nontrivial intersection remains. We loose at most $3k$ triangles and decrease the total vertex degree to

$$\sum_{p_i \in S} \delta_i - 2k$$

or less. Now using the same argument as in the proof of (i) we conclude that this is the maximum number of remaining triangles. We have then

$$|T| < 3k + \left(\sum_{p_i \in S} \delta_i - 2k\right) < k + n^2 < \binom{n}{2} + n^2$$

what completes the proof. ■

PROOF. [Theorem 5.1.1] By Lemma 5.1.2(b) we see that

$$x(n,t) > t - \frac{3}{2}n^2$$

for all $0 \leq t \leq \binom{n}{3}$. For fixed S and T with $|S| = n$ and $|T| = t$ we perform the following Bernoulli experiment. Remove every point of S with probability

$1-p$ for some $0 \leq p \leq 1$. Let x', t' and n' be the random variables for the number of crossing triangle pairs, number of triangles and number of points after the experiment, respectively. (A triangle survives if all three vertices survive and a crossing triangle pair survives if both triangles survive). We still have

$$x' > t' - \frac{3}{2}(n')^2$$

for any combination of values of these random variables, and so this equation also holds for the expectation, i.e.:

$$E(x') > E(t') - \frac{3}{2}E((n')^2).$$

We have $E(t') = p^3 t$ and $E(x') = p^6 x(S,T)$, since two crossing triangles are vertex-disjoint. Now n' is Bernoulli-distributed random variable, and so

$$E((n')^2) = E(n')^2 + \text{Var}(n') = n^2p^2 + np(1-p).$$

Choosing S and T with $x(n,t) = x(S,T)$ we obtain the inequality

$$p^6 x(n,t) > p^3 t - \frac{3}{2}\left(n^2p^2 + np(1-p)\right).$$

We will show later that for $\epsilon' = 1.0774$ we have

$$(\epsilon' - 1)n^2p^2 > np(1-p). \tag{5.1}$$

Then it follows

$$x(n,t) > p^{-3}t - \frac{3\epsilon'}{2}p^{-4}n^2. \tag{5.2}$$

The key idea is to find p which maximizes the right-hand side of (5.2). It is not hard to see that this is the case for p equal to

$$p_E = 2\epsilon' \, \frac{n^2}{t}.$$

To ensure that $p \leq 1$, we have to impose on t the condition

$$t \geq 2\epsilon' n^2 > c_0 n^2$$

with $c_0 = 2.155$. Then we obtain by (5.2)

$$x(n,t) > \frac{1}{32(\epsilon')^3}\frac{t^4}{n^6} > \frac{1}{41}\frac{t^4}{n^6}$$

and so $c_1 = 1/41$.

It remains to show (5.1). Obviously $\epsilon' > \frac{1}{2} + \frac{2}{\sqrt{12}}$, and so by $t \leq n^3/6$ we have

$$2\epsilon'(\epsilon' - 1) > \frac{1}{6} \geq \frac{t}{n^3}.$$

This is equivalent to

$$1 \leq n2\epsilon' \frac{n^2}{t}(\epsilon' - 1) = np_E(\epsilon' - 1)$$

and so by

$$np_E(1 - p_E) \leq np_E \leq n^2 p_E^2(\epsilon' - 1)$$

and the substitution $p = p_E$ we obtain (5.1). ■

DEY and EDELSBRUNNER show also in [DE94] upper bounds for $x(n, t)$ and $y(n, t)$ for t in the range

$$2n^2 \leq t \leq \binom{n}{3}.$$

While for $t = \binom{n}{3}$ the lower and upper bounds on $x(n, t)$ match asymptotically, we have for $t = O(n^2)$ a lower bound $O(n^2)$ and an upper bound $O(n^3)$ on $x(n, t)$. The authors of [DE94] conjecture that $x(n, t)$ is indeed cubic in n for $t = O(n^2)$. This would imply an upper bound $O(n^{5/2})$ on the number of halving triangles in $\mathbb{R}^3$. This result would be the best possible one when Lovász crossing lemma is used in the way as above, see [AW97].

There is also another proof of the bound $O(n^{8/3})$ which uses the probabilistic technique by KÁROLYI and WELZL [KW99]. Let T be a set of halving triangles in $\mathbb{R}^3$, $t = |T|$, and x the number of crossings between all line segments between two points in S and all triangles in T. By Lovász crossing lemma every segment between two points in S intersects at most $O(n^2)$ triangles, and since there are $\binom{n}{2}$ segments, we have $x = O(n^4)$. As next the authors show that $t \leq n^2 - n + x$. Using this equation as the basis for the probabilistic technique analogous to the one applied in the proof of Theorem 5.1.1 the authors obtain

$$t = O(n^{4/3}x^{1/3}).$$

In conjunction with the bound $x = O(n^4)$ this yields an upper bound $O(n^{8/3})$ on the number of halving triangles in $\mathbb{R}^3$.

5.2 Number of j-facets for neighborly, simplicial polytopes

Using a straightforward probabilistic analysis CLARKSON and SHOR [CS89] obtain an asymptotic upper bound on the number of $(\leq j)$-facets. They also obtain the exact number of the j-facets when S is the set of vertices of a convex polytope in $\mathbb{R}^3$. In following we show in Theorem 5.2.3 an easy generalization of this result for arbitrary d. More precisely, we obtain the numbers $g_k(S)$ for any $k \in \{0, \dots, n-d\}$, $d \geq 2$ and $n \geq d+1$, if S is a set of vertices of a convex, simplicial and neighborly polytope in $\mathbb{R}^d$. For the special case when such neighborly, simplicial polytope is a cyclic polytope this result is already given by Corollary 3.3.2.

The following key lemma of [CS89] gives the relationships between the quantities $g_j(S)$ and $u_r(S)$, where $u_r(S)$ is the expected number of facets of a convex hull of a random sample of size r of S.

Lemma 5.2.1 *[CS89, Lemma 2.1] Let $R \subseteq S$ be a random sample of size $r \geq 0$. Then*

$$\sum_{j=0}^{n-d} g_j(S) \binom{n-d-j}{r-d} = \binom{n}{r} u_r(S). \tag{5.3}$$

Lemma 5.2.1 has been used in [CS89] among others to prove that if S is in a convex position (and in general position) in $\mathbb{R}^3$, then $g_k(S) = 2\,(k+1)\,(n-k-2)$. This result is shown in [CS89] as follows. Because S is a set of vertices of a convex polytope, then by Upper Bound Theorem (see [Zie94]) we have $u_r(S) = 2\,(r-2)$ for $r = d, \dots, n$. Then (5.3) yields $n-d+1$ equations which are simultaneously solvable with the unique solution as stated above. Because every simplicial polytope in three-dimensional space is also neighborly, we obtain the result for $\mathbb{R}^3$ as a special case of the Theorem 5.2.3.

The following lemma has been communicated by KORTENKAMP.

Lemma 5.2.2 *Assume that P is a neighborly polytope in $\mathbb{R}^d$, and V' is a subset of the vertex set of P. Then conv(V') is also a neighborly polytope.*

PROOF. A set V_F of at most $\lfloor d/2 \rfloor$ points in V' is a vertex set of a face of P since P is neighborly. Furthermore, if F is a face of P such that its vertices are in V', then F is obviously also a face of conv(V'), and so we are done. ■

Theorem 5.2.3 *Let S be a vertex set of a simplicial, neighborly polytope. Then for every $k = 0, \ldots, n-d$ we have*

$$g_k(S) \quad = \quad 2\binom{k+\lfloor\frac{d}{2}\rfloor}{\lfloor\frac{d}{2}\rfloor}\binom{n-k-\lceil\frac{d}{2}\rceil}{\lfloor\frac{d}{2}\rfloor}$$

for odd d and

$$g_k(S) \quad = \quad \binom{k+\frac{d}{2}-1}{\frac{d}{2}-1}\binom{n-k-\frac{d}{2}}{\frac{d}{2}} + \binom{k+\frac{d}{2}}{\frac{d}{2}}\binom{n-k-\frac{d}{2}-1}{\frac{d}{2}-1}$$

if d is even.

