Labelled Modal Logics: Quantifiers

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Abstract. In previous work we gave an approach, based on labelled natural deduction, for formalizing proof systems for a large class of propositional modal logics that includes K, D, T, B, S4, S4.2, KD45, and S5. Here we extend this approach to quantified modal logics, providing formalizations for logics with varying, increasing, decreasing, or constant domains. The result is modular with respect to both properties of the accessibility relation in the Kripke frame and the way domains of individuals change between worlds. Our approach has a modular metatheory too; soundness, completeness and normalization are proved uniformly for every logic in our class. Finally, our work leads to a simple implementation of a modal logic theorem prover in a standard logical framework.

1 Introduction

Motivation

Modal logic is an active area of research in computer science and artificial intelligence: a large number of modal logics have been studied and new ones are frequently proposed. Each new logic demands, at a minimum, a semantics and a proof system, and a set of metatheorems relating them together. This development is often non-trivial and has not been systematized; i.e. the characterization of a new system may demand novel extensions of old techniques or even the invention of completely new ones.

The problem is particularly acute in the case of quantified modal logics (QMLs). Here difficulties not present in the propositional case [8,11,14] arise. Quantifiers introduce additional complexity to the range of possible semantics that might be appropriate for the logic of an intended application: we must not only specify

(i) properties of the accessibility relation in the Kripke frame, as in the propositional case, but also
(ii) how the domains of individuals change between worlds; for example, do the domains vary arbitrarily (varying domains), or do the same objects exist in every world (constant domains), or are objects possibly created (increasing domains) or destroyed (decreasing domains) when moving to accessible worlds?

These two choices can be made independently, which results in a two-dimensional space of possible QMLs.[1] In the past, this space has been explored piecemeal fashion and there has been a lack of uniformity in the presentation of proof systems and in the way their associated metatheoretic results, in particular completeness with respect to a Kripke semantics, are proved. Consider the following aspects.

First, different proof systems are employed. QMLs are typically given as Hilbert presentations, but the classical quantifier rules then automatically require the domains of a semantics to be increasing [8, p. 426], and this restricts the class of logics that are formalizable in a modular way. This problem can be solved by modifying either the proof system (e.g. by adopting the rules of free logic), or the semantics (e.g. by introducing truth value gaps), see [11,14]. These solutions, however, are imperfect in that none provides a general and uniform solution. For example, the rules of free logic don't provide modular completeness proofs since different proof strategies must be adopted for different conditions on the domains.

Second, incompleteness with respect to Kripke semantics is common. Simply adding quantifier rules to a complete propositional modal logic may not result in a QML complete with respect to the equivalent extension of the semantics. Moreover, minor changes to a complete QML, e.g. changing the conditions on the domains, can produce incompleteness. For instance, there are QMLs with the *Barcan Formula* (BF, $\forall x.\Box A \rightarrow \Box\forall x.A$) that are incomplete, while those without it are complete, and vice versa; e.g. $\text{QS4.2} + \text{BF}$ is incomplete although QS4.2 is complete [14].

Third, metatheoretic results are not proved in a uniform way. Often, even for related logics, completeness proofs or counter-examples must be devised ad hoc, using different mathematical techniques. For example, the standard canonical model technique fails for QS4.2, but we can prove completeness with respect to Kripke semantics by the subordination method [6, p. 175].

Quantified modal logics also raise special challenges when we begin actually to prove theorems with them. Many propositional modal logics are decidable, so proof search can be automated [7,8,29]. In the quantified case, however, even when we restrict ourselves to terms built from constants and variables, as is often done [14], and as we do here, the resulting modal logics are undecidable. Thus if we want to use them, it is desirable to have proof systems that support the interactive construction of proofs. Unfortunately, yet another problem with Hilbert systems is that they are (notoriously) unusable in practice. We know that natural deduction systems (or sequent systems) supporting proof under assumption are much more effective, but it is difficult to find such systems for modal logics, since the deduction theorem (at least as it is usually expressed) fails.[2] Moreover, even

[1] Other dimensions are possible, e.g. non-rigid designators [8,11]; we consider here only the rigid case.

[2] Natural deduction, sequent, and tableau systems for quantified modal logics can be found in, e.g., [7,8,29]; see Section 5 for a comparison with our work. Note also that more complex versions of the deduction theorem do hold, e.g. [7,19].

in the case where we wish to employ a semi-decision procedure, we still want that proofs have properties, such as the subformula property, which restrict the search space for proofs. Again, this is not the case for Hilbert presentations.

Context

This paper is a companion to [3]: we extend and generalize the results given there for labelled propositional modal logics to the quantified case and thereby provide solutions to the above problems. Let us first briefly summarize the approach and some of the results that we previously developed. In [3] we formalized natural deduction proof systems for propositional modal logics, based on the view of a logic as a Labelled Deductive System [9]; see [7,24] for similar approaches. We decomposed a modal logic into two interacting parts: a *base logic*, fixed for all modal logics, and a *relational theory*, different for each modal logic. In the base logic, we reason about formulae paired with labels; i.e. instead of $\vdash A$, we prove $\vdash w{:}A$, where $w{:}A$ is a *labelled formula*, w is an element of the set of possible worlds W in the Kripke frame, and $\vdash A$ iff $\forall w \in W(\vdash w{:}A)$. In the relational theory, we formalize the behavior of the accessibility relation R in the Kripke frame. *Relational formulae* $w\ R\ w'$ state that w accesses w'. This allows us, for instance, to formulate the behavior of modal operators like $\Box$ independent of the properties of R that are taken as providing the semantics; i.e. $\vdash w{:}\Box A$ iff $\vdash w'{:}A$ for all $w' \in W$ such that $\vdash w\ R\ w'$. As a consequence, we are able to give natural deduction introduction and elimination rules for $\Box$ that are fixed for all the logics we consider.

The main results that we established can be summarized as follows:

Soundness and completeness: We uniformly showed the soundness and completeness of all propositional modal logics formalizable in our framework with respect to the corresponding Kripke semantics.

Proof Theory: We explored tradeoffs in the formulation of the base logic and the relational theory. We showed, for example, that when the relational theory can be formulated as a set of Horn clauses (as opposed to a set of first or second-order axioms), then proofs are normalizing and there is a strong separation between the base logic and the relational theory, i.e. derivations in the base logic may depend on derivations in the relational theory, but not vice versa.

Implementation: We showed that the resulting proof systems can be implemented in a logical framework based on a minimal metalogic with higher-order quantification, e.g. Isabelle [18] or the Edinburgh LF [12]. We implemented our approach in Isabelle and the result is a simple and natural environment for interactive proof development in which it is possible to structure modal logics hierarchically, extending a logic with new properties to generate a new one, and having theorems inherited by these extensions.

Contribution

In this paper, we give a natural deduction presentation of QMLs that is modular in two dimensions, reflecting the two degrees of freedom discussed above. As before, it is based on a fixed base logic (now QK, for quantified K) where extensions are made by independently instantiating two separate theories: a relational theory (as before), and a *domain theory*, which formalizes the behavior of the domains of quantification. This second theory requires the introduction of *labelled terms*, $w{:}t$, expressing the existence of term t at world w. Thus $\vdash w{:}\forall x.A$ iff $\vdash w{:}A[t/x]$ for all t such that $\vdash w{:}t$. This formulation naturally suggests that we adopt the quantifier rules of free logic [5].[3]

By appropriate instantiation of these two theories, it is possible to present the predicate extensions — with varying, increasing, decreasing, or constant domains — of the propositional modal logics belonging to a large class which includes the Geach hierarchy and hence contains logics like K, D, T, B, S4, S4.2, KD45, and S5.

The metatheory of our QMLs is also modular. The use of explicit labels leads to a modular proof of soundness and completeness for all the logics we consider, which differs from the standard one: we provide a new kind of canonical model construction that accounts for the explicit formalization of labels, of the accessibility relation, and of the properties of the domains of quantification. This means that our presentations are sound and complete with respect to the appropriate Kripke semantics, and thus equivalent to the corresponding Hilbert systems only when these are themselves complete with respect to the same semantics. We also show that the proof theoretic results for labelled propositional modal logics carry over to QMLs. Hence, proof search may be restricted (proofs satisfy the subformula property) and the effectiveness of theorem proving can be improved. Moreover, given normalization for the natural deduction presentations, it is possible to give cut-free sequent systems for the same logics.[4]

Finally, we discuss tradeoffs in formalizations of the base logic and the theories extending it. We show not only that the results for labelled propositional modal logics carry over to QMLs, but also that new tradeoffs must be considered.

We do not discuss implementational aspects in this paper, since the work documented in [3] carries over directly to the quantified case. We have carried out such an implementation in the Isabelle logical framework [18] and the result is a simple and natural environment for interactive proof development. All the

[3] We show later that the previously mentioned problems for Hilbert-style QMLs based on free logic do not apply in our approach. There is also another important respect in which our approach differs from the standard ones based on free logic. In the latter, the existence of a term at a particular world is not an independent 'judgement' like $w{:}t$, but it is expressed by the atomic modal formula $E(t)$, which has to be explicitly considered in the completeness proof [11, p. 279].

[4] Normalizing natural deduction systems and cut-free sequent systems are closely related, e.g. [20,25]. The exact formalization of cut-free sequent calculi for labelled propositional modal logics is given in [4].

proofs of modal theorems given in this paper (e.g., at the end of Section 2) have been machine checked in our implementation.

Organization

The remainder of this paper is organized as follows. In Section 2 we introduce our approach to presenting QMLs by formalizing the base logic and the theories extending it. In Section 3 we show that our presentations of QMLs are sound and complete with respect to their intended semantics. In Section 4 we show that proofs normalize and we investigate tradeoffs in the formalizations of logics. In Section 5 we make comparisons with related work, and in Section 6 we draw conclusions.

