Labelled quantified modal logics

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Abstract. We present an approach to providing natural deduction style proof systems for a large class of quantified modal logics with varying, increasing, decreasing or constant domains of quantification. The systems we develop are modular both in the behavior of the accessibility relation and quantification relative to the semantics, and in the proofs of soundness and completeness relative to that semantics. Our systems also provide the basis of simple implementations of quantified modal logics in a standard logical framework theorem prover.

1 Introduction

Modal logic is an active area of research in computer science and artificial intelligence, as a language for formalizing and reasoning about, e.g., knowledge, belief, time, space, and other dynamic 'state oriented' properties. The principles of propositional modal logics (PMLs) are well understood, and the relationship between semantics and proof theory captured in general metatheorems which we can exploit in developing new systems. The situation is not, however, so well understood in the more complex case of quantified modal logics (QMLs), where arbitrary ingenuity may be needed to establish standard metatheoretic results for a new system [7,9,11].

The problems are the result of the new semantic possibilities introduced by quantifiers: in a QML we have to decide not only the properties of the accessibility relation between worlds in the Kripke frame but also how quantification in these worlds behaves. As we move from world to world, the domain of quantification might vary arbitrarily (varying domains), remain constant (constant domains), or expand (increasing domains) or shrink (decreasing domains). These two parts of the semantics are essentially independent of each other and define a two-dimensional space[1] which, in the past, has been explored piecemeal; i.e. there is no uniformity in the presentation of proof systems and the way metatheoretic results are proved.

[1] Even more complexity is possible, e.g. non-rigid designators [7,9], where the meaning of a name varies from world to world; we consider here only the rigid case.

There are several difficulties for a uniform approach. First, it may be difficult to find a suitable proof system for an intended semantics: QMLs are (like PMLs) usually presented as Hilbert systems. But the usual quantifier rules for Hilbert systems work only for increasing domains [7, p. 426]. This problem can sometimes be solved by adopting a more complex proof system (e.g. adopted from free logic) or even by redefining the principles of the semantics (e.g. by introducing truth value gaps). However these solutions are imperfect in that they are not uniform: different strategies for proving completeness are needed for different conditions on the domains [9,11]. Second, the relationship between the Kripke semantics of a PML and its quantified extension may not be simple. Just because a PML is complete with respect to a semantics it does not follow that the corresponding QML is complete with respect to the corresponding extension of the semantics. Moreover, minor changes to a complete QML, e.g. modifying the conditions on the domains, can produce incompleteness.[2]

Context and Contribution This paper is a companion to [4]; we extend and generalize the results given there for PMLs to the quantified case and thereby provide solutions to the above problems. Let us first briefly summarize the approach and some of the results that we previously developed. In [4] we formalized natural deduction proof systems for PMLs, based on the view of a logic as a Labelled Deductive System [8]. We decomposed a modal logic into two interacting parts: a *base logic*, fixed for all modal logics, and a *relational theory*, different for each modal logic. In the base logic, we reason about formulae paired with labels; instead of $\vdash A$, we prove $\vdash w{:}A$, where $w{:}A$ is a *labelled formula*, w is an element of the set of possible worlds W in the Kripke frame, and $\vdash A$ iff $\forall w \in W(\vdash w{:}A)$. In the relational theory, we formalize the behavior of the accessibility relation R in the Kripke frame. *Relational formulae* $w\ R\ w'$ state that w accesses w'. This allows us, for instance, to formulate the behavior of modal operators like $\Box$ independent of the properties of R in the frame taken as providing the semantics, i.e. $\vdash w{:}\Box A$ iff $\vdash w'{:}A$ for all $w' \in W$ such that $\vdash w\ R\ w'$. As a consequence, we are able to give natural deduction introduction and elimination rules for $\Box$ that are fixed for all the logics we consider. The main results that we established can be summarized as follows.
Soundness and completeness: we uniformly showed that all PMLs formalizable in our framework are sound and complete with respect to the corresponding Kripke semantics.
Proof theory: we explored tradeoffs in formulating the base logic and the relational theory. We showed, for example, that when the relational theory can be formulated as a set of Horn clauses (as opposed to a set of first or second-order axioms), then proofs are normalizing and there is a strict separation between the base logic and the relational theory, i.e. derivations in the base logic may depend on derivations in the relational theory, but not vice versa.

[2] There are, e.g., QMLs without the *Barcan Formula* (BF, $\forall x.\Box A \rightarrow \Box\forall x.A$) which are complete, where the introduction of BF produces incompleteness, or vice versa. For example QS4.2 + BF is incomplete even though QS4.2 is complete [11].

Implementation: we showed that the resulting proof systems can be implemented in a standard logical framework such as Isabelle [14] or the Edinburgh LF [10]. We implemented our approach in Isabelle and the result is a simple and natural environment for interactive proof development that supports hierarchical structuring: logics are structured by extension (enrichment with new rules) and theorems are inherited in extensions.

We give a natural deduction presentation of QMLs that is modular in two dimensions, reflecting the two degrees of freedom described above. As before, it is based on a fixed base logic (now QK, for quantified K) where extensions are made by independently instantiating two separate theories: a relational theory (as before), and a *domain theory*, which formalizes the behavior of the domains of quantification. This second theory requires the introduction of *labelled terms*, $w{:}t$, expressing the existence of term t at world w. Thus $\vdash w{:}\forall x.A$ iff $\vdash w{:}A[t/x]$ for all t such that $\vdash w{:}t$. This formulation naturally suggests adopting the quantifier rules of free logic [5], and we show later that the previously mentioned problems for Hilbert-style QMLs based on free logic do not apply in our approach. By appropriate instantiation of these two theories, it is possible to present the predicate extensions (with varying, increasing, decreasing, or constant domains) of the PMLs belonging to a large class, which includes the Geach hierarchy and hence contains logics like K, D, T, B, S4, S4.2, KD45, and S5.