PROOF. We use the idea of Sharir [Sha93] of rescaling g_k's and u_r's and applying the binomial inversion formula to obtain g_k's. To this end we put

$$g_k(S) = \frac{n!}{k!\,(n-k-d)!}\,\hat{g}_k, \quad k = 0, \ldots, n-d$$

and

$$\hat{u}_r = \frac{(r-d)!}{r!}\,u_r, \quad r = d, \ldots, n.$$

Now we can rewrite (5.3) as

$$\sum_{j=0}^{i}\binom{i}{j}\hat{g}_j = \hat{u}_{n-i}, \quad i = 0, \ldots, n-d.$$

In the next step we apply the binomial inversion formula [GKP89] and obtain after resubstituting

$$g_k(S) = \frac{n!}{d!\,k!\,(n-k-d)!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{u_{n-k+j}(S)}{\binom{n-k+j}{d}} \tag{5.4}$$

for $k = 0, \ldots, n-d$. By the Upper Bound Theorem and Lemma 5.2.2 the number $g_0(R)$, $R \subseteq S$, $|R| = r$ of the facets of the polytope $conv(R)$ with r vertices is

$$g_0(R) = \begin{cases} 2\binom{r-\lceil d/2\rceil}{\lfloor d/2\rfloor} & \text{if } d \text{ is odd,} \\ 2\binom{r-\lceil d/2\rceil}{\lfloor d/2\rfloor} - \binom{r-\lceil d/2\rceil-1}{\lfloor d/2\rfloor} & \text{if } d \text{ is even.} \end{cases}$$

Since $u_r(S) = g_0(R)$ for any $R \subseteq S$ with $|R| = r = d, \ldots, n$, we are able to "plug in" the values for u_r's into (5.4). After some tedious manipulations and using the formulas for sums of binomial coefficients stated in [GKP89] we obtain the result. ∎

5.3 Median intersections of pseudoline arrangements

A *pseudoline* is an x–monotone curve in $\mathbb{R}^2$. An *arrangement of pseudolines* is a finite set of pseudolines such that every two pseudolines are either disjoint or they meet in a single point and cross in this point. For an arrangement of n pseudolines in the plane, n even, a *median intersection* is an intersection p of two pseudolines such that there are $n/2 - 1$ pseudolines above p (and so below p). In this section we give a proof of a lower bound

$$n^{1+\Omega(1/\sqrt{\log n})} = ne^{\Omega\left(\sqrt{\log n}\right)}$$

on the maximum number of median intersections of an arrangement of n pseudolines due to KLAWE, PATERSON and PIPPENGER [KPP82]. This proof has been never carefully written up by the authors and is hard to understand from the handwritten copies which can be obtained from EDELSBRUNNER.

The connection to the number of k-sets is the following. We have shown in Section 2.2 that the complexity of a k-level in an arrangement of lines in $\mathbb{R}^2$ is essentially the same as the number of k-sets of a point set dual to the arrangement. Here, the number of median intersections gives a lower bound on the complexity of an $n/2$-"level" but in an arrangement of pseudolines. Therefore, if it could be shown that the constructed arrangement of pseudolines is stretchable (realisable) [BLVS$^+$93], we could obtain a lower bound on the number of halving lines of a planar point set. Such bound can be extended to lower bounds for arbitrary k, see [AW97, Ede87]. Recently, TÓTH proved by an explicit construction a lower bound on the maximum number of k-sets in $\mathbb{R}^2$ which is exactly $n^{1+\Omega(1/\sqrt{\log k})}$ [Tót99]. This bound can be also extended to higher dimensions. The constructions of TÓTH and the one in [KPP82] are different, so the question of realizability is still open.

Theorem 5.3.1 *For sufficiently large n, there is an arrangement of n pseudolines in $\mathbb{R}^2$ with*

$$n^{1+\Omega(1/\sqrt{\log n})}$$

median intersections.

PROOF. To show the above bound we construct an arrangement $\mathcal{A}_n$ of n pseudolines with many median intersections, where n is even. In the following we describe different "building blocks" of the construction, summarizing in an algorithm which creates a desired arrangement.

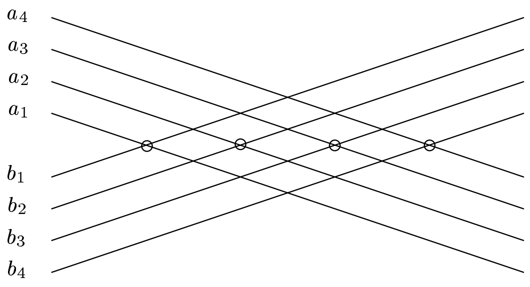

Figure 5.1: *A simple crossing of* 8 *pseudolines*

We start with an arrangement of two pseudolines which cross in one point. We call these pseudolines *pseudolines of order* 0, their set is denoted as $\mathcal{C}_0$. Let r be an even integer to be determined later. Each of the pseudolines in $\mathcal{C}_0$ is split into r pseudolines called *pseudolines of order* 1. Their set is denoted as $\mathcal{C}_1$. (These new pseudolines lie arbitrarily close to the original pseudoline). In general, for $k > 0$, a set $\mathcal{C}_k$ of *pseudolines of order* k is obtained by substituting each pseudoline in $\mathcal{C}_{k-1}$ by r new pseudolines.

For every intersection of two pseudolines we obtain after a split a part of arrangement of pseudolines as in Figure 5.1 (there for $r = 4$). For every even r, we call such part of an arrangement a *simple crossing of* $2r$ *pseudolines*. If an intersection of $2r$ pseudolines is obtained from a median intersection of two pseudolines, then we have r median intersections in the simple crossing after the split. In Figure 5.1 the (possible) median intersections are encircled.

In order to obtain more median intersections after a split, we introduce a modification of a simple crossing. Let us denote the $2r$ pseudolines in a simple crossing as $a_r, a_{r-1}, \ldots, a_1, b_1, b_2, \ldots, b_r$ in the order in they appear from the top to the bottom on the left border, as in Figure 5.1. Then a *twisted crossing of* $2r$ *pseudolines* is obtained from a simple crossing as follows (see Figure 5.2). We intersect the pseudoline pairs a_{i+1}, a_{i+2} for each $i = 1, 3, \ldots, r-3$ in the space below b_i, b_{i+1} and above b_{i+2}, b_{i+3}. Furthermore, for each $i = 1, 3, \ldots, r-1$ we intersect the pseudoline pairs b_i, b_{i+1} in the space just below a_{i+1} and possibly below a_{i+2}. We say that the pseudolines $a_1, \ldots, a_r$ have *odd phase* and the pseudolines $b_1, \ldots, b_r$ have *even phase*. If a twisted crossing of $2r$ pseudolines is obtained from a median intersection, then we have obviously $2r-1$ median intersections in the crossing after the split. In Figure 5.2 the (possible) median intersections are encircled.