2 A Modular Presentation of QMLs

We introduce a labelled natural deduction presentation of quantified modal logics, where we use labels to associate possible worlds with terms and formulae.

Let W be a set of *labels* and R a binary relation over W. If w and w' are labels, then $w\ R\ w'$ is a *relational formula* (*rwff*). If t is a constant c or a variable x, then $w{:}t$ is *labelled term* (*lterm*). If A is a modal formula built from atomic propositions (i.e. predicates applied to terms) and the connectives and quantifiers $\bot$, $\rightarrow$, $\Box$, $\forall$, then $w{:}A$ is a *labelled formula* (*lwff*). Lwffs over other connectives and quantifiers can be defined in the usual manner, e.g. $w{:}\Diamond A =_{def} w{:}(\Box(A \rightarrow \bot)) \rightarrow \bot$ and $w{:}\exists x.A =_{def} w{:}(\forall x.(A \rightarrow \bot)) \rightarrow \bot$.

Henceforth, we assume that the variable w ranges over labels, t ranges over terms, and A, B range over modal formulae. Further, let Γ, Δ and Θ be, respectively, arbitrary sets of lwffs, $\{w_1{:}A_1, \dots, w_n{:}A_n\}$, rwffs, $\{w_1\ R\ w_2, \dots, w_l\ R\ w_m\}$, and lterms, $\{w_1{:}t_1, \dots, w_j{:}t_j\}$. All metavariables may be annotated with subscripts or superscripts.

The rules given in Figure 1 determine QK, the base natural deduction system formalizing a labelled version of quantified K. The rules for $\forall$ are a labelled version of the rules of free logic [5], and, as in free logic, $w{:}\forall x.A \rightarrow \exists x.A$ is provable only under the assumption $w{:}t$, stating that the domain of quantification of w is non-empty (cf. Section 4). Note the symmetry between the rules for $\Box$ and those for $\forall$; this reinforces the role of $\Box$, and of modal logics in general, "as a replacement for the more powerful machinery of quantified classical logic, at least in some cases" [8, p. 377]. The same symmetry holds between the derived rules for $\Diamond$ and $\exists$ given in Figure 2.

Relational Theories

A family of QMLs (the *normal* QMLs) is obtained from the base logic QK by placing conditions on the accessibility relation R in the Kripke frame; e.g. we get the logic QT from QK by adding that R is reflexive, and then QS4 from

$$
\dfrac{\begin{matrix}[w_i{:}A \to \bot]\\ \vdots\\ w_j{:}\bot\end{matrix}}{w_i{:}A}\ \bot E
\qquad
\dfrac{\begin{matrix}[w{:}A]\\ \vdots\\ w{:}B\end{matrix}}{w{:}A \to B}\ \to I
\qquad
\dfrac{w{:}A \to B \quad w{:}A}{w{:}B}\ \to E
$$

$$
\dfrac{\begin{matrix}[w_i\ R\ w_j]\\ \vdots\\ w_j{:}A\end{matrix}}{w_i{:}\Box A}\ \Box I
\qquad
\dfrac{\begin{matrix}[w{:}t]\\ \vdots\\ w{:}A[t/x]\end{matrix}}{w{:}\forall x.A}\ \forall I
$$

$$
\dfrac{w_i{:}\Box A \quad w_i\ R\ w_j}{w_j{:}A}\ \Box E
\qquad
\dfrac{w{:}\forall x.A \quad w{:}t}{w{:}A[t/x]}\ \forall E
$$

In $\Box I$, w_j is different from w_i and does not occur in any assumption on which $w_j{:}A$ depends other than $w_i\ R\ w_j$. In $\forall I$, t does not occur in any assumption on which $w{:}A[t/x]$ depends other than $w{:}t$.

Fig. 1. The rules of QK

$$
\dfrac{w_j{:}A \quad w_i\ R\ w_j}{w_i{:}\Diamond A}\ \Diamond I
\qquad
\dfrac{w{:}A[t/x] \quad w{:}t}{w{:}\exists x.A}\ \exists I
$$

$$
\dfrac{w_i{:}\Diamond A \quad \begin{matrix}[w_j{:}A]\ [w_i\ R\ w_j]\\ \vdots\\ w_k{:}B\end{matrix}}{w_k{:}B}\ \Diamond E
\qquad
\dfrac{w_i{:}\exists x.A \quad \begin{matrix}[w_i{:}A[t/x]]\ [w_i{:}t]\\ \vdots\\ w_j{:}B\end{matrix}}{w_j{:}B}\ \exists E
$$

In $\Diamond E$, w_j is different from w_i and w_k, and does not occur in any assumption on which the upper occurrence of $w_k{:}B$ depends other than $w_j{:}A$ and $w_i\ R\ w_j$. In $\exists E$, t does not occur in any assumption on which the upper occurrence of $w_j{:}B$ depends other than $w_i{:}A[t/x]$ and $w_i{:}t$.

Fig. 2. The derived rules for $\Diamond$ and $\exists$

QT by further adding transitivity, cf. correspondence theory [26,27]. We formalize particular QMLs by extending QK with relational theories, which axiomatize properties of R. However, not all modal axioms can be axiomatized in a first-order setting (e.g. the McKinsey axiom $\Box\Diamond A \to \Diamond\Box A$, or the Löb axiom $\Box(\Box A \to A) \to \Box A$ of provability logic) and hence there is an important decision that we must make: Should our relational theories be axiomatized in higher-order logic (and thus allow the formalization of all normal modal logics), first-order logic, or some subset thereof? We showed in [3] that there are tradeoffs in formalization: different choices require different formalizations of the base modal logic and have different metatheoretic properties. In [3] we settled on those modal logics with accessibility relations axiomatizable in terms of Horn clauses, a choice we repeat here; we discuss the implications of this decision in Section 4.

We choose to admit precisely those properties of R that can be formulated as a collection of *relational rules*, i.e. (Horn) rules of the form

$$\frac{p_1 \; R \; q_1 \quad \cdots \quad p_m \; R \; q_m}{p_0 \; R \; q_0}$$

where $m \geq 0$, and the p_i and q_i are terms built from labels $w_1, \ldots, w_n$ and function symbols — some properties of R, e.g. seriality and convergency, can be expressed as relational rules only after the introduction of Skolem function constants; by the theorem on functional extensions [23, p. 55], the introduction of Skolem constants is a conservative extension, see [3]. A *relational theory* $\mathcal{T}$ is then a theory generated by a set of relational rules.

Each relational rule corresponds to a closed formula of the form

$$\forall w_1 \ldots \forall w_n (p_1 \; R \; q_1 \wedge \ldots \wedge p_m \; R \; q_m \to p_0 \; R \; q_0) \, .$$

In first-order logic, the addition of a Horn formula to a theory is equivalent to adding the corresponding rule. In our implementation of these logics, it is easiest to work with the rules; this allows us to carry out proofs in the relational theory without having to reason about the connectives and quantifiers of first-order logics.

Relational rules suffice to formalize the predicate extensions of the most common modal logics. Let i, j, m, n be natural numbers, and let $\Box^n$ (respectively $\Diamond^n$) stand for a sequence of n consecutive $\Box$s (respectively $\Diamond$s), e.g. $\Diamond^2\Box^3\Diamond^0 A$ is $\Diamond\Diamond\Box\Box\Box A$. Relational rules allow us to capture, among others, all those instances of the *generalized Geach axiom schema*

$$\Diamond^i \Box^m A \to \Box^j \Diamond^n A$$

for which if $m = n = 0$ then $i = j = 0$.[5] This axiom corresponds to the property

$$\forall w_1 \forall w_2 \forall w_3 (w_1 \; R^i \; w_2 \wedge w_1 \; R^j \; w_3 \to \exists w_4 (w_2 \; R^m \; w_4 \wedge w_3 \; R^n \; w_4)) \, ,$$

[5] This restriction is needed if one is considering relational theories without equality, as we do here, since there are instances of (i, j, m, n) that explicitly require equality between labels. The above statements are formally proved in [3].

Seriality, D: $\Box A \to \Diamond A$

$$\frac{}{w_i\ R\ f(w_i)}\ ser$$

Reflexivity, T: $\Box A \to A$

$$\frac{}{w_i\ R\ w_i}\ refl$$

Transitivity, 4: $\Box A \to \Box\Box A$

$$\frac{w_i\ R\ w_j \quad w_j\ R\ w_k}{w_i\ R\ w_k}\ trans$$

Euclideaness, 5: $\Diamond A \to \Box\Diamond A$

$$\frac{w_i\ R\ w_j \quad w_i\ R\ w_k}{w_j\ R\ w_k}\ eucl$$

Convergency, 2: $\Diamond\Box A \to \Box\Diamond A$

$$\frac{w_i\ R\ w_j \quad w_i\ R\ w_k}{w_j\ R\ g(w_i, w_j, w_k)}\ conv1 \qquad \frac{w_i\ R\ w_j \quad w_i\ R\ w_k}{w_k\ R\ g(w_i, w_j, w_k)}\ conv2$$

Where f and g are (Skolem) function constants.

Fig. 3. Some properties of R, characteristic axioms, and relational rules

where $w_1\ R^0\ w_2$ means $w_1 = w_2$, and $w_1\ R^{i+1}\ w_2$ means $\exists w_3(w_1\ R\ w_3 \wedge w_3\ R^i\ w_2)$. For instance, all of the properties given in Figure 3 correspond to instances of (i, j, m, n); e.g. transitivity and convergency are given by $(0, 2, 1, 0)$ and $(1, 1, 1, 1)$. We also present there the corresponding relational rules and characteristic modal axioms.