The metatheory of our QMLs is also modular. The use of explicit labels leads to a modular and uniform proof of soundness and completeness for all the logics we consider, which differs from the standard one: we provide a new kind of canonical model construction that accounts for the explicit formalization of labels, of the accessibility relation, and of the properties of the domains of quantification. This means that our presentations are sound and complete with respect to the appropriate Kripke semantics, and are thus equivalent to the corresponding Hilbert systems only when these are themselves complete with respect to the same semantics.

Due to lack of space, discussions and proofs have been omitted or considerably shortened; a detailed account is given in our technical report [3]. Furthermore, here we do not discuss proof theoretic results and implementational aspects, but point out that (i) the proof theoretic results for labelled PMLs carry over to QMLs, so that proof search may be restricted (normal proofs have the subformula property) and the effectiveness of theorem proving can be improved; (ii) our approach to implementing labelled modal logics in a logical framework carries over directly to the quantified case. In fact, all the proofs of modal theorems given in this paper (e.g., at the end of § 2) have been machine checked in our implementation of QMLs in the Isabelle logical framework.

2 A modular presentation of QMLs

We introduce a labelled natural deduction presentation of QMLs, where we use labels to associate possible worlds with terms and formulae. Let W be a set of *labels* and R a binary relation over W. If w and w' are labels, then $w\ R\ w'$

$$\dfrac{\begin{array}{c}[w_i{:}A \to \bot]\\ \vdots\\ w_j{:}\bot\end{array}}{w_i{:}A}\ \bot E \qquad \dfrac{\begin{array}{c}[w{:}A]\\ \vdots\\ w{:}B\end{array}}{w{:}A \to B} \to I \qquad \dfrac{w{:}A \to B \quad w{:}A}{w{:}B} \to E$$

$$\dfrac{\begin{array}{c}[w_i\ R\ w_j]\\ \vdots\\ w_j{:}A\end{array}}{w_i{:}\Box A}\ \Box I \qquad \dfrac{w_i{:}\Box A \quad w_i\ R\ w_j}{w_j{:}A}\ \Box E \qquad \dfrac{\begin{array}{c}[w{:}t]\\ \vdots\\ w{:}A[t/x]\end{array}}{w{:}\forall x.A}\ \forall I \qquad \dfrac{w{:}\forall x.A \quad w{:}t}{w{:}A[t/x]}\ \forall E$$

In $\Box I$, w_j is different from w_i and does not occur in any assumption on which $w_j{:}A$ depends other than $w_i\ R\ w_j$. In $\forall I$, t does not occur in any assumption on which $w{:}A[t/x]$ depends other than $w{:}t$.

Fig. 1. The rules of QK

is a *relational formula* (*rwff*). If t is a constant c or a variable x, then $w{:}t$ is *labelled term* (*lterm*). If A is a modal formula built from atomic propositions (i.e. predicates applied to terms) and the connectives and quantifiers $\bot$, $\to$, $\Box$, $\forall$, then $w{:}A$ is a *labelled formula* (*lwff*). We can define the other connectives and quantifiers in the usual manner, e.g. $\neg A$ as $A \to \bot$, $\Diamond A$ as $\neg\Box\neg A$, and $\exists x.A$ as $\neg\forall x.\neg A$. Henceforth, we assume that the variable w ranges over labels, t ranges over terms, and A, B range over modal formulae. Further, let $\Gamma = \{w_1{:}A_1, \ldots, w_n{:}A_n\}$, $\Delta = \{w_1\ R\ w_2, \ldots, w_l\ R\ w_m\}$ and $\Theta = \{w_1{:}t_1, \ldots, w_j{:}t_j\}$ be arbitrary sets of lwffs, rwffs and lterms, respectively. All metavariables may be annotated with subscripts or superscripts.

The rules given in Fig. 1 determine QK, the base natural deduction system formalizing quantified K. The rules for $\forall$ are a labelled version of the rules of free logic [5], and, as in free logic, $w{:}\forall x.A \to \exists x.A$ is provable only under the assumption $w{:}t$, stating that the domain of quantification of w is non-empty. Note the symmetry between the rules for $\Box$ and those for $\forall$; this reinforces the role of $\Box$, and of modal logics in general, "as a replacement for the more powerful machinery of quantified classical logic, at least in some cases" [7, p. 377].