As indicated above, we start with an arrangement of two pseudolines and split

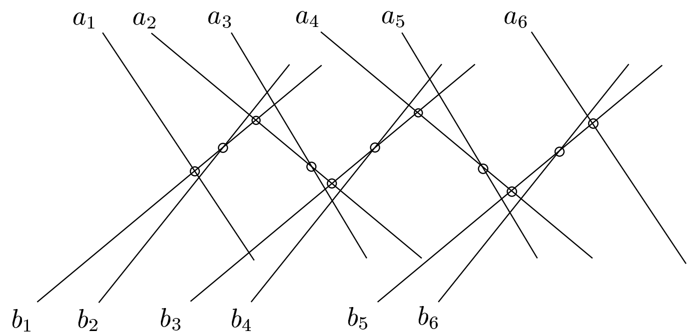

Figure 5.2: *A twisted crossing of* 12 *pseudolines*

recursively each pseudoline until we have n pseudolines. Simultaneously we substitute each median intersection of two pseudolines of order k by a twisted crossing of $2r$ pseudolines of order $k+1$. Every other (non-median) intersection of two pseudolines of order k is turned into a simple crossing. We have to ensure the condition that every two pseudolines in an arrangement intersect at most once. This condition may be violated if too many median intersections of a pseudoline are turned into twisted crossings. Assume that a pseudoline L of order k participates in j median intersections, which all are turned into twisted crossings. By the definition of a median intersection L "enters" alternatively from above and from below into median intersections. Thus, the pseudolines obtained from L will have alternatively odd and even phases in the twisted crossings they participate in. For $r = 4$ this is shown in Figure 5.3. (Note that for every even $r \geq 4$ such an arrangement represents an "odd-even transposition sort" discussed by Knuth in [Knu92]). It is not hard to see that after r twisted crossings every two pseudolines obtained from L have intersected each other and so these pseudolines cannot participate in any more twisted crossings. Thus, if $j > r$, then only the r first median intersections of L can be turned into twisted crossings.

Summarizing, we obtain the following algorithm for constructing $\mathcal{A}_n$.

1. Start with an arrangement of two pseudolines.
2. Split recursively each pseudoline into r pseudolines, until there is a pseudoline with more than r median intersections. In each step, every median intersection is turned into a twisted crossing and a non-median intersection is turned into a simple crossing. Let k be the final order of the pseudolines.

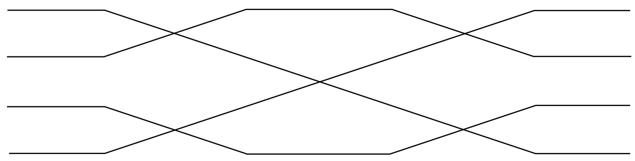

Figure 5.3: *A sequence of odd and even phases of 4 pseudolines*

(Obviously for some values of n there is no $\mathcal{A}_n$. But it is not hard to see that if n' is the largest even integer smaller than n such that $\mathcal{A}_{n'}$ exists, then we may add $n - n'$ pseudolines to $\mathcal{A}_{n'}$ without changing the number of median intersections. The asymptotic lower bound derived below will not change by this operation.)

In the following we determine r and bound from below the maximum number of median intersections in $\mathcal{A}_n$. All logarithms have basis 2. First, observe (Figure 5.2) that each pseudoline with j median intersections is split into pseudolines which have at most $3j$ intersections each (with a little more effort it can be shown that $3j$ can be interchanged with $2j$) and so $3^k > r$. Using the fact that we have $n = 2r^k$ pseudolines and that $k > \log r / \log 3$, we obtain

$$\log 2 + \frac{\log^2 r}{\log 3} \quad > \quad \log n,$$

and

$$\log r > \sqrt{(\log n - 1)\log 3}.$$

This yields

$$r > (n-1)^{\sqrt{\log 3}/\sqrt{\log n - 1}} = n^{\Omega(1/\sqrt{\log n})}.$$

By the properties of a twisted crossing, the number $t(n)$ of median intersections of the pseudolines of order k is at least $t(n) = (2r-1)^k = 2^k r^k (1 - 2/r)^k$. Therefore $t(n)/n \geq 2^{k-1}(1-2/r)^k$. We obtain

$$t(n)/n > \frac{1}{2}\, 2^{\log r/\log 3}(1-2/r)^k = \frac{1}{2}\, r^{1/\log 3} - \mathrm{o}(r^{1/\log 3}),$$

and so $t(n)/n = n^{\Omega(1/\sqrt{\log n})}$. Consequently, we obtain an arrangement of pseudolines with $n^{1+\Omega(1/\sqrt{\log n})}$ median intersections. ■

Note that it not possible to improve the lower bound for this construction by such modifications as choosing different r's in each step (for example $r_i =$

3^i+1 for $i=1,\ldots,k$). This is due to the fact that $t(n)/n$ is not greater than the maximum number of median intersections of a (final) pseudoline. The last quantity is bounded by $3r$. As already $t(n)/n = \Omega(r^{1/\log 3})$, we could only improve the constants in $\Omega(1/\sqrt{\log n})$ for this construction, but not the asymptotic behavior of the lower bound.

Chapter 6

Known bounds on the numbers of k-sets and j-facets

In this section we give a historical outline of the known lower and upper bounds on the numbers of k-sets, j-sets, $(\leq k)$-sets and $(\leq j)$-facets. We classify these results according to the dimension. The symbols used here are defined in Section 2.1.

The following facts are important to interpret the results. First, the bounds for the j-facets can be applied to the k-sets according to (3.2), Theorem 3.10.1 and Corollary 3.4.2. Subsequently, lower bounds on the maximum number of halving hyperplanes ($k = n/2$) imply lower bounds on the numbers of k-sets for $k \neq n/2$ [Ede87, AW97] in any dimension. For example, a lower bound $\Omega(n \log n)$ on $e_{1/2}(2, n)$ implies $e_k(2, n) = \Omega(n \log(k + 1))$ for any $k \in \{1, \ldots, n - 1\}$. An analogous statement holds for the j-facets.

As can be seen in Table 6.1, the gap between upper and lower bounds on the maximum number of j-facets or k-sets becomes greater with higher dimension. Opposite to this, there are asymptotically tight bounds on the maximum numbers of $(\leq k)$-sets and maximum numbers of $(\leq j)$-facets for any dimension, see Table 6.2.

dim.	Symbol	Bound	Source
2	$e_k(2,n)$	$\Omega(n \log k)$	[ELSS73, EW85]
2	$e_k(2,n)$	$ne^{\Omega(\sqrt{\log k})}$	[Tót99]*
2	$e_{1/2}(2,n)$	$O(n\sqrt{n})$	[Lov71]
2	$e_k(2,n)$	$O(n\sqrt{k})$	[ELSS73]
2	$e_k(2,n)$	$O(n\sqrt{k})$	[EW85]
2	$e_k(2,n)$	$O(n\sqrt{k}/\log^*(k+1))$	[PSS92]
2	$e_k(2,n)$	$O(nk^{1/3})$	[Dey98]*
3	$e_{1/2}(3,n)$	$\Omega(n^2 \log n)$	[Ede87]
3	$e_{1/2}(3,n)$	$n^2e^{\Omega(\sqrt{\log n})}$	[Tót99]*
3	$e_k(3,n)$	$O(n^2k)$	[CSY87]
3	$e_k(3,n)$	$O(nk^4)$	[CP86]
3	$g_{1/2}(3,n)$	$O(n^{3-1/343})$	[BFL90]
3	$g_{1/2}(3,n)$	$O(n^{8/3}\log^{5/3} n)$	[ACE$^+$91]
3	$g_{1/2}(3,n)$	$O(n^{8/3}\log^{2/3} n)$	[Epp93]
3	$g_{1/2}(3,n)$	$O(n^{8/3})$	[DE94]*
3	$g_j(3,n)$	$O(n(j+1)^{5/3})$	[AACS98]*
$d>3$	$e_{1/2}(d,n)$	$\Omega(n^{d-1}\log n)$	[Ede87]
$d>3$	$e_{1/2}(d,n)$	$n^{d-1}e^{\Omega(\sqrt{\log n})}$	[Tót99]*
$d>3$	$g_{1/2}(d,n)$	$O(n^{d-\varepsilon}),\quad \varepsilon>0$	[BFL90, ŽV92], [DE94], [ABFK92]*