The QML $L = \mathrm{QK} + \mathcal{T}$ is obtained by extending QK with a given relational theory $\mathcal{T}$; this extension is represented by the horizontal arrows in Figure 4. We adopt the convention of naming the logic $\mathrm{QK}+\mathcal{T}$ as QKAx, where Ax is a string consisting of the standard names of the characteristic axioms corresponding to the relational rules generating $\mathcal{T}$; e.g. QKT4 identifies the logic also known as QS4. Various combinations of relational rules define therefore predicate extensions of propositional modal logics, including QK, QD, QT, QB, QS4, QS4.2, QKD45, and QS5.

Domain Theories

So far, we have made no commitments about the relationship between the domains of quantification in the different worlds; in this case we say that the domains of $\mathrm{QK} + \mathcal{T}$ are *varying*. We can then place constraints on them; e.g. requiring that, when we move from a world to another world accessible from it, objects persist (the domains are *increasing*), are not created (and possibly deleted, i.e. the domains are *decreasing*), or stay the same (the domains are both increasing and decreasing, i.e. *constant*). The conditions for increasing and decreasing domains can be formalized by the (Horn) rules *ID* and *DD* respectively:

$$\frac{w_i\ R\ w_j \quad w_i{:}t}{w_j{:}t}\ ID \qquad \frac{w_i\ R\ w_j \quad w_j{:}t}{w_i{:}t}\ DD$$

$$
\begin{tikzcd}
\mathrm{QK} + \mathcal{D} \arrow[r, "\text{accessibility}", "\text{relation}"'] & \mathrm{QK} + \mathcal{T} + \mathcal{D} \\
\mathrm{QK} \arrow[u, "\text{domains}"] \arrow[r, "\text{accessibility}", "\text{relation}"'] & \mathrm{QK} + \mathcal{T} \arrow[u, "\text{domains}"']
\end{tikzcd}
$$

Fig. 4. A hierarchy of labelled QMLs

Different combinations of these rules define different QMLs: the logic $L = \mathrm{QK} + \mathcal{T} + \mathcal{D}$ is obtained by extending $\mathrm{QK} + \mathcal{T}$ with a given theory of the domains of quantification (or *domain theory*, for short) $\mathcal{D}$, generated by a subset of $\{ID, DD\}$; this extension is represented by the vertical arrows in Figure 4. This yields the two-dimensional uniformity of the proof system motivated in the introduction. (Uniformity of soundness and completeness is discussed in Section 3.)

$$
\begin{tikzcd}
 & \mathrm{QKT4.c} & \\
\mathrm{QKT4.i} \arrow[ur, "DD"] & & \mathrm{QKT4.d} \arrow[ul, "ID"'] \\
 & \mathrm{QKT4} \arrow[ul, "ID"] \arrow[ur, "DD"'] &
\end{tikzcd}
$$

Fig. 5. QKT4.l

We extend the above convention and name the logic $\mathrm{QK} + \mathcal{T} + \mathcal{D}$ as $\mathrm{QK}Ax.l$, where l represents the conditions imposed on the domains. We write $\mathrm{QK}Ax$ when $\mathcal{D}$ is empty, as done above; $\mathrm{QK}Ax$.i (respectively $\mathrm{QK}Ax$.d) when $\mathcal{D}$ is generated by *ID* (respectively *DD*); $\mathrm{QK}Ax$.c when $\mathcal{D}$ is generated by *ID* and *DD*.[6] We can therefore specify one of four related QMLs simply by instantiating $\mathcal{D}$; e.g., as shown in Figure 5, we can specify QKT4 (QS4) with domains that are varying (QKT4), increasing (QKT4.i), decreasing (QKT4.d), or constant (QKT4.c).

[6] We consider constant domains only for worlds connected by the accessibility relation. *'Fully' constant domains*, where all worlds, even unconnected ones, share the same domain, can be formalized by the rule

$$\frac{w_i{:}t}{w_j{:}t}$$

from which both *ID* and *DD* can be derived.

This is *not* the case in Hilbert systems for QMLs, where the domains are committed to being increasing, since the classical rules for $\forall$ automatically enforce the *Converse Barcan Formula* (CBF),

$$\Box\forall x.A \rightarrow \forall x.\Box A\,,$$

which corresponds to the increasing domains condition [8, p. 426]. Constant domains are then obtained by further adding as an axiom the *Barcan Formula* (BF),

$$\forall x.\Box A \rightarrow \Box\forall x.A\,,$$

which corresponds to the decreasing domains condition. Hilbert-style proof systems for QMLs with varying domains can be given by substituting the classical quantifier rules with the rules of free logic, as done by Garson in [11]. However, Garson also shows that his completeness proof fails for some QMLs, e.g. QB; we return to this at the end of Section 3.

Some particular QMLs with varying domains can also be formalized by systems that keep the classical quantifier rules, e.g. by using free variables as disguised universal quantifiers and restricting the necessitation rule to closed sentences [15], or by adopting a semantics with truth value gaps [14]. However, none of these techniques provides uniform proof systems (or semantics) since it is not clear how to generalize them to other QMLs. For a detailed discussion of the limits of these systems see [11].

Definition 1. Let φ be an lwff, an rwff, or an lterm. An *L-derivation of φ from Γ, Δ, Θ* is a tree formed using the rules in L, ending with φ and depending only on Γ, Δ, Θ. We write $\Gamma, \Delta, \Theta \vdash_L \varphi$ when φ can be so derived. φ is a *theorem* of L, $\vdash_L \varphi$, if it is L-derivable when Γ, Δ, Θ are all empty.

Fact 2. Due to the separations enforced between the base logic, the relational theory, and the domain theory, we have that:

(i) $\Gamma, \Delta, \Theta \vdash_L w_i\ R\ w_j$ iff $\Delta \vdash_L w_i\ R\ w_j$.
(ii) $\Gamma, \Delta, \Theta \vdash_L w{:}t$ iff $\Delta, \Theta \vdash_L w{:}t$.

That is, while lwffs are derived from lwffs, rwffs *and* lterms, i.e. $\Gamma, \Delta, \Theta \vdash_L w{:}A$, (i) rwffs are derived from rwffs *alone*, and (ii) lterms are derived from rwffs and lterms *but not* from lwffs.

In comparison, note that in approaches based on *semantic embedding*, also called *semantics-based translations*, e.g. [1,13,17], a first-order modal formula is translated into a formula in first-order predicate logic and shown to be true (or false) in a first-order theory formalizing the semantics of the modalities and quantification domains. However, with these translations all structure is lost as relations, predicates, and terms are flattened into formulae of predicate logic, and derivations of lwffs are mingled with derivations of rwffs and lterms.

As an example of a derivation, we show that CBF is a theorem of (any extension of) QK.i:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[w{:}\Box\forall x.A]^3 \quad [w\,R\,w_1]^1}{w_1{:}\forall x.A}\,\Box E \qquad \dfrac{[w\,R\,w_1]^1 \quad [w{:}t]^2}{w_1{:}t}\,ID}{w_1{:}A[t/x]}\,\forall E}{w{:}\Box A[t/x]}\,\Box I^1}{w{:}\forall x.\Box A}\,\forall I^2}{w{:}\Box\forall x.A \to \forall x.\Box A}\,{\to} I^3$$

Note that we associate discharged assumptions with rule applications.[7] In a similar manner, we can prove BF in QK.d,

$$\vdash_{\mathrm{QK.d}} w{:}\forall x.\Box A \to \Box\forall x.A\,,$$

and other theorems usually considered in standard texts:

$$\vdash_{\mathrm{QKB.i}} w{:}\forall x.\Box A \to \Box\forall x.A \tag{1}$$
$$\vdash_{\mathrm{QK.d}} w{:}\Diamond\exists x.A \to \exists x.\Diamond A \tag{2}$$
$$\vdash_{\mathrm{QK.i}} w{:}\exists x.\Diamond A \to \Diamond\exists x.A \tag{3}$$
$$\vdash_{\mathrm{QK.i}} w{:}\Diamond\forall x.A \to \forall x.\Diamond A \tag{4}$$
$$\vdash_{\mathrm{QK.i}} w{:}\exists x.\Box A \to \Box\exists x.A \tag{5}$$

Some remarks. *ID* and *DD* are interderivable when the rule

$$\frac{w_i\,R\,w_j}{w_j\,R\,w_i}$$

is present, i.e. when the accessibility relation is *symmetric* (symmetry corresponds to the modal axiom B: $A \to \Box\Diamond A$). (1) shows that a QML with a symmetric accessibility relation and with increasing domains (QKB.i) validates BF, and has therefore constant domains; similarly we can show that CBF is a theorem of QKB.d. By (2) and (3), $\Diamond\exists x.A$ and $\exists x.\Diamond A$ are equivalent in QK.c; by analysis of normal form proofs (cf. Section 4), we can show that they are equivalent only in systems with constant domains. Similarly, we can show that, as is the case in Hilbert systems, the converses of (4) and (5) are not provable even when *DD* is added as a rule.