Relational theories

Different QMLs are obtained from the base logic QK by placing conditions on the accessibility relation R in the Kripke frame; e.g. we get the logic QT from QK by adding that R is reflexive, and then QS4 from QT by further adding transitivity. We formalize particular QMLs by extending QK with relational theories, which axiomatize properties of R. However, not all modal axioms can be axiomatized in a first-order setting (e.g. the McKinsey axiom $\Box\Diamond A \to \Diamond\Box A$, or the Löb axiom $\Box(\Box A \to A) \to \Box A$) and hence there is an important decision that we must make: Should our relational theories be axiomatized in higher-order logic (and

Transitivity, 4: $\Box A \rightarrow \Box\Box A$

$$\frac{w_i\ R\ w_j \quad w_j\ R\ w_k}{w_i\ R\ w_k}\ \textit{trans}$$

Euclideaness, 5: $\Diamond A \rightarrow \Box\Diamond A$

$$\frac{w_i\ R\ w_j \quad w_i\ R\ w_k}{w_j\ R\ w_k}\ \textit{eucl}$$

Reflexivity, T: $\Box A \rightarrow A$

$$\frac{}{w_i\ R\ w_i}\ \textit{refl}$$

Convergency, 2: $\Diamond\Box A \rightarrow \Box\Diamond A$

$$\frac{w_i\ R\ w_j \quad w_i\ R\ w_k}{w_j\ R\ g(w_i, w_j, w_k)}\ \textit{conv1} \qquad \frac{w_i\ R\ w_j \quad w_i\ R\ w_k}{w_k\ R\ g(w_i, w_j, w_k)}\ \textit{conv2}$$

Fig. 2. Some properties of R, axioms, and relational rules (g is a Skolem function)

thus allow the formalization of all quantified modal logics), first-order logic, or some subset thereof? We showed in [4] that there are tradeoffs in formalization: different choices require different formalizations of the base modal logic and have different metatheoretic properties (see also [3]). In [4] we settled on those logics with an accessibility relation axiomatizable in terms of Horn clauses, a choice we repeat here. We choose to admit precisely those properties of R that can be formulated by a collection of *relational rules*, i.e. (Horn) rules of the form

$$\frac{p_1\ R\ q_1 \quad \cdots \quad p_m\ R\ q_m}{p_0\ R\ q_0}$$

where $m \geq 0$, and p_i and q_i are terms built from labels $w_1, \ldots, w_n$ and function symbols.[3] A *relational theory* $\mathcal{T}$ is a theory generated by a set of such rules.

Relational rules suffice to formalize the predicate extensions of the most common PMLs, for example those in the Geach hierarchy. Examples of some common characteristic axioms and the corresponding relational rules are given in Fig. 2. The QML $L = \mathrm{QK} + \mathcal{T}$ is obtained by extending QK with a given relational theory $\mathcal{T}$; this extension is represented by the horizontal arrows in Fig. 3(a). We adopt the convention of naming the logic $\mathrm{QK}+\mathcal{T}$ as QKAx, where Ax is a string consisting of the standard names of the characteristic axioms corresponding to the relational rules generating $\mathcal{T}$; e.g. QKT4 identifies the logic also known as QS4. Various combinations of relational rules define therefore different QMLs, including QK, QD, QT, QB, QS4, QS4.2, QKD45 and QS5.

Domain theories

So far, we have made no commitments to the relationship between the domains of quantification in the different worlds, i.e. the domains of $\mathrm{QK}+\mathcal{T}$ are *varying.* We can impose semantic conditions on them by, e.g., requiring that, when moving to

[3] Some properties of R, e.g. seriality and convergency, can be expressed as relational rules only after the introduction of Skolem function constants. By the theorem on functional extensions [15, p. 55], the introduction of such constants is a conservative extension; see [3,4].

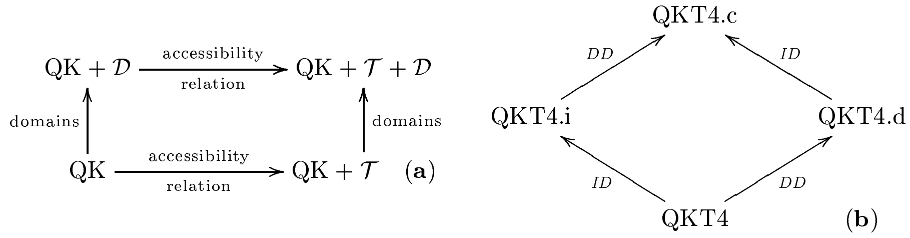

Fig. 3. A hierarchy of labelled QMLs (a); QKT4.l (b)

an accessible world, objects persist (the domains are *increasing*), are not created (and possibly deleted, i.e. the domains are *decreasing*), or stay the same (the domains are *constant*). The conditions for increasing and decreasing domains are formalized by the (Horn) rules *ID* and *DD* respectively:

$$\frac{w_i \; R \; w_j \quad w_i{:}t}{w_j{:}t}\; ID \qquad\qquad \frac{w_i \; R \; w_j \quad w_j{:}t}{w_i{:}t}\; DD$$

Different combinations of these rules define different QMLs: the logic $L = \mathrm{QK} + \mathcal{T} + \mathcal{D}$ is obtained by extending $\mathrm{QK} + \mathcal{T}$ with a given theory $\mathcal{D}$ of the domains of quantification (or *domain theory*, for short), generated by a subset of $\{ID, DD\}$; this extension is represented by the vertical arrows in Fig. 3(a). This yields the two-dimensional uniformity of the proof system motivated in the introduction. (Uniform proofs of soundness and completeness are given in § 3.) We extend the above convention and name the logic $\mathrm{QK} + \mathcal{T} + \mathcal{D}$ as QK$Ax.l$, where l represents the conditions imposed on the domains. We write QKAx when $\mathcal{D}$ is empty, as done above; QKAx.i (respectively QKAx.d) when $\mathcal{D}$ is generated by *ID* (respectively *DD*); QKAx.c when $\mathcal{D}$ is generated by *ID* and *DD*. We can thus specify one of four related QMLs simply by instantiating $\mathcal{D}$; as shown in Fig. 3(b), we can specify QKT4 (QS4) with domains that are varying (QKT4), increasing (QKT4.i), decreasing (QKT4.d), or constant (QKT4.c).