Table 6.1: *A compilation of known lower and upper bounds for k-sets and j-facets. Currently best bounds are marked with *.*

dim.	Symbol	Bound	Source
2	$E_k(2,n)$	$\geq 3\binom{k+1}{2}$ for $k \leq n/3$	[EHSS89]*
2	$\sum_{i\in K} e_i(S)$	$O(n\,(\sum_{i\in K} i)^{1/2})$	[Wel86]
2	$\sum_{i\in K} e_i(S)$	$O(n\,(\sum_{i\in K} i)^{1/3})$	[AAH+98]*
2	$E_k(2,n)$	$= nk$ for $k < n/2$	[AG86]
2	$E_k(2,n)$	$= nk$	[Pec85]*
3	$E_k(3,n)$	$\Omega(nk^2)$	[Ede87]*
3	$E_k(3,n)$	$O(nk^5)$	[CP86]
3	$E_k(3,n)$	$= k^2n - k(k-1)(2k+5)/3$ for $2k \leq n-2$	[Wel99]*
3	$G_j(3,n)$	$= 2\left(\binom{j+2}{2}n - 2\binom{j+3}{3}\right)$ for $2j \leq n-4$	[Wel99]*
$d > 1$	$G_j(d,n)$	$\Theta(n^{\lfloor d/2\rfloor}(j+1)^{\lceil d/2\rceil})$	[CS89]*

Table 6.2: *A compilation of known lower and upper bounds for $(\leq k)$-sets and $(\leq j)$-facets. Currently best bounds are marked with *.*

6.1 Results for the planar case

6.1.1 k-sets and j-facets

The planar case has received the most extensive treatment in the literature. LOVÁSZ wrote the very first paper [Lov71] on k-sets. It is dedicated to induced halving segments, and its main result is the bound $g_{1/2}(n) = O(n\sqrt{n})$. The paper introduces the important Lovász crossing lemma [AW97]. The refinement of this lemma appeared in [ELSS73] by ERDŐS, LOVÁSZ, SIMMONS and STRAUS, where the result of LOVÁSZ is generalized for j-facets, yielding $g_j(2,n) = O(n\sqrt{j+1})$. In addition to this result, a lower bound $\Omega(n \log n)$ on the maximum number of j-facets and k-sets is given. This lower bound could be recently improved by TÓTH to $ne^{\Omega(\sqrt{\log k})}$ [Tót99], see also Section 5.3.

Although the proofs in [ELSS73] are mainly geometrical, the whole problem is treated in the settings of graph theory. To this end authors define directed *k-graphs* $G_k(S)$, whose edges are k-segments for S and vertices are points in S incident to any k-segment. On this basis a number of graph-theoretical results is shown. Some theorems treat the connectivity of the graphs $G_k(S)$. Furthermore it is shown that each component of $G_k(S)$ has an oriented Euler circuit, because at each vertex of G_k the number of incoming edges is equal to the number of outgoing edges. There are also some results on the number of vertices of a graph $G_k(S)$. In particular, it is shown that all points on the convex hull of S are vertices of every $G_k(S)$; that every point of S is a vertex of $G_{(n-2)/2}(S)$; that $G_k(S)$ has at least $k < (n-2)/2$ vertices if $k < (n-2)/2$. Yet another result is that the degree of the vertices of $G_k(S)$ does not exceed $2\,k+2$. As far as we know, none of these properties has been used in later papers, but the graph-theoretical approach has been also applied in [PSS92].

The interest in halfspace range searching problems lead to a rediscovery of the problem by EDELSBRUNNER and WELZL in [EW85]. Using a different technique as the one in [ELSS73], namely a combinatorial technique of allowable sequences [AW97], they obtain the same upper bound on the maximum number k-sets as in [ELSS73]. Also the same lower bound as in [ELSS73] is shown in the paper. In addition to this, some applications of k-sets in combinatorial geometry are discussed.

In [PSS92] PACH, STEIGER and SZEMERÉDI improve the upper bound for $e_k(2,n)$ to $O(n\sqrt{k}/\log^*(k+1))$. This result is obtained by "a slightly stronger blend of geometry and combinatorics", i.e. the proofs are indeed non-trivial.

The currently best upper bound on the maximum number $e_k(2, n)$ of planar k-sets is $O(nk^{1/3})$ and has been obtained by DEY [Dey98]. In addition, he gives bounds on the complexity of k-levels of pseudolines. DEY uses in his proof the notion of concave chains above an upper $(k-1)$-level in an arrangement of n lines [AACS98]. Another component of the proof is a probabilistic technique of SZÉKELY, also used in Section 5.1. ARONOV and WELZL [AAH+98] could give an alternative proof of this bound by exploiting the identities for the number of crossings of k-segments.

6.1.2 $(\leq k)$-sets and $(\leq j)$-facets

A good transition from the world of k-sets to the world of $(\leq k)$-sets is the following result of WELZL [Wel86]. For any $K \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, we have

$$\sum_{i \in K} e_i(S) \leq O(n\,(\sum_{i \in K} i)^{1/2}).$$

The proof has been obtained by refining the analysis of the maximum number of so-called allowable k-sets of an allowable sequence [EW85]. Using the identities showed in [AAH+98] this result can be immediately improved to

$$\sum_{i \in K} e_i(S) \leq O(n\,(\sum_{i \in K} i)^{1/3}).$$

For $\mathbb{R}^2$, the exact maximum numbers of $(\leq k)$-sets and $(\leq k)$-facets are known. The number of $(\leq k)$-sets of a convex set is nk. Thus, to show that $E_k(2, n) = nk$ it remains to prove that $E_k(2, n) \leq nk$. The first step towards this goal has been made by GOODMAN and POLLACK [GP93]. They show that $E_k(2, n) \leq 2nk - 2k^2 - k$ for $k < n/2$. The proof uses the technique of allowable sequences. This result could be improved to optimal $E_k(2, n) = nk$ for $k < n/2$ by ALON and GYOERI [AG86]. Again, allowable sequences have been used. At the same time an easy, geometric proof for $E_k(2, n) \leq nk$ for all $k \in \{1, \ldots, n-1\}$ was given by PECK [Pec85].

Another lower bound is obtained in [EHSS89] by EDELSBRUNNER, HASAN, SEIDEL and SHEN. It is $E_k(2, n) \geq E_k(S) \geq 3\binom{k+1}{2}$ for any point set S and $k < n/3$. The second inequality is tight for $k \leq n/3$. The proof technique are also allowable sequences.