3 Soundness and completeness of Labelled QMLs

Definition 3. A *model* for a QML L is a tuple $\mathfrak{M} = (\mathfrak{W}, \mathfrak{R}, \mathfrak{D}, \mathfrak{q}, \mathfrak{a})$, where $\mathfrak{W}$ is a non-empty set of worlds; $\mathfrak{R} \subseteq \mathfrak{W} \times \mathfrak{W}$; $\mathfrak{D}$ is a set of objects; $\mathfrak{q}$ is a mapping

[7] Note also that the assumption $w{:}t$ is discharged by the application of $\forall I$. CBF is not a theorem of QK, because *ID* is missing and the application of $\forall I$ at world w cannot discharge $w_1{:}t$; a formal proof of this can be given by exploiting the results on proof normalization discussed in Section 4, to show that there is no normal proof (and, a fortiori, no proof at all) of CBF in QK.

assigning to each member w of $\mathfrak{W}$ some subset of $\mathfrak{D}$, the *domain of quantification* of w; $\mathfrak{a}$ is a function interpreting the terms and predicate letters by assigning to them the corresponding kind of intensions with respect to $\mathfrak{W}$ and $\mathfrak{D}$. $\mathfrak{a}(w,t)$ is an element of $\mathfrak{D}$, and for a predicate letter P of arity n, $\mathfrak{a}(w,P)$ is a set of ordered n-tuples, $\langle a_1,\ldots,a_n\rangle$, where each $a_i \in \mathfrak{D}$. We say that $\mathfrak{M}$ has some property of binary relations iff $\mathfrak{R}$ has that property. Moreover, for every $w_i, w_j \in \mathfrak{W}$ such that $(w_i,w_j) \in \mathfrak{R}$, the domains of $\mathfrak{M}$ are: *increasing* iff $\mathfrak{q}(w_i) \subseteq \mathfrak{q}(w_j)$; *decreasing* iff $\mathfrak{q}(w_i) \supseteq \mathfrak{q}(w_j)$; *constant* iff $\mathfrak{q}(w_i) = \mathfrak{q}(w_j)$. Otherwise, the domains are *varying*.

Note that we only consider rigid designators [8,11], where $\mathfrak{a}$ is such that $\mathfrak{a}(w_i,t) = \mathfrak{a}(w_j,t)$ for all $w_i, w_j \in \mathfrak{W}$. Moreover, our models do not contain functions corresponding to possible Skolem functions in the signature; when such constants are present, the appropriate Skolem expansion of the model is required [28, p. 137].

Call the ordered triple (Γ,Δ,Θ) a *proof context* (*pc*). We write $w{:}A \in (\Gamma,\Delta,\Theta)$ when $w{:}A \in \Gamma$; $w\ R\ w' \in (\Gamma,\Delta,\Theta)$ when $w\ R\ w' \in \Delta$; and $w{:}t \in (\Gamma,\Delta,\Theta)$ when $w{:}t \in \Theta$. Moreover, we say that a label w *occurs in* the pc (Γ,Δ,Θ) and, continuing our slight notational abuse, we write $w \in (\Gamma,\Delta,\Theta)$ if there exists an A such that $w{:}A \in \Gamma$, or a w' such that $w\ R\ w' \in \Delta$ or $w\ R\ w' \in \Delta$, or a t such that $w{:}t \in \Theta$. $t \in (\Gamma,\Delta,\Theta)$ is defined analogously.

We now define truth for ground lterms, rwffs and lwffs, where truth for lterms indicates definedness, and truth for rwffs indicates accessibility. Quantifiers are treated in each world as ranging over the domain of that world only.

Definition 4. We define a ground lterm, rwff or lwff φ to be *true* in a model $\mathfrak{M}$, in symbols $\models^{\mathfrak{M}} \varphi$, as follows. First we ensure, as is standard [16,23], that we have a name for each object in the domain $\mathfrak{D}$ of $\mathfrak{M}$ by extending, if necessary, the class of terms with a new constant c_o for each $o \in \mathfrak{D}$, and then extending $\mathfrak{a}$ so that $\mathfrak{a}(w,\mathrm{c}_o) = o$. Then we define $\models^{\mathfrak{M}}$ to be the smallest relation satisfying:

$$
\begin{array}{ll}
\models^{\mathfrak{M}} w{:}t & \text{iff } \mathfrak{a}(w,t) \in \mathfrak{q}(w) \\
\models^{\mathfrak{M}} w_i\ R\ w_j & \text{iff } (w_i,w_j) \in \mathfrak{R} \\
\models^{\mathfrak{M}} w{:}P(t_1,\ldots,t_n) & \text{iff } \langle \mathfrak{a}(w,t_1),\ldots,\mathfrak{a}(w,t_n)\rangle \in \mathfrak{a}(w,P) \\
\models^{\mathfrak{M}} w{:}A \to B & \text{iff } \models^{\mathfrak{M}} w{:}A \text{ implies } \models^{\mathfrak{M}} w{:}B \\
\models^{\mathfrak{M}} w{:}\Box A & \text{iff for all } w_i,\ \models^{\mathfrak{M}} w\ R\ w_i \text{ implies } \models^{\mathfrak{M}} w_i{:}A \\
\models^{\mathfrak{M}} w{:}\forall x.A & \text{iff for all } t,\ \models^{\mathfrak{M}} w{:}t \text{ implies } \models^{\mathfrak{M}} w{:}A[t/x]
\end{array}
$$

By extension, $\models^{\mathfrak{M}} (\Gamma,\Delta,\Theta)$ means that $\models^{\mathfrak{M}} \varphi$ for all $\varphi \in (\Gamma,\Delta,\Theta)$; $\Gamma,\Delta,\Theta \models^{\mathfrak{M}} \varphi$ means that $\models^{\mathfrak{M}} (\Gamma,\Delta,\Theta)$ implies $\models^{\mathfrak{M}} \varphi$ in the model $\mathfrak{M}$; and $\Gamma,\Delta,\Theta \models \varphi$ means that $\Gamma,\Delta,\Theta \models^{\mathfrak{M}} \varphi$ for all models $\mathfrak{M}$.

Note that, of course, $\not\models^{\mathfrak{M}} w{:}\bot$ for every w. Moreover, truth for lwffs is related to the standard truth relation for unlabelled quantified modal logics by observing that $\models^{\mathfrak{M}} w{:}A$ iff $\models^{\mathfrak{M}}_w A$.

The explicit embedding of properties of the models, and the possibility of explicitly reasoning about them, via lterms and rwffs, require us to consider soundness and completeness also for lterms and rwffs, where we show that $\Delta \vdash_L w_i\ R\ w_j$ iff $\Delta \models w_i\ R\ w_j$, and that $\Delta,\Theta \vdash_L w{:}t$ iff $\Delta,\Theta \models w{:}t$.

Definition 5. The QML $L = \mathrm{QK} + \mathcal{T} + \mathcal{D}$ is *sound* iff

(i) $\Delta \vdash_L w_i \, R \, w_j$ implies $\Delta \vDash w_i \, R \, w_j$,
(ii) $\Delta, \Theta \vdash_L w{:}t$ implies $\Delta, \Theta \vDash w{:}t$, and
(iii) $\Gamma, \Delta, \Theta \vdash_L w{:}A$ implies $\Gamma, \Delta, \Theta \vDash w{:}A$.

L is *complete* iff the converses hold.

Lemma 6. *$L = \mathrm{QK} + \mathcal{T} + \mathcal{D}$ is sound.*

Proof. Soundness follows by induction on the structure of the L-derivations. Consider an arbitrary model $\mathfrak{M}_L = (\mathfrak{W}_L, \mathfrak{R}_L, \mathfrak{D}_L, \mathfrak{q}_L, \mathfrak{a}_L)$ for the logic L. The base cases, e.g. $w{:}A \in (\Gamma, \Delta, \Theta)$, are trivial. There is a step case for each inference rule of L, and we only treat *conv1* and *conv2* (as an example involving Skolem functions), *ID*, and $\Box I$ as representative cases; the cases for the other rules follow analogously.[8]

Assume that $\mathfrak{R}_L$ is convergent and consider applications of the rules *conv1* and *conv2*

$$\dfrac{\begin{matrix}\Pi_1 & \Pi_2 \\ w_i \, R \, w_j & w_i \, R \, w_k\end{matrix}}{w_j \, R \, g(w_i, w_j, w_k)}\ \mathit{conv1} \qquad\qquad \dfrac{\begin{matrix}\Pi_1 & \Pi_2 \\ w_i \, R \, w_j & w_i \, R \, w_k\end{matrix}}{w_k \, R \, g(w_i, w_j, w_k)}\ \mathit{conv2}$$

where Π_1 and Π_2 are the derivations $\Delta_1 \vdash_L w_i \, R \, w_j$ and $\Delta_2 \vdash_L w_i \, R \, w_k$, with $\Delta = \Delta_1 \cup \Delta_2$. Recall that for convergency the signature of the relational theory is conservatively extended with a ternary Skolem function constant g, and a function g is also added to the model. Assume $\vDash^{\mathfrak{M}_L} \Delta$. Then, from the induction hypotheses we obtain $\vDash^{\mathfrak{M}_L} w_i \, R \, w_j$ and $\vDash^{\mathfrak{M}_L} w_i \, R \, w_k$, i.e. $(w_i, w_j) \in \mathfrak{R}_L$ and $(w_i, w_k) \in \mathfrak{R}_L$. Since $\mathfrak{R}_L$ is convergent, we conclude $\vDash^{\mathfrak{M}_L} w_j \, R \, g(w_i, w_j, w_k)$ and $\vDash^{\mathfrak{M}_L} w_k \, R \, g(w_i, w_j, w_k)$ by Definition 4.