This is *not* the case in Hilbert systems for QMLs, where the domains are committed to being increasing, since the classical rules for $\forall$ automatically enforce the *Converse Barcan Formula* (CBF), $\Box\forall x.A \to \forall x.\Box A$, which corresponds to the increasing domains condition [7, p. 426]. Constant domains are then obtained by further adding as an axiom the *Barcan Formula* (BF), $\forall x.\Box A \to \Box\forall x.A$, which corresponds to the decreasing domains condition. Hilbert-style proof systems for QMLs with varying domains can be given by substituting the classical quantifier rules with the rules of free logic, as done by Garson in [9]. However, Garson also shows that his completeness proof is not general, and fails for some QMLs, e.g. QB; we return to this at the end of § 3.

Definition 1. An *L-derivation* of an lwff, rwff or lterm φ from Γ, Δ, Θ, is a tree formed using the rules in L, ending with φ and depending only on Γ, Δ, Θ. We write $\Gamma, \Delta, \Theta \vdash_L \varphi$ when φ can be so derived. φ is a *theorem* of L, $\vdash_L \varphi$, if it is L-derivable when Γ, Δ, Θ are all empty.

Fact 2. *Due to the separations enforced between the base logic, the relational theory, and the domain theory, we have (i)* $\Gamma, \Delta, \Theta \vdash_L w_i \, R \, w_j$ *iff* $\Delta \vdash_L w_i \, R \, w_j$, *and (ii)* $\Gamma, \Delta, \Theta \vdash_L w{:}t$ *iff* $\Delta, \Theta \vdash_L w{:}t$.

That is, while lwffs are derived from lwffs, rwffs *and* lterms, i.e. $\Gamma, \Delta, \Theta \vdash_L w{:}A$, (i) rwffs are derived from rwffs *alone*, and (ii) lterms are derived from rwffs and lterms *but not* from lwffs. In comparison, note that in approaches based on *semantic embedding*, also called *semantics-based translations*, e.g. [1,13], a formula of quantified modal logic is translated into a formula in first-order predicate logic and shown to be true (or false) in a first-order theory formalizing the semantics of the modalities and quantification domains. However, with these translations all structure is lost as relations, predicates and terms are flattened into formulae of first-order logic, and derivations of lwffs are mingled with derivations of rwffs and lterms.

As an example of a derivation, we show that CBF is a theorem of (any extension of) QK.i:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[w{:}\Box\forall x.A]^3 \quad [w \, R \, w_1]^1}{w_1{:}\forall x.A}\,\Box E \qquad \dfrac{[w \, R \, w_1]^1 \quad [w{:}t]^2}{w_1{:}t}\,ID}{w_1{:}A[t/x]}\,\forall E}{w{:}\Box A[t/x]}\,\Box I^1}{w{:}\forall x.\Box A}\,\forall I^2}{w{:}\Box\forall x.A \to \forall x.\Box A}\,{\to} I^3$$

Note that we associate discharged assumptions with rule applications.[4] In a similar manner, we can prove BF in QK.d, (1) below, and other common examples:

(1)	$\vdash_{\mathrm{QK.d}} w{:}\forall x.\Box A \to \Box\forall x.A$	(2)	$\vdash_{\mathrm{QKB.i}} w{:}\forall x.\Box A \to \Box\forall x.A$
(3)	$\vdash_{\mathrm{QK.d}} w{:}\Diamond\exists x.A \to \exists x.\Diamond A$	(4)	$\vdash_{\mathrm{QK.i}} w{:}\exists x.\Diamond A \to \Diamond\exists x.A$

Some remarks are in order. *ID* and *DD* are interderivable when the rule $\dfrac{w_i \, R \, w_j}{w_j \, R \, w_i}$ is present, i.e. when the accessibility relation is *symmetric* (symmetry corresponds to the modal axiom B: $A \to \Box\Diamond A$). (2) shows that a QML with a symmetric accessibility relation and with increasing domains (QKB.i) validates BF, and has therefore constant domains; similarly we can show that CBF is a theorem of QKB.d. By (3) and (4), $\Diamond\exists x.A$ and $\exists x.\Diamond A$ are equivalent in QK.c.

3 Soundness and completeness of labelled QMLs

Definition 3. A *model* for a QML L is a tuple $\mathfrak{M} = (\mathfrak{W}, \mathfrak{R}, \mathfrak{D}, \mathfrak{q}, \mathfrak{a})$, where $\mathfrak{W}$ is a non-empty set of worlds; $\mathfrak{R} \subseteq \mathfrak{W} \times \mathfrak{W}$; $\mathfrak{D}$ is a set of objects; $\mathfrak{q}$ is a mapping

[4] CBF is not a theorem of QK, because *ID* is missing and the application of $\forall I$ at world w cannot discharge $w_1{:}t$. A formal proof of this can be given by exploiting the results on proof normalization discussed in [3], to show that there is no normal proof (and, a fortiori, no proof at all) of CBF in QK.