6.2 Results for $\mathbb{R}^3$ and higher dimensions

6.2.1 k-sets and j-facets

EDELSBRUNNER gives in Section 3.5 of [Ede87] a lower bound $e_{1/2}(d,n) = \Omega(n^{d-1}\log n)$ which is an elaboration of an idea of SEIDEL. In more detail, EDELSBRUNNER shows that if $\Omega(g(n))$ is a lower bound for the maximum number of halving hyperplanes in $\mathbb{R}^d$, then $\Omega(n\,g(n))$ is the lower bound for the maximum number of halving hyperplanes in $\mathbb{R}^{d+1}$. Using this result and his planar bound $e_{1/2}(2,n) = ne^{\Omega(\sqrt{\log n})}$, TÓTH gives a lower bound of $n^{d-1}e^{\Omega(\sqrt{\log n})}$ on the maximum number of having hyperplanes in $\mathbb{R}^d$ [Tót99].

COLE, SHARIR and YAP [CSY87] obtain the very first non-trivial upper bound $O(n^2k)$ for $e_k(3,n)$. CHAZELLE and PREPATA show in [CP86] that $e_k(3,n) = O(nk^5)$. More exactly, they show that the number of $(\leq k)$-sets in $\mathbb{R}^3$ is bounded by $O(nk^5)$. From this result they follow that for any k, $1 \leq k \leq n$ and any constant $\epsilon > 0$ there exists an integer $i \in [(1-\epsilon)k, k]$ such that the number of i-sets is $O(nk^4)$.

The trivial upper bound $O(n^3)$ on the maximum number $g_{1/2}(3,n)$ of halving triangles could be improved to $O(n^{3-1/343})$ by BÁRÁNY, FÜREDI and LOVÁSZ [BFL90]. To this end, the authors proved the planar version of the colored Tverberg's theorem [ŽV92] and a so-called covering theorem. A specialized version of it says that for every set S of n points in the plane there exists a set A with $|A| < n^{1-1/343}$ which covers all but at most $O(n^{3-1/343})$ triangles with vertices in S. From this result, the upper bound is easily obtained by the Lovász crossing lemma for $\mathbb{R}^3$ [AW97]. This result could be improved to $g_{1/2}(3,n) = O(n^{8/3}\log^{5/3} n)$ by ARONOV, CHAZELLE, EDELSBRUNNER, GUIBAS, SHARIR and WENGER [ACE+91]. Essentially, the authors showed that for any set S on n points of $\mathbb{R}^2$ and $n^{3-\alpha}$ triangles with vertices in S there exists a point of $\mathbb{R}^2$ contained in at least $n^{3-3\,\alpha}/(2^9\log^5 n)$ of the triangles. Also here an application of the Lovász crossing lemma yields the desired result. EPPSTEIN could further improve the bound on $g_{1/2}(3,n)$ to $O(n^{8/3}\log^{2/3} n)$ [Epp93]. He uses similar approach as [ACE+91], but obtains a better lower bound on the number of triangles in the plane sharing a common point. More precisely, this lower bound is $\Omega(m/n + m^3/(n^6\log^2 n))$, where m is the total number of planar triangles with vertices in a set of n points.

DEY and EDELSBRUNNER showed in [DE94] the currently best upper bound $O(n^{8/3})$ for the maximum number $g_{1/2}(3,n)$ of halving triangles. The main result says that for every set S of n points in $\mathbb{R}^3$ and every set of t triangles with vertices in S there are $\Omega(t^4/n^6)$ crossing triangle pairs. In Section 5.1 we gave a probabilistic proof of this theorem (with improved constants). From this theorem it can be easily followed that there is a line in $\mathbb{R}^3$ which crosses $\Omega(t^3/n^6)$ of such triangles. A straightforward application of the Lovász crossing lemma for $\mathbb{R}^3$ gives the upper bound.

AGARWAL, ARONOV, CHAN and SHARIR [AACS98] show the currently best upper bound $O(n(j+1)^{5/3})$ on the number of j-facets in $\mathbb{R}^3$. They use a new technique which is a combination of the analysis given by CLARKSON and SHOR [CS89] and a probabilistic technique. Also an alternative proof using MATOUŠEK's Shallow Cutting Lemma [Mat92] is given.

ŽIVALJEVIĆ and VREĆICA showed in [ŽV92] the multicolored Tverberg's theorem for all dimensions. With this result they provide the missing link in the proof of BÁRÁNY, FÜREDI and LOVÁSZ [BFL90] of the upper bound $g_{1/2}(d,n) = O(n^{d-\epsilon})$, for some $\epsilon > 0$, on the number of halving facets in $\mathbb{R}^d$. It is currently the best result for dimensions greater 3. The proof of the multicolored Tverberg's theorem relies heavily on algebraic topology. The same upper bound is also proved by ALON, BÁRÁNY, FÜREDI and KLEITMAN in [ABFK92], where the multicolored Tverberg's theorem is assumed.

6.2.2 $(\leq k)$-sets and $(\leq j)$-facets

The very first bound $E_k(3,n) = O(nk^5)$ on the number of $(\leq k)$-sets has been obtained by CHAZELLE and PREPATA [CP86]. CLARKSON and SHOR [CS89] showed an upper bound $O(n^{\lfloor d/2 \rfloor}(j+1)^{\lceil d/2 \rceil})$ on the maximum number $G_j(d,n)$ of $(\leq j)$-facets in $\mathbb{R}^d$, as $j/n \to 0$. By Corollary 3.4.2 it follows that $O(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil})$ is also an upper bound for the maximum number $E_k(d,n)$ of $(\leq k)$-sets in $\mathbb{R}^d$. The authors point out that both bounds are asymptotically tight, as especially cyclic polytopes realize the bounds. See Section 5.2 for another consequence of the results in [CS89].

For $\mathbb{R}^3$ also *exact* maximum numbers of $(\leq k)$-sets and $(\leq j)$-facets are known (for a certain range of j or k). For $\mathbb{R}^3$ and $j \leq n/4 - 2$, HAR-PELED, SEIDEL and WELZL [AAH+98] showed that the maximum number of $(\leq j)$-sets and the maximum number of $(\leq j)$-facets is obtained when S is a set of vertices of a convex polytope. This result is improved by WELZL [Wel99] by showing that the vertex set of a convex polytope provides the maximum number of $(\leq j)$-facets for $2j \leq n-4$. See also [LIN94] for exact results on a related problem (in a dual setting) for $\mathbb{R}^3$ and $\mathbb{R}^4$.

6.3 Results for special cases and variations

A set S of n points in $\mathbb{R}^d$ is *δ-dense* if the ratio of the maximum and minimum distance is at most $\delta\, n^{1/d}$, for a $\delta > 0$. The set S is called simply *dense* if δ is a constant. For this special choice of S EDELSBRUNNER, VALTR and WELZL [EVW97] gave an upper bound $O(n^{7/6})$ for the number of halving lines in $\mathbb{R}^2$ (the bound given in [EVW97] is $O(n^{5/4}/\log^* n)$, but using the result of DEY [Dey98] the above bound follows immediately). This result is better than the bound for the general case; it uses the notion of k-set polytopes, see Section 3.2. Furthermore, also the construction from [EW85] for the lower bound $e_{1/2}(2, n) = \Omega(n \log n)$ could be adapted in [EVW97] so it holds for dense point sets, too. For $\mathbb{R}^3$ the authors obtain $O(n^{7/3})$ and for $d > 3$ they state $O(n^{d-2/d})$ as upper bounds on the number of halving facets of dense point sets.