Assume that $\mathfrak{M}_L$ is increasing and consider an application of the rule *ID*

$$\dfrac{\begin{matrix}\Pi_1 & \Pi_2 \\ w_i \, R \, w_j & w_i{:}t\end{matrix}}{w_j{:}t}\ \mathit{ID}$$

where Π_1 and Π_2 are the derivations $\Delta_1 \vdash_L w_i \, R \, w_j$ and $\Delta_2, \Theta \vdash_L w_i{:}t$, with $\Delta = \Delta_1 \cup \Delta_2$. Assume $\vDash^{\mathfrak{M}_L} \Delta$ and $\vDash^{\mathfrak{M}_L} \Theta$. Then, from the induction hypotheses we obtain $\vDash^{\mathfrak{M}_L} w_i \, R \, w_j$ and $\vDash^{\mathfrak{M}_L} w_i{:}t$. Since $\mathfrak{M}_L$ is increasing, we conclude $\vDash^{\mathfrak{M}_L} w_j{:}t$ by Definition 4.

Consider an application of the rule $\Box I$

$$\dfrac{\begin{matrix}[w \, R \, w_i] \\ \Pi \\ w_i{:}A\end{matrix}}{w{:}\Box A}\ \Box I$$

[8] The case for $\forall$ can be easily obtained from the one for $\Box$ by exploiting the symmetry between the rules for $\Box$ and $\forall$.

where Π is the L-derivation $\Gamma, \Delta_1, \Theta \vdash_L w_i{:}A$, with $\Delta_1 = \Delta \cup \{w\ R\ w_i\}$. By the induction hypothesis, $\Gamma, \Delta_1, \Theta \vdash_L w_i{:}A$ implies $\Gamma, \Delta_1, \Theta \models w_i{:}A$. Assume $\models^{\mathfrak{M}_L} (\Gamma, \Delta, \Theta)$. Considering the restriction on the application of $\Box I$, we can extend Δ to $\Delta' = \Delta \cup \{w\ R\ w'\}$ for an arbitrary $w' \notin (\Gamma, \Delta, \Theta)$, and assume $\models^{\mathfrak{M}_L} \Delta'$. Since $\models^{\mathfrak{M}_L} \Delta'$ implies $\models^{\mathfrak{M}_L} \Delta_1$, from the induction hypothesis we obtain $\models^{\mathfrak{M}_L} w_i{:}A$, that is $\models^{\mathfrak{M}_L} w'{:}A$ for an arbitrary $w' \notin (\Gamma, \Delta, \Theta)$ such that $\models^{\mathfrak{M}_L} w\ R\ w'$. We conclude $\models^{\mathfrak{M}_L} w{:}\Box A$ by Definition 4. $\Box$

Completeness follows by a Henkin-style proof, where a canonical model $\mathfrak{M}_L^C = (\mathfrak{W}_L^C, \mathfrak{R}_L^C, \mathfrak{D}_L^C, \mathfrak{q}_L^C, \mathfrak{a}_L^C)$ is built to show the following implications, which are the contrapositives of the conditions in Definition 5.

$$\Delta \nvdash_L w_i\ R\ w_j \text{ implies } \Delta \not\models^{\mathfrak{M}_L^C} w_i\ R\ w_j \tag{6}$$

$$\Delta, \Theta \nvdash_L w{:}t \text{ implies } \Delta, \Theta \not\models^{\mathfrak{M}_L^C} w{:}t \tag{7}$$

$$\Gamma, \Delta, \Theta \nvdash_L w{:}A \text{ implies } \Gamma, \Delta, \Theta \not\models^{\mathfrak{M}_L^C} w{:}A \tag{8}$$

In particular, given the presence of labelled formulae and explicit assumptions on the relations between the labels and their domains of quantification (i.e. Δ and Θ), in our version of the Lindenbaum lemma (Lemma 8 below) we consider a 'global' saturated set of labelled formulae, where consistency is also checked against the additional assumptions in Δ and Θ, instead of the usual saturated sets of unlabelled formulae. Moreover, given a logic $L = \mathrm{QK} + \mathcal{T} + \mathcal{D}$ and a proof context (Γ, Δ, Θ), we consider the deductive closure Δ_L of Δ under L, i.e.

$$\Delta_L =_{def} \{w_i\ R\ w_j \mid \Delta \vdash_L w_i\ R\ w_j\}\,,$$

and the deductive closure $\Theta_{L,\Delta}$ of Θ under L with respect to Δ, i.e.

$$\Theta_{L,\Delta} =_{def} \{w{:}t \mid \Delta, \Theta \vdash_L w{:}t\}\,.$$

Definition 7. (Γ, Δ, Θ) is *saturated* iff

(i) (Γ, Δ, Θ) is consistent, i.e. $\Gamma, \Delta, \Theta \nvdash_L w{:}\bot$ for every w;
(ii) $\Delta = \Delta_L$ and $\Theta = \Theta_{L,\Delta}$;
(iii) for every lwff $w{:}A$, either $w{:}A \in \Gamma$ or $w{:}\neg A \in \Gamma$;
(iv) for every w, if $\Gamma, \Delta, \Theta \vdash_L w{:}t$ implies $\Gamma, \Delta, \Theta \vdash_L w{:}A[t/x]$ for every t, then $\Gamma, \Delta, \Theta \vdash_L w{:}\forall x.A$; and
(v) for every w, if $\Gamma, \Delta, \Theta \vdash_L w\ R\ w_i$ implies $\Gamma, \Delta, \Theta \vdash_L w_i{:}B$ for every w_i, then $\Gamma, \Delta, \Theta \vdash_L w{:}\Box B$.

In the Lindenbaum lemma for first-order logic, a saturated set of formulae is inductively built by adding for every formula $\neg\forall x.A$ a *witness* to its truth, namely a formula $\neg A[c/x]$ for some new individual constant c. This ensures that the set is ω-complete, a property equivalent to condition (iv) in Definition 7. A similar procedure applies here not only for every lwff $w{:}\neg\forall x.A$, but also for every lwff $w{:}\neg\Box A$ (cf. condition (v) in Definition 7). That is, together with $w{:}\neg\Box A$, we consistently add $v{:}\neg A$ and $w\ R\ v$ for some new v, which acts as a 'witness world' to the truth of $w{:}\neg\Box A$. This ensures that the saturated pc (Γ, Δ, Θ) is such that $w{:}\Box B \in (\Gamma, \Delta, \Theta)$ iff $w\ R\ w_i \in (\Gamma, \Delta, \Theta)$ implies $w_i{:}B \in (\Gamma, \Delta, \Theta)$ for every w_i, as shown in Lemma 9 below.[9]

Lemma 8. *Every consistent pc (Γ, Δ, Θ) can be extended to a saturated pc.*

Proof. [Sketch] We first extend the language of the logic L with infinitely many new constants for witness terms and witness worlds. Systematically let t range over the original terms, s range over the new constants for witness terms, and r range over both. Analogously, let w range over labels, v range over the new constants for witness worlds, and u range over both. All these may be subscripted. Let $l_1, l_2, \ldots$ be an enumeration of all lwffs in the extended language. Starting from $(\Gamma_0, \Delta_0, \Theta_0) = (\Gamma, \Delta, \Theta)$, we inductively build a sequence of consistent pcs by defining $(\Gamma_{i+1}, \Delta_{i+1}, \Theta_{i+1})$ to be:

- $(\Gamma_i, \Delta_i, \Theta_i)$, if $(\Gamma_i \cup \{l_{i+1}\}, \Delta_i, \Theta_i)$ is inconsistent; else
- $(\Gamma_i \cup \{l_{i+1}\}, \Delta_i, \Theta_i)$, if l_{i+1} is neither $u{:}\neg\Box A$ nor $u{:}\neg\forall x.A$; else
- $(\Gamma_i \cup \{u{:}\neg\forall x.A, u{:}\neg A[s/x]\}, \Delta_i, \Theta_i \cup \{u{:}s\})$, for an $s \notin (\Gamma_i \cup \{u{:}\neg\forall x.A\}, \Delta_i, \Theta_i)$, if l_{i+1} is $u{:}\neg\forall x.A$; else
- $(\Gamma_i \cup \{u{:}\neg\Box A, v{:}\neg A\}, \Delta_i \cup \{u\ R\ v\}, \Theta_i)$, for a $v \notin (\Gamma_i \cup \{u{:}\neg\Box A\}, \Delta_i, \Theta_i)$, if l_{i+1} is $u{:}\neg\Box A$.

A saturated pc is then $(\Gamma^*, \Delta^*, \Theta^*) =_{def} (\bigcup_{i\geq 0} \Gamma_i, (\bigcup_{i\geq 0} \Delta_i)_L, (\bigcup_{i\geq 0} \Theta_i)_{L,\Delta})$. $\Box$

Lemma 9. *Let $(\Gamma^*, \Delta^*, \Theta^*)$ be a saturated pc.*

(i) *$\Gamma^*, \Delta^*, \Theta^* \vdash_L \varphi$ iff $\varphi \in (\Gamma^*, \Delta^*, \Theta^*)$, where φ is an lterm, rwff or lwff.*
(ii) *$u{:}A \to B \in (\Gamma^*, \Delta^*, \Theta^*)$ iff $u{:}A \in (\Gamma^*, \Delta^*, \Theta^*)$ implies $u{:}B \in (\Gamma^*, \Delta^*, \Theta^*)$.*
(iii) *$u_i{:}\Box B \in (\Gamma^*, \Delta^*, \Theta^*)$ iff for all u_j, $u_i\ R\ u_j \in (\Gamma^*, \Delta^*, \Theta^*)$ implies $u_j{:}B \in (\Gamma^*, \Delta^*, \Theta^*)$.*
(iv) *$u{:}\forall x.A \in (\Gamma^*, \Delta^*, \Theta^*)$ iff for all r, $u{:}r \in (\Gamma^*, \Delta^*, \Theta^*)$ implies $u{:}A[r/x] \in (\Gamma^*, \Delta^*, \Theta^*)$.*