assigning to each member w of $\mathfrak{W}$ some subset of $\mathfrak{D}$, the *domain of quantification* of w; $\mathfrak{a}$ is function interpreting the terms and predicate letters by assigning them the corresponding kind of intensions with respect to $\mathfrak{W}$ and $\mathfrak{D}$. $\mathfrak{a}(w,t)$ is an element of $\mathfrak{D}$, and for an n-ary predicate letter P, $\mathfrak{a}(w,P)$ is a set of ordered n-tuples, $\langle a_1,\dots,a_n\rangle$, where each $a_i \in \mathfrak{D}$. Moreover, for every $w_i, w_j \in \mathfrak{W}$ such that $(w_i,w_j) \in \mathfrak{R}$, the domains of $\mathfrak{M}$ are: *increasing* iff $\mathfrak{q}(w_i) \subseteq \mathfrak{q}(w_j)$; *decreasing* iff $\mathfrak{q}(w_i) \supseteq \mathfrak{q}(w_j)$; *constant* iff $\mathfrak{q}(w_i) = \mathfrak{q}(w_j)$. Otherwise, the domains are *varying*.

Note that we only consider rigid designators [7,9], where $\mathfrak{a}$ is such that $\mathfrak{a}(w_i,t) = \mathfrak{a}(w_j,t)$ for all $w_i, w_j \in \mathfrak{W}$. Moreover, our models do not contain functions corresponding to possible Skolem functions in the signature; when such constants are present, the appropriate Skolem expansion of the model is required [16, p. 137].

Call the ordered triple (Γ,Δ,Θ) a *proof context* (*pc*). We write $w{:}A \in (\Gamma,\Delta,\Theta)$ when $w{:}A \in \Gamma$; $w\ R\ w' \in (\Gamma,\Delta,\Theta)$ when $w\ R\ w' \in \Delta$; and $w{:}t \in (\Gamma,\Delta,\Theta)$ when $w{:}t \in \Theta$. We say that a label w *occurs in* (Γ,Δ,Θ), and, continuing our slight notational abuse, write $w \in (\Gamma,\Delta,\Theta)$, if there exists an A such that $w{:}A \in \Gamma$, or a w' such that $w\ R\ w' \in \Delta$ or $w\ R\ w' \in \Delta$, or a t such that $w{:}t \in \Theta$. $t \in (\Gamma,\Delta,\Theta)$ is defined analogously. We now define truth for ground lterms, rwffs and lwffs, where truth for lterms indicates definedness, truth for rwffs indicates accessibility, and quantifiers are treated in each world as ranging over the domain of that world only.

Definition 4. We define a ground lterm, rwff or lwff φ to be *true* in a model $\mathfrak{M}$, in symbols $\models^{\mathfrak{M}} \varphi$, as follows. First, we ensure that we have a name for each object in the domain $\mathfrak{D}$ of $\mathfrak{M}$ by extending, if necessary, the class of terms with a new constant c_o for each $o \in \mathfrak{D}$, and then extending $\mathfrak{a}$ so that $\mathfrak{a}(w,\mathrm{c}_o) = o$, as is standard [12,15]. Then, we define $\models^{\mathfrak{M}}$ to be the smallest relation satisfying:

$$\begin{array}{ll}
\models^{\mathfrak{M}} w{:}t & \text{iff } \mathfrak{a}(w,t) \in \mathfrak{q}(w) \\
\models^{\mathfrak{M}} w_i\ R\ w_j & \text{iff } (w_i,w_j) \in \mathfrak{R} \\
\models^{\mathfrak{M}} w{:}P(t_1,\dots,t_n) & \text{iff } \langle \mathfrak{a}(w,t_1),\dots,\mathfrak{a}(w,t_n)\rangle \in \mathfrak{a}(w,P) \\
\models^{\mathfrak{M}} w{:}A \to B & \text{iff } \models^{\mathfrak{M}} w{:}A \text{ implies } \models^{\mathfrak{M}} w{:}B \\
\models^{\mathfrak{M}} w{:}\Box A & \text{iff for all } w_i,\ \models^{\mathfrak{M}} w\ R\ w_i \text{ implies } \models^{\mathfrak{M}} w_i{:}A \\
\models^{\mathfrak{M}} w{:}\forall x.A & \text{iff for all } t,\ \models^{\mathfrak{M}} w{:}t \text{ implies } \models^{\mathfrak{M}} w{:}A[t/x]
\end{array}$$

By extension, $\models^{\mathfrak{M}} (\Gamma,\Delta,\Theta)$ means that $\models^{\mathfrak{M}} \varphi$ for all $\varphi \in (\Gamma,\Delta,\Theta)$; $\Gamma,\Delta,\Theta \models^{\mathfrak{M}} \varphi$ means that $\models^{\mathfrak{M}} (\Gamma,\Delta,\Theta)$ implies $\models^{\mathfrak{M}} \varphi$ in the model $\mathfrak{M}$; and $\Gamma,\Delta,\Theta \models \varphi$ means that $\Gamma,\Delta,\Theta \models^{\mathfrak{M}} \varphi$ for all models $\mathfrak{M}$.

Truth for lwffs containing other connectives and quantifiers can be defined in the usual manner, e.g. $\models^{\mathfrak{M}} w{:}\exists x.A$ iff $\not\models^{\mathfrak{M}} w{:}\forall x.\neg A$, since $\not\models^{\mathfrak{M}} w{:}\bot$ for every w. The explicit embedding of properties of the models, and the possibility of explicitly reasoning about them, via lterms and rwffs, require us to consider soundness and completeness results also for lterms and rwffs. We define:

Definition 5. The QML $L = \mathrm{QK} + \mathcal{T} + \mathcal{D}$ is *sound* iff (i) $\Delta \vdash_L w_i\ R\ w_j$ implies $\Delta \models w_i\ R\ w_j$, (ii) $\Delta,\Theta \vdash_L w{:}t$ implies $\Delta,\Theta \models w{:}t$, and (iii) $\Gamma,\Delta,\Theta \vdash_L w{:}A$ implies $\Gamma,\Delta,\Theta \models w{:}A$. L is *complete* iff the converses hold.