So-called allowable sequences [Ede87, AW97] allow the extraction of the combinatorial properties of planar sets of points and also provide upper bounds on the number of k-sets. Assume that every allowable sequence can occur with the same probability. What is then the average (i.e. expected) number of allowable k-sets of allowable sequences? This question is raised and solved by EDELMAN [Ede92]. By means of combinatorial analysis he obtains for this quantity the formula

$$\frac{(2k-1)(2(n-k)-1)}{2^{2n-5}} \binom{2k-2}{k-1} \binom{2(n-k-1)}{n-k-1}.$$

As most allowable sequences are not realizable, the corresponding value for geometric k-sets is probably different. By an application of Stirling's formula Edelman obtains $(4/\pi)\, n + O(1)$ as the expected number of allowable $n/2$-sets for even n.

In [BS94] BÁRÁNY and STEIGER give a host of results concerning the expected number $g_j^*(d, n, P)$ of j-facets when S is a random sample of n points from a probability distribution P on $\mathbb{R}^d$. The authors show that there are probability distributions for which $g_j^*(2, n, P)$ are large. They construct a distribution P for which $g_j^*(2, n, P) = \Omega(n \log n)$. For a spherically symmetric probability distribution P they obtain the relation $g_j^*(2, n, P) = O(n)$. For P being an uniform probability distribution on a convex body $K \subseteq \mathbb{R}^2$ two cases can be distinguished. First, if j is a constant fraction of n, then $g_j^*(2, n, P) = \Theta(n)$. Otherwise, if j is a constant, then $g_j^*(2, n)$ is asymptotically the expected number of vertices of the convex hull of S. For higher dimensions they obtain the bound $g_j^*(d, n, P) = O(n^{d-1})$ for a spherically symmetric probability distribution P.

AGARWAL, ARONOW, CHAN and SHARIR [AACS98] extend the notion of a k-level to arrangements of n line segments in $\mathbb{R}^2$ and n triangles in $\mathbb{R}^3$ in the following way. For a collection on n triangles in $\mathbb{R}^3$ and $k \in \{0, \ldots, n-1\}$, the *k-level* is the set of all points which lie on a triangle and have exactly k other triangles below them (i.e., the relatively open vertical downward-directed ray emerging from such a point intersects exactly k triangles). For the segments in $\mathbb{R}^2$, the definition is analogous. Under the assumption that segments or triangles are in general position, the complexity of a k-level is the number of *inner* vertices, i.e. vertices which are incident to three distinct triangles (or two distinct segments). The authors show that the complexity of a k- level of an arrangement of line segments in $\mathbb{R}^2$ is $O(n\sqrt{k}\alpha(n/k))$, and that the complexity of a single level in an arrangement of n triangles in $\mathbb{R}^3$ is $O(n^2k^{5/6}\alpha(n/k))$. Here $\alpha(t)$ denotes the inverse Ackermann function.

Further extensions with corresponding results are described in [AW97]. They comprise $(\leq k)$-facets from arrangements of curves and surfaces and k-sets and $(\leq k)$-facets of bicolored sets of points.

Chapter 7

Conclusion

In this thesis we have studied k-sets, j-facets and the generalizations of these objects, the (i, j)-partitions. The contributions comprise results in combinatorial geometry presented in Chapter 3 as well as results in computational geometry (Chapter 4). This justifies our remark in Chapter 1 that k-sets and related objects are a perfect example of the symbiosis of these two research areas.

7.1 Combinatorial geometry

By generalizing both the k-sets and the j-facets we have obtained objects which turn out to be well suitable for studies in combinatorial geometry: the (i, j)-partitions. It is possible that careful studies of these objects will bring an exact answer to the k-set problem; it would be the case if the current approaches for solving this problem aimed at objects which cannot be considered separately (i.e. the k-sets). Summarizing, the concept of (i, j)-partitions seems to be the right one to refine our understanding of the phenomena related to the k-sets.

Speaking about more tangible results, we have successfully refined the technique of continuous motions of points to the degree that it is possible to obtain information about the combinatorial structure of the matrix D of the numbers of (i, j)-partitions. The most important application of this technique are the generalized Dehn-Sommerville relations "modulo a constant" shown in Theorem 3.9.1. They demonstrate that the combinatorics behind the technique of continuous motions of points is closely related to the fundamental linear relations among the entries of the matrix D. As the generalized Dehn-Sommerville relations come from arrangements of hyperplanes, there is some hope that the technique of continuous motions can be also applied in the world of polytopes and hyperplane arrangements, for example to prove a generalized Upper Bound Theorem for hyperplane arrangements [Mul93]. Another example of the impact of this technique are the proofs of the relations between the e-vector and the g-vector in Section 3.10.

Another notion which turned out to be very fruitful are k-set polytopes studied in Section 3.2. Since the combinatorics of the convex polytopes is well studied, one can hope to gain even more insights into the combinatorics of the (i, j)-partitions via the bijection stated in Theorem 3.2.1. One example of such an application of k-set polytopes are the relations among the numbers of (i, j)-partitions stated in Theorem 3.2.3.

Of course there remain open problems. The most promising one is to show that the constant in the generalized Dehn-Sommerville relations "modulo a constant" is independent of n, d, i and j and is 0 (see Theorem 3.9.1). This would be equivalent to showing the generalized Dehn-Sommerville relations for the whole possible index range (Conjecture 3.9.6).

In Section 3.3 we have derived formulas for the number of (i, j)-partitions of cyclic polytopes. Unfortunately, these formulas are quite complicated. It would be very useful to find a simpler representation for them. This might also help to solve the problem described above (see discussion before Conjecture 3.9.6).

Another promising direction of research is to find a characterization of the redundancy of the matrix D, i.e. to answer the question: which rows of D are determined by other rows and how to compute them? (Conjecture 3.8.4). This would extend the results in Sections 3.10 and 3.11.

Also k-set polytopes deserve further studies. One interesting and tangible aspect would be to determine the change of the f-vector (the vector with the numbers of i-faces) of a k-set polytope due to a mutation. Furthermore, we know that faces of highest dimension of such a polytope are not simplicial (see Theorem 3.2.8). It would be interesting to learn more about the numbers of these faces depending on the numbers of their vertices.

7.2 Computational geometry

The notion of a k-set polytope turned out to be prolific also in computational geometry. In the latter field they give rise to two new output-sensitive algorithms for enumeration of k-sets of a point set in general position (based on the technique of reverse search). These algorithms are a serious and practical alternative to the previously known enumeration algorithms, since they are easy to implement, need only a few memory and allow parallelism.

We have also studied the notion of polytopal neighbors of a k-set and of combinatorial neighbors of a k-set, obtaining some interesting results. Another noteworthy notion are the affine k-sets; their study show that the 1-skeleton of a k-set polytope restricted to the vertices corresponding to the affine k-sets might be disconnected.

The extension of the work in this direction could comprise obtaining an algorithm for point sets which are not in general position. Also not much is known about the average and maximum number of the horizon ridges of a k-cell; bounds on these values could have impact on the bounds on the running time of our first enumeration algorithm. Finally, implementing our algorithms for experimental purposes is worth a deep thought.

Also interesting might be algorithms which enumerate (i, j)-partitions of a point set for an arbitrary $i \in \{0, \ldots, d\}$. While for k-sets k-set polytopes help to set up output-sensitive enumeration algorithms, for $i > 0$ the situation becomes complicated, since an i-face of a k-set polytope corresponds to an $(i+1, j)$-partition for $j \in \{k-i+1, \ldots, k-1\}$ (Theorem 3.2.1). Thus, enumeration of all i-faces of a k-set polytope does not yield an output-sensitive algorithm for enumeration of $(i-1, j)$-partitions.