Proof. (i) follows immediately by definition and Fact 2. We only treat (iv); the proof of (iii) can be easily obtained from the proof of (iv) by exploiting the symmetry between $\Box$ and $\forall$, and (ii) follows analogously. For the left-to-right direction of (iv) suppose that $u{:}\forall x.A \in (\Gamma^*, \Delta^*, \Theta^*)$. Then, by (i), $\Gamma^*, \Delta^*, \Theta^* \vdash_L u{:}\forall x.A$, and, by $\forall E$, $\Gamma^*, \Delta^*, \Theta^* \vdash_L u{:}r$ implies $\Gamma^*, \Delta^*, \Theta^* \vdash_L u{:}A[r/x]$ for all r. By (i), conclude $u{:}r \in (\Gamma^*, \Delta^*, \Theta^*)$ implies $u{:}A[r/x] \in (\Gamma^*, \Delta^*, \Theta^*)$ for all r. For the converse, suppose that $w{:}\forall x.A \notin (\Gamma^*, \Delta^*, \Theta^*)$. Then $u{:}\neg\forall x.A \in (\Gamma^*, \Delta^*, \Theta^*)$ and, by the construction of $(\Gamma^*, \Delta^*, \Theta^*)$, there exists an r such that $u{:}r \in (\Gamma^*, \Delta^*, \Theta^*)$ and $u{:}A[r/x] \notin (\Gamma^*, \Delta^*, \Theta^*)$. $\Box$

[9] In the standard completeness proof for unlabelled modal logics, $\mathfrak{W}^C_L$ is defined to be the set of all saturated sets, and it is possible to show that if $w \in \mathfrak{W}^C_L$ and $\neg\Box A \in w$, then there is a $w' \in \mathfrak{W}^C_L$ accessible from w such that $\neg A \in w'$.

Definition 10. Given a saturated pc $(\Gamma^*, \Delta^*, \Theta^*)$, we define the *canonical model* $\mathfrak{M}_L^C$ for the logic L as follows:

- $\mathfrak{W}_L^C = \{u \mid u \in (\Gamma^*, \Delta^*, \Theta^*)\}$;
- $(u_i, u_j) \in \mathfrak{R}_L^C$ iff $u_i\ R\ u_j \in \Delta^*$;
- $\mathfrak{a}(u, r) = r$, and $\langle r_1, \dots, r_n \rangle \in \mathfrak{a}(u, P)$ iff $u{:}P(r_1, \dots, r_n) \in \Gamma^*$, for P an n-ary predicate;
- $\mathfrak{q}(u) = \{\mathfrak{a}(u, r) \mid u{:}r \in \Theta^*\}$;
- $\mathfrak{D} = \bigcup_{u \in (\Gamma^*, \Delta^*, \Theta^*)} \mathfrak{q}(u)$.

The standard definition of $\mathfrak{R}_L^C$, i.e. $(u_i, u_j) \in \mathfrak{R}_L^C$ iff $\{A \mid \Box A \in u_i\} \subseteq u_j$, is not applicable in our setting, since $\{A \mid \Box A \in u_i\} \subseteq u_j$ does *not* imply $\vdash_L u_i\ R\ u_j$. We would therefore lose completeness for rwffs, since there would be cases, e.g. if $L = \mathrm{QK}$ and $\Delta = \{\}$, where $\nvdash_L u_i\ R\ u_j$ but $(u_i, u_j) \in \mathfrak{R}_L^C$ and thus $\models^{\mathfrak{M}_L^C} u_i\ R\ u_j$. Hence, we instead define $(u_i, u_j) \in \mathfrak{R}_L^C$ iff $u_i\ R\ u_j \in \Delta^*$; note that therefore $u_i\ R\ u_j \in \Delta^*$ implies $\{A \mid \Box A \in u_i\} \subseteq u_j$.[10]

The deductive closures of Δ^* and Θ^* ensure not only completeness for rwffs and lterms, but also that the conditions on $\mathfrak{R}_L^C$ and $\mathfrak{D}_L^C$ are satisfied, so that $\mathfrak{M}_L^C$ is really a model for L. For example, it is easy to show that if $\mathcal{T}$ includes *conv1* and *conv2*, then $\mathfrak{R}_L^C$ is convergent.

Fact 11. We immediately have that:

(i) $u_i\ R\ u_j \in (\Gamma^*, \Delta^*, \Theta^*)$ iff $\Delta^* \models^{\mathfrak{M}_L^C} u_i\ R\ u_j$.
(ii) $u{:}r \in (\Gamma^*, \Delta^*, \Theta^*)$ iff $\Delta^*, \Theta^* \models^{\mathfrak{M}_L^C} u{:}r$.

Let the *degree* of an lwff be the number of times $\bot$, $\rightarrow$, $\Box$, and $\forall$ occur in it.

Lemma 12. *$u{:}A \in (\Gamma^*, \Delta^*, \Theta^*)$ iff $\Gamma^*, \Delta^*, \Theta^* \models^{\mathfrak{M}_L^C} u{:}A$.*

Proof. [Sketch] By induction on the degree of $u{:}A$. We treat only the case when $u{:}A$ is $u{:}\Box B$; the other cases follow analogously.[11] Assume $u{:}\Box B \in \Gamma^*$. Then, by Lemma 9, $u\ R\ u_i \in \Delta^*$ implies $u_i{:}B \in \Gamma^*$, for all u_i. Hence, by the induction hypothesis and Fact 11, we obtain $\Gamma^*, \Delta^*, \Theta^* \models^{\mathfrak{M}_L^C} u_i{:}B$ for all u_i such that $\Gamma^*, \Delta^*, \Theta^* \models^{\mathfrak{M}_L^C} u\ R\ u_i$, and thus $\Gamma^*, \Delta^*, \Theta^* \models^{\mathfrak{M}_L^C} u{:}\Box B$ by Definition 4. For the converse, assume $u{:}\neg\Box B \in \Gamma^*$. Then, by Lemma 9, $u\ R\ u_i \in \Delta^*$ and $u_i{:}\neg B \in \Gamma^*$, for some u_i. Hence, by the induction hypothesis and Fact 11, we obtain $\Gamma^*, \Delta^*, \Theta^* \models^{\mathfrak{M}_L^C} u\ R\ u_i$ and $\Gamma^*, \Delta^*, \Theta^* \models^{\mathfrak{M}_L^C} u_i{:}\neg B$, and thus $\Gamma^*, \Delta^*, \Theta^* \models^{\mathfrak{M}_L^C} u{:}\neg\Box B$ by Definition 4. □

It is now a simple matter to show (6), (7) and (8), and thus prove that:

Lemma 13. *$L = \mathrm{QK} + \mathcal{T} + \mathcal{D}$ is complete.*

[10] As a further comparison with the standard definition, note also that in the canonical model the label u can be identified with the set of formulae $\{A \mid u{:}A \in \Gamma^*\}$.

[11] As above, the case for $u{:}\forall x.A$ can be easily obtained from the case for $\Box$ by exploiting the symmetry between $\Box$ and $\forall$.

By Lemma 6 and Lemma 13 we immediately have that:

Theorem 14. *$L = \mathrm{QK} + \mathcal{T} + \mathcal{D}$ is sound and complete.*

Some remarks and comparisons are in order. Our proof is modular: the same method applies uniformly to every logic L. As explained previously, this is *not* the case for the completeness proof of unlabelled QMLs based on free logic [11,14]. Garson himself points out that his proof "lacks generality" [11, pp. 280-281], since (i) it does not work for systems with constant domains, and (ii) it is not general with respect to the underlying propositional modal logic (although there are tricks one can use to overcome the difficulties for particular systems). As we have shown, none of these problems applies in our approach.

Most importantly, being complete, our QMLs are adequate presentations of the Kripke semantics, and are thus equivalent to the corresponding Hilbert systems only when these are themselves complete. For example, by the incompleteness results in [14], QKT42.i is equivalent to (the Hilbert system for) QS4.2 since they are both complete with respect to reflexive, transitive and convergent Kripke models with increasing domains; on the other hand, QKT42.c is not equivalent to (the Hilbert system for) QS4.2 + BF, since the latter is incomplete.

4 Normalization

Soundness and (when attainable) completeness are minimal requirements for proof systems. In this section we show that derivations have additional properties: derivations of lwffs can be reduced to a normal form that does not contain unnecessary detours and satisfies a subformula property. This provides us with positive results, such as alternative proofs of the consistency of our logics and restricted search space for proofs. It also allows us to establish negative results, such as how incompleteness can arise; we show how analysis of normal forms provides a basis for investigating tradeoffs in formalizations. To reduce notational overhead, we follow, where possible, Prawitz [20,21].

Definition 15. A *maximal lwff* in a derivation is an lwff that is both the conclusion of an introduction rule and the major premise of an elimination rule.

A maximal lwff constitutes a detour in a derivation, and we remove it by (finitely many applications of) *proper reductions.* Three possible configurations (for $\rightarrow$, $\Box$, and $\forall$) result in a maximal lwff in a derivation. As examples, we give the proper reductions for $\Box$ and $\forall$.

$$\dfrac{\dfrac{\begin{array}{c}[w_i\ R\ w_j]^1\\ \Pi_1\\ w_j{:}A\end{array}}{w_i{:}\Box A}\,\Box I^1 \qquad \begin{array}{c}\Pi_2\\ w_i\ R\ w_k\end{array}}{w_k{:}A}\,\Box E \quad \rightsquigarrow \quad \begin{array}{c}\Pi_2\\ w_i\ R\ w_k\\ \Pi_1[w_k/w_j]\\ w_k{:}A\end{array} \tag{9}$$

$$\dfrac{\dfrac{\begin{array}{c}[w{:}t_i]^1\\ \Pi_1\\ w{:}A[t_i/x]\end{array}}{w{:}\forall x.A}\,\forall I^1 \quad \begin{array}{c}\Pi_2\\ w{:}t_j\end{array}}{w{:}A[t_j/x]}\,\forall E \quad \rightsquigarrow \quad \begin{array}{c}\Pi_2\\ w{:}t_j\\ \Pi_1[t_j/t_i]\\ w{:}A[t_j/x]\end{array} \tag{10}$$

where $\Pi[\alpha/\beta]$ is obtained from Π by systematically substituting α for β, with a suitable renaming of the variables to avoid clashes. Note that we only show the part of the derivation where the reduction actually takes place; the missing parts remain unchanged. Note also that Π_2 is empty when (9) and (10) are QK-derivations, since the relational theory and the domain theory are both empty.