Theorem 6. *$L = \mathrm{QK} + \mathcal{T} + \mathcal{D}$ is sound and complete.*

We only sketch the main proof ideas; see [3] for a full account. Soundness follows by induction on the structure of the L-derivations. The base cases, e.g. $w{:}A \in (\Gamma, \Delta, \Theta)$, are trivial, and there is a step case for each inference rule of L. As an example, we give the case for $\Box I$; the cases for the other rules follow analogously. Let $\mathfrak{M}_L$ be an arbitrary model for L and consider an application of $\Box I$

$$\dfrac{\begin{matrix}[w\ R\ w_i]\\ \Pi \\ w_i{:}A\end{matrix}}{w{:}\Box A}\ \Box I$$

where Π is the L-derivation $\Gamma, \Delta_1, \Theta \vdash_L w_i{:}A$, with $\Delta_1 = \Delta \cup \{w\ R\ w_i\}$. By the induction hypothesis, $\Gamma, \Delta_1, \Theta \vdash_L w_i{:}A$ implies $\Gamma, \Delta_1, \Theta \models w_i{:}A$. Assume $\models^{\mathfrak{M}_L} (\Gamma, \Delta, \Theta)$. By the restriction on the application of $\Box I$, we can extend Δ to $\Delta' = \Delta \cup \{w\ R\ w'\}$ for an arbitrary $w' \notin (\Gamma, \Delta, \Theta)$, and assume $\models^{\mathfrak{M}_L} \Delta'$. Since $\models^{\mathfrak{M}_L} \Delta'$ implies $\models^{\mathfrak{M}_L} \Delta_1$, the induction hypothesis yields $\models^{\mathfrak{M}_L} w_i{:}A$, i.e. $\models^{\mathfrak{M}_L} w'{:}A$ for an arbitrary $w' \notin (\Gamma, \Delta, \Theta)$ such that $\models^{\mathfrak{M}_L} w\ R\ w'$. We conclude $\models^{\mathfrak{M}_L} w{:}\Box A$ by Definition 4.

Completeness follows by a Henkin-style proof, where a canonical model $\mathfrak{M}_L^C = (\mathfrak{W}_L^C, \mathfrak{R}_L^C, \mathfrak{D}_L^C, \mathfrak{q}_L^C, \mathfrak{a}_L^C)$ is built to show the contrapositives of the conditions in Definition 5, i.e. (i') $\Delta \nvdash_L w_i\ R\ w_j$ implies $\Delta \not\models^{\mathfrak{M}_L^C} w_i\ R\ w_j$, (ii') $\Delta, \Theta \nvdash_L w{:}t$ implies $\Delta, \Theta \not\models^{\mathfrak{M}_L^C} w{:}t$, and (iii') $\Gamma, \Delta, \Theta \nvdash_L w{:}A$ implies $\Gamma, \Delta, \Theta \not\models^{\mathfrak{M}_L^C} w{:}A$. In particular, given the presence of labelled formulae and explicit assumptions on the relations between the labels and their domains of quantification (i.e. Δ and Θ), in our version of the Lindenbaum lemma (Lemma 8 below) we consider a 'global' saturated set of labelled formulae, where consistency is also checked against the additional assumptions in Δ and Θ, instead of the usual saturated sets of unlabelled formulae. Moreover, given a logic $L = \mathrm{QK} + \mathcal{T} + \mathcal{D}$ and a proof context (Γ, Δ, Θ), we consider the deductive closure Δ_L of Δ under L, and the deductive closure $\Theta_{L,\Delta}$ of Θ under L with respect to Δ, i.e.

$$\Delta_L =_{def} \{w_i\ R\ w_j \mid \Delta \vdash_L w_i\ R\ w_j\} \quad \text{and} \quad \Theta_{L,\Delta} =_{def} \{w{:}t \mid \Delta, \Theta \vdash_L w{:}t\}\,.$$

Definition 7. (Γ, Δ, Θ) is *saturated* iff (i) (Γ, Δ, Θ) is consistent, i.e. $\Gamma, \Delta, \Theta \nvdash_L w{:}\bot$ for every w; (ii) $\Delta = \Delta_L$ and $\Theta = \Theta_{L,\Delta}$; (iii) for every lwff $w{:}A$, either $w{:}A \in \Gamma$ or $w{:}\neg A \in \Gamma$; (iv) for every w, if $\Gamma, \Delta, \Theta \vdash_L w{:}t$ implies $\Gamma, \Delta, \Theta \vdash_L w{:}A[t/x]$ for every t, then $\Gamma, \Delta, \Theta \vdash_L w{:}\forall x.A$; and (v) for every w, if $\Gamma, \Delta, \Theta \vdash_L w\ R\ w_i$ implies $\Gamma, \Delta, \Theta \vdash_L w_i{:}B$ for every w_i, then $\Gamma, \Delta, \Theta \vdash_L w{:}\Box B$.