Finally, enumerating special classes of k-sets deserve a closer look. An interesting candidate are the corner cuts studied in [OS99]. The applications of corner cuts in $\mathbb{R}^2$ and $\mathbb{R}^3$ appear in combinatorial number theory and in computer vision.

List of symbols

Symbol	Page	Meaning
S	7	set of n points in $\mathbb{R}^d$
$\mathrm{aff}(P)$	7	affine hull of P
$\mathcal{A}(H)$	7	arrangement of hyperplanes in a set H
$\mathbf{o}$	7	origin of $\mathbb{R}^d$
M_d	7	moment curve in $\mathbb{R}^d$
$P_{n,d}$	7	cyclic polytope with n vertices in $\mathbb{R}^d$
$\langle \mathbf{v}, \mathbf{w} \rangle$	7	scalar product of vectors $\mathbf{v}$, $\mathbf{w}$
$\mathrm{rank}(\mathbf{A})$	7	rank of the matrix $\mathbf{A}$
$\mathrm{lp}(m, n)$	7	time for solving a linear program
$[i = j]$	7	1 if $i = j$, 0 otherwise
$\mathcal{D}$	8	paraboloidal (duality) transformation
$\mathcal{T}$	9	polar (duality) transformation
$e_k(S)$	11	number of k-sets of S
$\mathbf{e}(S)$	11	e-vector
$g_j(S)$	13	number of j-facets of S
$\mathbf{g}(S)$	13	g-vector
$\mathbf{b}$	14	base point of a hyperplane arrangement
$V_j(H)$	14	set of vertices of level j in the arrangement $\mathcal{A}(H)$
ℓ_k	16	k-level
$\lvert\ell_k\rvert$	18	complexity of the k-level
$D_{i,j}(S)$	21	number of (i, j)-partitions of S
D	21	matrix of the numbers of (i, j)-partitions
$X_k(S)$	22	set $\left\{\sum_{p\in T} p \mid T \in \binom{S}{k}\right\}$
$\mathcal{Q}_k(S)$	22	k-set polytope of S
$f_{l,k}(S)$	22	number of l-faces of $Q_k(S)$
$v(T)$	29	sum $\sum_{p\in T} p$
$T(v)$	29	set $T \in \binom{S}{k}$ such that $\sum_{p\in T} p = v$
$h_{i,j}$	47	number of j-facets with subindex i
$(\mathrm{U})_i$	51	i-th entry of the covector U
$T_{d,i,j}(a, b)$	55	expression $\binom{a}{d+1-i-j+b}\binom{d+1-a}{j-b}$
S_I	58	initial point set of a continuous transformation
S_F	58	final point set of a continuous transformation
$D(S, x, y)$	58	polynomial $\sum_{j=0}^{n}\sum_{i=0}^{d} D_{i,j}(S)x^i y^j$
$c_{m,l}$	59	# (m, l)-mutations $-$ # $(d + 1 - m, l)$-mutations

Symbol	Page	Meaning
$\overline{h}_{i,j}$	62	linear combination of $D_{i',j'}$'s
f	83	local search function
Adj	83	adjacency oracle
δ	83	max. degree of a node in the graph G
U	85	$\mathbb{R}^d$ embedded in $\mathbb{R}^{d+1}$
$\mathcal{A}_{\text{sp}}$	85	spherical arrangement
$h(q)$	85	central hyperplane in $\mathbb{R}^{d+1}$ with n.v. $(q - \mathbf{o})$
$h_{\text{sp}}(q)$	85	intersection of $h(q)$ with the sphere S^d
$h_{\text{aff}}(q)$	85	intersection of $h(q)$ with U
$c(T)$	85	k-cell corresponding to a k-set T
$R(c)$	86	set of horizon ridges of a k-cell c
$\mathcal{A}_{\text{aff}}$	94	arrangement of hyperplanes $h_{\text{aff}}(p)$, $p \in S$
$x(S,T)$	98	number of crossing triangle pairs of triangles in T
$y(S,T)$	98	max. number of triangles in T crossed by a single edge
$u_r(S)$	102	expected # of facets of a random sample of r points in S
$\mathcal{C}_k$	105	set of pseudolines of order k

Bibliography

[AACS98] Pankaj K. Agarwal, Boris Aronov, Timothy M. Chan, and Micha Sharir. On levels in arrangements of lines, segments, planes and triangles. *Discrete Comput. Geom.*, 19:315–331, 1998.

[AAH+98] Artur Andrzejak, Boris Aronov, Sariel Har-Peled, Raimund Seidel, and Emo Welzl. Results on k-sets and j-facets via continuous motions. In *Proc. 14th Annu. ACM Sympos. Comput. Geom. (SoCG'98)*, pages 192–199, 1998.

[ABFK92] Noga Alon, Imre Bárány, Zoltán Füredi, and Daniel J. Kleitman. Point selections and weak ε-nets for convex hulls. *Combin., Probab. Comput.*, 1(3):189–200, 1992.

[ACE+91] Boris Aronov, Bernard Chazelle, Herbert Edelsbrunner, Leonidas J. Guibas, Micha Sharir, and Rephael Wenger. Points and triangles in the plane and halving planes in space. *Discrete Comput. Geom.*, 6(5):435–442, 1991.

[AF96] David Avis and Komei Fukuda. Reverse search for enumeration. *Disc. Applied Math.*, 65:21–46, 1996.

[AF99] Artur Andrzejak and Komei Fukuda. Optimization over k-set polytopes and efficient k-set enumeration. In *Proceedings of 6th International Workshop on Algorithms and Data Structures (WADS'99)*, LNCS 1663, pages 1–12. Springer-Verlag, 1999.

[AG86] Noga Alon and Ervin Györi. The number of small semispaces of a finite set of points in the plane. *J. Comb. Theory, Ser. A*, 41:154–157, 1986.

[AK95] Edoardo Amaldi and Viggo Kann. The complexity and approximability of finding maximum feasible subsystems of linear relations. *Theoretical Computer Science*, 147(1–2):181–210, 1995.

[AM95] Pankaj K. Agarwal and Jiří Matoušek. Dynamic half-space range reporting and its applications. *Algorithmica*, 13(4):325–345, April 1995.

[AMdS94] Pankaj K. Agarwal, Jiří Matoušek, Mark de Berg, and Otfried Schwarzkopf. Constructing levels in arrangements and higher order voronoi diagrams. In *Proc. 10th Annu. ACM Sympos. Comput. Geom. (SoCG'94)*, pages 67–75, June 1994.

[Aur91] Franz Aurenhammer. Voronoi diagrams - a survey of a fundamental geometric data structure. *ACM Computing Surveys*, 23(3), 1991.

[AW97] Artur Andrzejak and Emo Welzl. k-sets and j-facets - A tour of discrete geometry. In preparation, 1997.

[BFL90] Imre Bárány, Zoltán Füredi, and László Lovász. On the number of halving planes. *Combinatorica*, 10(2):175–183, 1990.

[BLVS+93] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, White Neil, and Günter M. Ziegler. *Oriented Matroids*. Cambridge U. Press, Encyclopedia of Mathematics, 1993.

[BMFN97] Adrian Brüngger, Ambros Marzetta, Komei Fukuda, and Jürg Nievergelt. The parallel search bench zram and its applications. *Annals of Operations Research, to appear*, July 1997.

[BS94] Imre Bárány and William Steiger. On the expected number of k-sets. *Discrete Comput. Geom.*, 11(3):243–263, 1994.

[Cha98] Timothy M. Chan. Random sampling, halfspace range reporting, and construction of $(\leq k)$-levels in three dimensions. In *IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 586–595, 1998.