Definition 16. A derivation is in *normal form* (is a *normal derivation*) iff it contains no maximal lwffs.

Following Prawitz, it is easy to show that each reduction reduces a suitable well-formed measure on derivations. Hence, the reduction process must eventually terminate with a derivation free of maximal lwffs. We have:

Lemma 17. *Every derivation of $w{:}A$ from Γ, Δ, Θ in $\mathrm{QK} + \mathcal{T} + \mathcal{D}$ reduces to a derivation in normal form.*

Proof (Sketch). First, note that derivations in the Horn theories $\mathcal{T}$ and $\mathcal{D}$ cannot introduce maximal lwffs. Then consider a derivation Π of $w{:}A$ from Γ, Δ, Θ in $\mathrm{QK} + \mathcal{T} + \mathcal{D}$. Any lwff $w_i{:}B$ in Π is the root of a tree of rule applications leading back to assumptions; we call the lwffs in this tree other than $w{:}A$ the *side lwffs* of $w{:}A$. Then, from the set of maximal lwffs of Π pick some $w_i{:}B$ that has the highest degree and has maximal lwffs only of lower degree as side lwffs. Let Π' be the reduction of Π at $w_i{:}B$. Π' is also a derivation of $w{:}A$ from Γ, Δ, Θ in $\mathrm{QK} + \mathcal{T} + \mathcal{D}$ and no new maximal lwff as large, or larger than $w_i{:}B$ has been introduced. Hence, by a finite number of similar reductions we obtain a derivation of $w{:}A$ from Γ, Δ, Θ in $\mathrm{QK} + \mathcal{T} + \mathcal{D}$ containing no maximal lwffs. □

We can now exploit Lemma 17 to show that derivations in $L = \mathrm{QK} + \mathcal{T} + \mathcal{D}$ have a well-defined structure. First, for any derivation in $L = \mathrm{QK} + \mathcal{T} + \mathcal{D}$, we can strictly separate derivations involving lwffs, rwffs, and lterms (cf. Fact 2):

1. the derivation of an lwff can depend on the derivation of an rwff (via an application of $\Box E$), but not vice versa;
2. the derivation of an lwff can depend on the derivation of an lterm (via an application of $\forall E$), but not vice versa;
3. the derivation of an lterm can depend on the derivation of an rwff (via an application of *ID* or *DD*), but not vice versa.

As a consequence, any derivation of an lwff is structured as a central derivation in the base logic 'decorated' with (i) subderivations in the relational theory, which attach onto the central derivation through instances of $\Box E$, and (ii) subderivations in the domain theory, which attach onto the central derivation through instances of $\forall E$. Moreover, the structure of the central derivation in L, when in normal form, can be further characterized by identifying particular sequences of lwffs, which Prawitz calls *branches*, *paths*, and *segments* [21, pp. 249–250], and showing that in these sequences there is an ordering on inferences. By exploiting this ordering, we can show, directly analogous with [21, p. 251], a subformula property for all extensions of QK.

Definition 18. The notion of *subformula* is defined inductively by: (i) A is a subformula of A; (ii) if $B \to C$ is a subformula of A, then so are B and C; (iii) if $\Box B$ is a subformula of A, then so is B; (iv) if $\forall x.B$ is a subformula of A, then so is $B[t/x]$ for all t. Given a derivation $\Gamma, \Delta, \Theta \vdash w_i{:}A$, let S be the set of subformulae of the formulae in $\{C \mid w_k{:}C \in \Gamma \cup \{w_i{:}A\}\}$, i.e. S is the set consisting of the subformulae of the assumptions Γ and of the goal $w_i{:}A$. We say that $\Gamma, \Delta, \Theta \vdash w_i{:}A$ satisfies the *subformula property* iff for all lwffs $w_j{:}B$ in the derivation (i) $B \in S$; or (ii) B is an assumption $B' \to \bot$ discharged by an application of $\bot E$, where $B' \in S$; or (iii) B is an occurrence of $\bot$ obtained by $\to E$ from an assumption $B' \to \bot$ discharged by an application of $\bot E$, where $B' \in S$.

Summarizing, we have:

Theorem 19. *For every derivation Π in $L = \mathrm{QK} + \mathcal{T} + \mathcal{D}$, there is a normal form derivation Π' that is strictly partitioned and satisfies the subformula property.*

From this theorem, standard corollaries follow; for example, our systems are consistent since there is no introduction rule for $\bot$. We can also exploit the existence of normal forms to design equivalent cut-free sequent systems and automate proof search. This was done in [4] for our labelled propositional systems.[12]

However, in exchange for this extra structure there are limits to the generality of the formulation: the properties in Theorem 19 depend on design decisions we have made, in particular, the use of Horn theories. This, of course, limits what we can formalize in comparison to a semantic embedding in first-order logic. There are tradeoffs in the possible formalizations: if we remove these limitations by introducing first-order theories (of the accessibility relation and of the domains of quantification), then, in general, to achieve complete presentations we must give up the properties in Theorem 19. In particular, we must give up the ability to partition derivations so that reasoning can be factored into interacting theories, and instead retreat to systems where derivations arbitrarily mix lwffs, rwffs and lterms. Such liberalized systems essentially amount to a direct formalization (embedding) of the semantics in first-order logic.

[12] We also showed there that in the cut-free sequent systems for certain propositional modal logics, e.g., K and T, we can bound applications of the contraction rule and thus show decidability. This will not be the case for quantified modal logics (since we cannot bound the use of universally quantified subformulae), but still the existence of partitioned normal forms allows us to restrict the search space during theorem proving.

To illustrate this, we first briefly review the tradeoffs for extensions to first-order relational theories discussed in [3]. Then we consider problems that appear only in the quantified case, namely the tradeoffs in formalizations of QMLs with first-order domain theories.

Consider an extension of the relational theory to a full first-order theory; this theory consists of standard first-order proof rules for reasoning about relational formulae built by using the connectives $\emptyset$ (falsum), $\supset$ (implies), All (for all).[13] Such an extension is needed when one wants to capture properties of the accessibility relation that cannot be expressed as Horn relational rules, e.g. to capture *irreflexivity* and *connectedness* we extend the theory with the rules

$$\frac{}{\mathsf{All}\, w(\sim (w\ R\ w))}\ \mathit{irrefl}$$

and

$$\frac{}{\mathsf{All}\, w_i \mathsf{All}\, w_j \mathsf{All}\, w_k((w_i\ R\ w_j \cap w_i\ R\ w_k) \supset (w_j\ R\ w_k \cup w_k\ R\ w_j))}\ \mathit{conn}$$

where, as usual, $\sim$ (not), $\cap$ (and), and $\cup$ (or) are defined in terms of $\emptyset$ and $\supset$.

It is easy to show that the logics obtained by simply adding a first-order relational theory to QK possess the properties in Theorem 19. However, these logics are in general *not* complete. We have investigated this problem in detail in [3], where, by analysis of normal form proofs, we have shown that the addition of *conn* to K does not suffice to prove the modal axiom

$$\neg\Box(\Box A \to B) \to \Box(\Box B \to A)\,.$$

Since this axiom corresponds to *conn* [26,27], this implies that there are extensions of K (and, a fortiori, QK) with first-order relational theories that are not complete with respect to the corresponding semantics.

Completeness can be restored by giving up the separation we enforced between the base logic and the relational theory, and identifying $\bot$ with $\emptyset$. That is, $\bot$ should not only propagate between worlds (this propagation is embodied in the rule $\bot E$), but also between base logic and the relational theory in either direction.[14] This is best achieved by adding the rules

$$\frac{w{:}\bot}{\emptyset} \quad \text{and} \quad \frac{\emptyset}{w{:}\bot} \tag{11}$$

[13] We use different connectives to avoid confusion with the connectives in modal formulae.

[14] Note that if, on the other hand, we restrict the propagation of $\bot$ by requiring that all the lwffs in $\bot E$ have the same label, i.e.

$$\begin{array}{c}[w{:}A \to \bot]\\ \vdots \\ \dfrac{w{:}\bot}{w{:}A}\end{array}$$

we obtain systems that possess interesting paraconsistency properties but are inadequate for presenting modal logics [3].

By doing this, however, we lose the normalization and separation properties of Theorem 19 in exchange for systems that are essentially equivalent to semantic embeddings of modal logics in first-order logic, see [3].

Before showing that similar tradeoffs must be considered for QMLs with first-order domain theories, let us discuss when and why such theories might be of interest. As we have shown above, all the properties commonly considered, i.e. that the domains are increasing, decreasing or constant, can be easily axiomatized by Horn clauses. However, in some particular applications of QMLs, we might want to consider more complicated properties. For example, we might want to state explicitly that some object does not exist in a world w. Or we might want to refine the increasing domains property by specifying the size of the increase, e.g. that there are at least n 'new' objects. Such properties require a full first-order domain theory. Analogous with the case of relational theories, it is not conceptually difficult, although notationally cumbersome, to introduce first-order (or even higher-order) domain theories.[15] We just need to introduce a standard first-order natural deduction system for reasoning about labelled terms built using the connectives $\emptyset$ (falsum), $\supset$ (implies), All (for all); lterms over other connectives, e.g. $\sim$ (not), $\cap$ (and), $\cup$ (or), Ex (exists), and corresponding rules are defined as usual.