In the Lindenbaum lemma for first-order logic, a saturated set of formulae is inductively built by adding for every formula $\neg\forall x.A$ a *witness* to its truth, namely a formula $\neg A[c/x]$ for some new individual constant c. This ensures that the set is ω-complete, a property equivalent to condition (iv) in Definition 7. A similar procedure applies here not only for every lwff $w{:}\neg\forall x.A$, but also for every lwff $w{:}\neg\Box A$ (cf. (v) in Definition 7); i.e. together with $w{:}\neg\Box A$, we consistently add $v{:}\neg A$ and $w\ R\ v$ for some new v, which acts as a 'witness world' to the truth of $w{:}\neg\Box A$. This ensures that the saturated pc (Γ, Δ, Θ) is such that $w{:}\Box B \in \Gamma$ iff $w\ R\ w_i \in \Delta$ implies $w_i{:}B \in \Gamma$ for every w_i, as shown in Lemma 9 below.[5]

Lemma 8. *Every consistent pc* (Γ, Δ, Θ) *can be extended to a saturated pc.*

Proof (Sketch). We first extend the language of the logic L with infinitely many new constants for witness terms and witness worlds. Let s range over the new constants for witness terms, and r range over the original terms and over the new constants. Analogously, let v range over the new constants for witness worlds, and u range over labels and over the new constants. All these may be subscripted. Let $l_1, l_2, \ldots$ be an enumeration of all lwffs in the extended language. Starting from $(\Gamma_0, \Delta_0, \Theta_0) = (\Gamma, \Delta, \Theta)$, we inductively build a sequence of consistent pcs by defining $(\Gamma_{i+1}, \Delta_{i+1}, \Theta_{i+1})$ to be:

- $(\Gamma_i, \Delta_i, \Theta_i)$, if $(\Gamma_i \cup \{l_{i+1}\}, \Delta_i, \Theta_i)$ is inconsistent; else
- $(\Gamma_i \cup \{l_{i+1}\}, \Delta_i, \Theta_i)$, if l_{i+1} is neither $u{:}\neg\Box A$ nor $u{:}\neg\forall x.A$; else
- $(\Gamma_i \cup \{u{:}\neg\forall x.A, u{:}\neg A[s/x]\}, \Delta_i, \Theta_i \cup \{u{:}s\})$, for an $s \notin (\Gamma_i \cup \{u{:}\neg\forall x.A\}, \Delta_i, \Theta_i)$, if l_{i+1} is $u{:}\neg\forall x.A$; else
- $(\Gamma_i \cup \{u{:}\neg\Box A, v{:}\neg A\}, \Delta_i \cup \{u\ R\ v\}, \Theta_i)$, for a $v \notin (\Gamma_i \cup \{u{:}\neg\Box A\}, \Delta_i, \Theta_i)$, if l_{i+1} is $u{:}\neg\Box A$.

A saturated pc is then $(\Gamma^*, \Delta^*, \Theta^*) =_{def} (\bigcup_{i\geq 0} \Gamma_i, (\bigcup_{i\geq 0} \Delta_i)_L, (\bigcup_{i\geq 0} \Theta_i)_{L,\Delta})$.

Lemma 9. *Let* $(\Gamma^*, \Delta^*, \Theta^*)$ *be a saturated pc.*

(i) $\Gamma^*, \Delta^*, \Theta^* \vdash_L \varphi$ *iff* $\varphi \in (\Gamma^*, \Delta^*, \Theta^*)$, *where* φ *is an lterm, rwff or lwff.*
(ii) $u{:}A \to B \in \Gamma^*$ *iff* $u{:}A \in \Gamma^*$ *implies* $u{:}B \in \Gamma^*$.
(iii) $u_i{:}\Box B \in \Gamma^*$ *iff for all* u_j, $u_i\ R\ u_j \in \Delta^*$ *implies* $u_j{:}B \in \Gamma^*$.
(iv) $u{:}\forall x.A \in \Gamma^*$ *iff for all* r, $u{:}r \in \Theta^*$ *implies* $u{:}A[r/x] \in \Gamma^*$.

Definition 10. Given a saturated pc $(\Gamma^*, \Delta^*, \Theta^*)$, we define the *canonical model* $\mathfrak{M}_L^C$ for the logic L as follows: $\mathfrak{W}_L^C = \{u \mid u \in (\Gamma^*, \Delta^*, \Theta^*)\}$; $(u_i, u_j) \in \mathfrak{R}_L^C$ iff $u_i\ R\ u_j \in \Delta^*$; $\mathfrak{a}(u, r) = r$, and $\langle r_1, \ldots, r_n\rangle \in \mathfrak{a}(u, P)$ iff $u{:}P(r_1, \ldots, r_n) \in \Gamma^*$, for P an n-ary predicate; $\mathfrak{q}(u) = \{\mathfrak{a}(u, r) \mid u{:}r \in \Theta^*\}$; $\mathfrak{D} = \bigcup_{u\in(\Gamma^*,\Delta^*,\Theta^*)} \mathfrak{q}(u)$.