[CP86] Bernard Chazelle and Franco P. Preparata. Halfspace range search: an algorithmic application of k-sets. *Discrete Comput. Geom.*, 1:83–93, 1986.

[CS89] Kenneth L. Clarkson and Peter W. Shor. Applications of random sampling in computational geometry II. *Discrete Comput. Geom.*, 4(5):387–421, 1989.

[CSY87] Richard Cole, Micha Sharir, and Chee K. Yap. On k-hulls and related problems. *SIAM J. Comput.*, 16(1):61–77, 1987.

[DE94] Tamal K. Dey and Herbert Edelsbrunner. Counting triangle crossings and halving planes. *Discrete Comput. Geom.*, 12(3):281–289, 1994.

[Dey98] Tamal K. Dey. Improved bounds for planar k-sets and related problems. *Discrete Comput. Geom.*, 19:373–382, 1998.

[Ede87] Herbert Edelsbrunner. *Algorithms in Combinatorial Geometry*. Springer-Verlag, 1987.

[Ede92] Paul H. Edelman. On the average number of k-sets. *Discrete Comput. Geom.*, 8(2):209–213, 1992.

[EHSS89] Herbert Edelsbrunner, Nany Hasan, Raimund Seidel, and Xiaojun Shen. Circles through two points that always enclose many points. *Geom. Dedicata*, 32(1):1–12, 1989.

[ELSS73] Paul Erdős, László Lovász, A. Simmons, and E. G. Straus. Dissection graphs of planar point sets. In *A Survey of Combinatorial Theory (Sympos. Colorado State Univ.)*, pages 139–149. North-Holland, 1973.

[Epp93] David Eppstein. Improved bounds for intersecting triangles and halving planes. *J. Comb. Theory Series A*, 62:176–182, 1993.

[ERv93] Hazel Everett, Jean-Marc Robert, and Marc van Kreveld. An optimal algorithm for the $(\leq k)$-levels, with applications to separation and transversal problems. In *Proc. 9th Annu. ACM Sympos. Comput. Geom. (SoCG'93)*, pages 38–46, 1993.

[EVW97] Herbert Edelsbrunner, Pavel Valtr, and Emo Welzl. Cutting dense point sets in half. *Discrete Comput. Geom.*, 17:243–255, 1997.

[EW85] Herbert Edelsbrunner and Emo Welzl. On the number of line separations of a finite set in the plane. *J. Comb. Theory, Ser. A*, 38:15–29, 1985.

[EW86] Herbert Edelsbrunner and Emo Welzl. Constructing belts in two-dimensional arrangements with applications. *SIAM J. Comput.*, 15(1):271–284, 1986.

[GH97] Tor Gullikson and Arne Hole. On the number of linearly separable subsets of finite sets in $\mathbb{R}^d$. *Discrete Comput. Geom.*, 18:463–472, 1997.

[GJ97] Jacob E. Goodman and O'Rourke Joseph, editors. *Handbook of Discrete and Computational Geometry*. CRC Press, Boca Raton, New York, 1997.

[GKP89] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete Mathematics*. Addison-Wesley, 1989.

[GP93] Jacob E. Goodman and Richard Pollack. Allowable sequences and order types in discrete and computational geometry. In János Pach, editor, *New trends in discrete and computational geometry*, pages 103–134. Springer-Verlag, 1993.

[KM93] Nikolai M. Korneenko and Horst Martini. Hyperplane Approximations and Related Topics. In János Pach, editor, *New trends in discrete and computational geometry*, pages 135–161. Springer-Verlag, 1993.

[Knu92] Donald E. Knuth. *Axioms and Hulls*. LNCS 606. Springer-Verlag, 1992.

[Kor95] Ulrich Kortenkamp. Pseudosphärenarrangements zu orientierten matroiden. Master's thesis, Westfällische Wilhelms-Universität Münster, 1995.

[KPP82] Maria M. Klawe, Mike Paterson, and Nicholas Pippenger. Inversions with $n^{1+\Omega(1/\sqrt{\log n})}$ transpositions at the median. Unpublished manuscript, 1982.

[KW99] Gyula Károlyi and Emo Welzl. Crossing-free segments and triangles in point configurations. Unpublished manuscript, 1999.

[Lin94] Johann Linhart. The upper bound conjecture for arrangements of half-spaces. *Beitr. Algebra Geom.*, 35(1):29–35, 1994.

[Lov71] László Lovász. On the number of halving lines. *Ann. Univ. Sci. Budapest, Eötvös, Sect. Math.*, 14:107–108, 1971.

[Mat92] Jiří Matoušek. Reporting points in halfspaces. *Computational Geometry: Theory and Applications*, 2:169–186, 1992.

[Mul91] Ketan Mulmuley. On Levels in Arrangements and Voronoi Diagrams. *Discrete Comput. Geom.*, 6:307–338, 1991.

[Mul93] Ketan Mulmuley. Dehn-Sommerville relations, upper bound theorem, and levels in arrangements. In *Proc. 9th Annu. ACM Sympos. Comput. Geom. (SoCG'93)*, pages 240–246. ACM Press, 1993.

[Mul94] Ketan Mulmuley. *Computational Geometry: An Introduction Through Randomized Algorithms*. Prentice Hall, 1994.

[OS99] Shmuel Onn and Bernd Sturmfels. Cutting corners. *Advances in Applied Mathematics*, 23:29–48, 1999.

[OSS95] Thomas Ottmann, Sven Schuierer, and Subbiah Soundaralakshmi. Enumerating Extreme Points in Higher Dimensions. In *Proc. 12th Annu. Sympos. Theoret. Aspects of Computer Science (STACS'95)*, LNCS 900, pages 562–570. Springer-Verlag, 1995.

[Pec85] G. W. Peck. On 'k-sets' in the plane. *Discrete Math.*, 56:73–74, 1985.

[PSS92] János Pach, William Steiger, and Endre Szemerédi. An upper bound on the number of planar k-sets. *Discrete Comput. Geom.*, 7(2):109–123, 1992.

[Ric92] Jürgen Richter-Gebert. *New construction methods for oriented matroids*. PhD thesis, KTH Stockholm, 1992.

[Sha79] Robert W. Shannon. Simplicial cells in arrangements of hyperplanes. *Geometriae Dedicata*, 8:179–187, 1979.

[Sha93] Micha Sharir. k-sets and random hulls. *Combinatorica*, 13(4):483–495, 1993.

[Tót99] Géza Tóth. Point sets with many k-sets. Submitted to *Discrete Comput. Geom.*, 1999.

[Tve66] Helge Tverberg. A generalization of Radon's Theorem. *J. London. Math. Soc.*, 41:123–128, 1966.

[Wel86] Emo Welzl. More on k-sets of finite sets in the plane. *Discrete Comput. Geom.*, 1:95–100, 1986.

[Wel99] Emo Welzl. Entering and leaving j-facets. Unpublished manuscript, 1999.

[YKII88] Peter Yamamoto, Kazuhiko Kato, Keiko Imai, and Hiroshi Imai. Algorithms for vertical and orthogonal L_1 linear approximation of points. In *Proc. 4th Annu. ACM Sympos. Comput. Geom. (SoCG'88)*, pages 352–361, 1988.

[Zie94] Günter M. Ziegler. *Lectures on Polytopes*. Springer-Verlag, 1994.

[ŽV92] Rade T. Živaljević and Siniša T. Vrećica. The colored Tverberg's problem and complexes of injective functions. *J. Comb. Theory, Ser. A*, 61(2):309–318, 1992.

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