The particular properties of the domains are then added as axioms (or rules) directly in their full form. For example, to state that the domain of each world contains at least one term we add the rule

$$\frac{}{w{:}\mathsf{Ex}\, x.x}\ \textit{non-empty}$$

Of course, the non-emptyness of the domains is a property expressible as a Horn rule: we can express it as

$$\frac{}{w{:}c(w)}$$

where c is a Skolem function constant. However, it is interesting to consider it in its full (unskolemized) form, since even this very simple property gives rise to a tradeoff between expressivity, completeness and metatheoretic properties of our systems.

From free logic [5] we know that *non-empty* corresponds to the axiom

$$w{:}\forall x.A \rightarrow \exists x.A \tag{12}$$

(cf. also Section 2). Therefore there should be a proof of it in the extension of QK with a first-order domain theory $\mathcal{D}_F$ containing *non-empty* as the only property. Moreover, since normalization in QK + $\mathcal{D}_F$ can be easily shown by extending

[15] Note that the possibility of expressing complicated properties of the domains of quantification in our systems provides another advantage of our approach with respect to Hilbert-style axiomatizations, since it is often difficult, if not impossible, to give axioms corresponding to such properties in Hilbert systems.

Lemma 17, if there is a proof of (12), then there is a normal one. But reasoning backwards from (12), we see that we need a proof of $w{:}t$ from *non-empty*:

$$\dfrac{\dfrac{\dfrac{[\forall x.A]^1 \quad w{:}t}{w{:}A[t/x]}\;\forall E \qquad w{:}t}{w{:}\exists x.A}\;\exists I}{w{:}\forall x.A \to \exists x.A}\;\to I^1$$

However, such a proof cannot exist: we can only use *non-empty* as the major premise in an application of (the derived rule) $\mathsf{Ex}\,E$,

$$\dfrac{w_i{:}\mathsf{Ex}\,x.x \qquad \begin{array}{c}[w_i{:}t_i]\\ \vdots \\ w_j{:}t_j\end{array}}{w_j{:}t_j}\;\mathsf{Ex}\,E$$

which has the side condition that t_i must not occur in $w_i{:}\mathsf{Ex}\,x.x$, in $w_j{:}t_j$, or in any assumption on which the upper occurrence of $w_j{:}t_j$ depends other than $w_i{:}t_i$. In particular, $w_j{:}t_j$ cannot be $w_i{:}t_i$, and we cannot derive $w{:}t$ by *non-empty* and $\mathsf{Ex}\,E$. Hence (12) is not provable in $\mathrm{QK}+\mathcal{D}_F$. As a consequence, $\mathrm{QK}+\mathcal{D}_F$ is not complete with respect to its corresponding semantics (in which (12) is a valid formula). Thus we have:

Theorem 20. *There are systems $\mathrm{QK}+\mathcal{T}+\mathcal{D}_F$ that are incomplete with respect to the corresponding Kripke models with first-order theories of the domains of quantification.*

As in the case of first-order relational theories, we can restore completeness by giving up the separations in our systems. Specifically, we need again rules that allow us to propagate, in either direction, inconsistency (falsum) between the base logic and the theory extending it. The addition of the rules

$$\dfrac{w_i{:}\bot}{w_j{:}\emptyset} \qquad \text{and} \qquad \dfrac{w_j{:}\emptyset}{w_i{:}\bot} \tag{13}$$

allows us to mingle derivations of lwffs with derivations of lterms, and we can then derive rules to prove (12) as follows:

$$\dfrac{\dfrac{}{w{:}\mathsf{Ex}\,x.x}\;\textit{non-empty} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[w{:}\forall x.A]^1 \quad [w{:}t]^2}{w{:}A[t/x]}\;\forall E \qquad [w{:}t]^2}{w{:}\exists x.A}\;\exists I}{w{:}\forall x.A \to \exists x.A}\;\to I^1}{}}{w{:}\forall x.A \to \exists x.A}\;\mathsf{Ex}\,E^2_{\mathit{lwff}}$$

Note that we use a derived rule of the domain theory, $\mathsf{Ex}\,E_{\mathit{lwff}}$, to infer an lwff; the derivation of $\mathsf{Ex}\,E_{\mathit{lwff}}$ requires using the rules in (13). Hence, to restore completeness not only have we lost partitioned derivations, but also the other good metatheoretic properties in Theorem 19, in exchange for a system in which, like in semantic embedding, derivations of lwffs are mingled with derivations of rwffs and lterms. Such a system does not seem to offer any advantages over semantic embedding in first-order logic (where there is no separation at all), and provides no essential alternative to this better known approach.[16]

5 Related Work

In motivating our work in the introduction, we described various problems that arise in traditional approaches to QMLs based on Hilbert formalizations, and throughout the paper we have argued that they are not encountered in our approach. We now compare our work with approaches based on sequent, or tableau, systems, and on embeddings of modal logics in first-order logic.

Fitting, for example, introduces cut-free sequent systems for quantified modal logics in [7,8]; see also [29]. Fitting first gives 'standard' systems for non-symmetric logics with increasing domains, and then, in order to capture the other conditions, he extends his calculi by introducing *prefixes*. These allow him to formulate sequent systems for a class of modal logics (including symmetric logics like S5) with varying, increasing, or constant domains. In prefixed systems, the different properties of the domains are expressed by imposing different side conditions on the applicability of the quantifier rules; analogously, the properties of the accessibility relation require different side conditions on the rules for the modalities. The main disadvantage of these systems, apart from the fact that they don't capture decreasing domains, is that their formalizations often require considerable ingenuity, and that the rules for the modalities and quantifiers can be quite awkward, since they carry side conditions on the complete set of assumptions.[17] As a consequence, unlike our approach which leads to simple implementations, these systems cannot be directly formalized in standard logical frameworks such as Isabelle [18] or the Edinburgh LF [12].

Our work is closely related to approaches based on semantic embeddings, e.g. [1,13,17]. In these approaches, a formula of quantified modal logic is translated into a formula of first-order predicate logic, and shown to be true (or false) in a theory formalizing the semantics of the modalities and domains of quantification. For example, $\Box(A \wedge B)$ would be translated into a formula equivalent to

$$\forall w.\, R(0, w) \rightarrow (A(w) \wedge B(w))$$

where there would be additional axioms characterizing the accessibility relation R and the domains of quantification. Ohlbach [17], for example, provides a

[16] In fact, by defining a suitable mapping between derivations, we can show that the above system is essentially equivalent to the usual semantic embedding of QMLs in first-order logic; see [3] where details are given for the propositional case.

[17] See Avron's discussion of degrees of impurity of natural deduction systems in [2, §5.5]. Note also that prefixed sequent systems and prefixed tableau systems could be modified to cover decreasing domains. To do so, however, one should replace the standard quantifier rules with rules similar to ours.

general framework for carrying out such translations and reasoning about their soundness and completeness; translations are defined by morphisms on formulae and these are shown sound and complete by providing morphisms on interpretations.

Our work differs from embedding based approaches with respect to the nature of the translations, the metatheoretic properties that hold, and how they are proved. First, we separate, rather than combine, reasoning about relations, predicates and terms (see Fact 2 and Theorem 19). In the semantic embedding approach there is no formal distinction between lwffs, rwffs and lterms or separation between relational and first-order reasoning. Second, rather than using interpretation morphisms and building on top of the semantics of first-order logic, we directly define deductive systems for our QMLs and show, using a parameterized canonical model construction that these systems are sound and complete. Finally, our proofs have normal forms with the subformula property (again, see Theorem 19), while in the translation approach the normal forms are those of derivations in first-order logic.

An approach related to semantic embeddings has been considered by Gabbay in his book introducing Labelled Deductive Systems [9], and further developed for modal logics, in parallel with our work, by Russo [22]. Russo extends the analysis of QMLs given by Gabbay, by giving quantifier rules based on free logic similar to ours. However, her systems are based on multiple-conclusion rules that operate on configurations. Most importantly, they have no separation between base logic, theory of the accessibility relation, and theory of the domains of quantification: there is only one falsum, like our systems based on first-order theories with a 'unique' falsum, i.e. with rules (11) and (13).

Finally, Gabbay [9] shows that his approach solves one standard problem in modal theorem proving, namely that when the domains are not constant, the skolemization of $\Diamond\exists x.A$ and $\exists x.\Diamond A$ should yield different formulae. By exploiting normalization results, it is easy to prove that our systems are equally able to show that $\Diamond\exists x.A$ and $\exists x.\Diamond A$ are not equivalent unless the domains are constant (cf. also (2) and (3) in Section 2).

6 Conclusion

We have given a modular presentation of a large class of quantified modal logics, including QK, QD, QT, QB, QS4, QS4.2, QKD45, and QS5, all with varying, increasing, decreasing, or constant domains. Our approach is modular with respect to both properties of the accessibility relation in the Kripke frame and the way domains of individuals change between worlds. We also have a modular metatheory: soundness, completeness and normalization are proven uniformly for every logic in our class. Finally, our approach lends itself well to implementation in standard logical frameworks. We have implemented our approach in the Isabelle Logical Framework and the result is a simple and natural environment for interactive proof development that supports hierarchical structuring: quantified modal logics are structured by extension (enrichment with new rules), and theorems are inherited in extensions.

acknowledgements We thank the anonymous referees for helpful comments.

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