The standard definition of $\mathfrak{R}_L^C$, i.e. $(u_i, u_j) \in \mathfrak{R}_L^C$ iff $\{A \mid \Box A \in u_i\} \subseteq u_j$, is not applicable in our setting, since $\{A \mid \Box A \in u_i\} \subseteq u_j$ does *not* imply $\vdash_L u_i\ R\ u_j$. We would therefore lose completeness for rwffs, since there would be cases, e.g. if $L = \mathrm{QK}$ and $\Delta = \{\}$, where $\nvdash_L u_i\ R\ u_j$ but $(u_i, u_j) \in \mathfrak{R}_L^C$ and thus $\models^{\mathfrak{M}_L^C} u_i\ R\ u_j$. Hence, we instead define $(u_i, u_j) \in \mathfrak{R}_L^C$ iff $u_i\ R\ u_j \in \Delta^*$; note that therefore $u_i\ R\ u_j \in \Delta^*$ implies $\{A \mid \Box A \in u_i\} \subseteq u_j$. The deductive closures of Δ^* and Θ^* ensure not only completeness for rwffs and lterms, but also that the conditions on $\mathfrak{R}_L^C$ and $\mathfrak{D}_L^C$ are satisfied, so that $\mathfrak{M}_L^C$ is really a model for L.

We immediately have (i) $u_i\ R\ u_j \in \Delta^*$ iff $\Delta^* \models^{\mathfrak{M}_L^C} u_i\ R\ u_j$, and (ii) $u{:}r \in \Theta^*$ iff $\Delta^*, \Theta^* \models^{\mathfrak{M}_L^C} u{:}r$. By induction on the size of $u{:}A$, we can then prove that

[5] In the standard completeness proof for unlabelled modal logics, $\mathfrak{W}_L^C$ is defined to be the set of all saturated sets, and it is possible to show that if $w \in \mathfrak{W}_L^C$ and $\neg\Box A \in w$, then there is a $w' \in \mathfrak{W}_L^C$ accessible from w such that $\neg A \in w'$.

$u{:}A \in \Gamma^*$ iff $\Gamma^*, \Delta^*, \Theta^* \models^{\mathfrak{M}_L^C} u{:}A$, and thus conclude the proof of completeness, by showing (i'), (ii') and (iii') above.

Some remarks and comparisons are in order. Our proof is modular: the same method applies uniformly to every logic L. As explained previously, this is *not* the case for the completeness proof of unlabelled QMLs based on free logic [9,11]. Garson himself points out that his proof "lacks generality" [9, pp. 280-281], since (i) it does not work for systems with constant domains, and (ii) it is not general with respect to the underlying PML (although there are tricks one can use to overcome the difficulties for particular systems). As we have shown, none of these problems applies in our approach. Most importantly, being complete, our QMLs are adequate presentations of the Kripke semantics, and are thus equivalent to the corresponding Hilbert systems only when these are themselves complete with respect to the same semantics. For example, by the incompleteness results in [11], QKT42.i is equivalent to the Hilbert system for QS4.2 since they are both complete with respect to reflexive, transitive and convergent Kripke models with increasing domains; on the other hand, QKT42.c is not equivalent to the Hilbert system for QS4.2 + BF, since the latter is incomplete.

4 Related Work

In the introduction we described various problems that arise in traditional approaches to QMLs based upon Hilbert formalizations, and throughout the paper we have argued how these problems are solved in our approach. We now compare our work with approaches based on sequent or tableau systems, and on embeddings of modal logics in first-order logic.

Fitting introduces cut-free sequent systems for QMLs in [6,7], giving 'standard' systems for non-symmetric logics with increasing domains, and then, to capture the other conditions, introducing *prefixes*. These allow him to formulate sequent systems for a class of QMLs (including symmetric logics like S5) with varying, increasing, or constant domains. In prefixed systems, the different properties of the domains are expressed by imposing different side conditions on the applicability of the quantifier rules; analogously, the properties of the accessibility relation require different side conditions on the rules for the modalities. The main disadvantage of these systems, apart from the fact that they do not capture decreasing domains, is that their formalizations often requires considerable ingenuity, and the rules for the modalities and quantifiers can be awkward, carrying side conditions on the complete set of assumptions.[6] Thus, unlike our approach, which leads to simple implementations, these systems cannot be easily formalized in standard logical frameworks such as Isabelle or the Edinburgh LF.

Our work is closely related to approaches based on semantic embeddings, e.g. [1,8,13], where a formula of quantified modal logic is translated into a for-

[6] See Avron's discussion of degrees of impurity of natural deduction systems in [2, §5.5]. Note also that prefixed sequent systems and prefixed tableau systems could be modified to cover decreasing domains. To do so, however, one would need to replace the standard quantifier rules with rules similar to ours.

mula in first-order predicate logic and shown to be true (or false) in a first-order theory formalizing the semantics of the modalities and domains of quantification. For example, $\Box(A \wedge B)$ is translated into some first-order formula like $\forall w.\, R(0,w) \rightarrow (A(w) \wedge B(w))$, and there may be additional axioms characterizing the accessibility relation R and the domains of quantification. Ohlbach [13], for example, provides a general framework for carrying out such translations and reasoning about their soundness and completeness; translations are defined by morphisms on formulae and these are shown sound and complete by providing morphisms on interpretations. Our work differs from embedding based approaches in the nature of the translations, the metatheoretic properties that hold, and how they are proved. First, we separate, rather than combine, reasoning about relations, predicates and terms (cf. Fact 2). With semantic embedding there is no formal distinction between lwffs, rwffs and lterms or separation between relational and first-order reasoning. Second, rather than using interpretation morphisms and building on top of the semantics of first-order logic, we directly define deductive systems for QMLs and show their completeness by a parameterized canonical model construction. Finally, as shown in [3], the normal form of a derivation in our approach preserves important extra structure compared to the normal form of a derivation in the translation approach.

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