# On the Rectilinear Crossing Number of Complete Graphs

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## Abstract

We prove a lower bound of $0.3288\binom{n}{4}$ for the rectilinear crossing number $\overline{\mathrm{cr}}(K_n)$ of a complete graph on $n$ vertices, or in other words, for the minimum number of convex quadrilaterals in any set of $n$ points in general position in the Euclidean plane. As we see it, the main contribution of this paper is not so much the concrete numerical improvement over earlier bounds, as the novel method of proof, which is not based on bounding $\overline{\mathrm{cr}}(K_n)$ for some small $n$.

## 1 Introduction

The *rectilinear crossing number* $\overline{\mathrm{cr}}(G)$ of a graph $G$ is the minimal number of crossings in any straight-edge drawing of $G$ in the Euclidean plane. We refrain here from attempting to give an overview of the vast area of crossing numbers and their applications in discrete geometry and theoretical computer science, and refer the interested reader to the survey article by Pach and Tóth [8], to Sections 4.3 and 4.4 of Matoušek's recent textbook [6], and to the extensive on-line bibliography by Vrťo [7].

Here, we consider the rectilinear crossing number of complete graphs. Even for this very specific set of examples, the exact values of $\overline{\mathrm{cr}}(K_n)$ are only known up to $n = 12$ (see [1]), and even the precise asymptotics remain to be elucidated.

First asymptotic bounds were given by Jensen [5] and by Singer [9] in the early seventies, and in recent years, the problem has been studied with renewed interest [2, 3, 1]. The currently best bounds of

$$(1)\qquad 0.3115\binom{n}{4} < \overline{\mathrm{cr}}(K_n) < 0.3807\binom{n}{4}$$

(for sufficiently large $n$) were found by Aichholzer, Aurenhammer, and Krasser [1] in 2002.

Even if one does not know or care about crossing numbers, the problem is a natural question about point configurations: What is the minimum number of convex quadrilaterals (4-tuples in convex position) in any set of $n$ points in general position in the Euclidean plane (here, *general position* means that no three points are collinear)?

It will be convenient to reformulate this in terms of probabilities. Let $P$ be a set of $n$ points in general position in the Euclidean plane. We define

$$(2)\qquad \square(P) := \frac{|\{Q \in \binom{P}{4} : Q \text{ in convex position}\}|}{\binom{n}{4}},$$

and $\square(n) := \min_{|P|=n} \square(P) = \overline{\mathrm{cr}}(K_n)/\binom{n}{4}$.

By double counting those pairs $(p, Q)$ for which $p \in P$ and $Q \in \binom{P\setminus p}{4}$ is in convex position, we see that $\square(n)$ is a monotonically increasing sequence (and obviously bounded from above by 1). Therefore, the limit

$$(3)\qquad \square^* := \lim_{n\to\infty} \square(n).$$

exists. It is called the *rectilinear crossing constant for complete graphs.*

Let us mention that the rectilinear crossing number of complete graphs is also related to the *k-set* problem: For integer $j$, let $e_j = e_j(P)$ be the number of *$j$-edges* of $P$, i.e. of ordered pairs $(p, q)$, $p, q \in P$, $p \neq q$, such that there are exactly $j$ points of $P$ to the left of the oriented line through $p$ and $q$ (note

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that $e_j$ is the number of $(j+1)$-sets of $P$). Then the expected number of edges (or vertices) of the convex hull of a random 4-tuple from $P$ is

$$\Box(P) + 3 = \frac{1}{\binom{n}{4}} \sum_j \binom{j}{2} e_j(P), \tag{4}$$

so $\Box(P)$ can be thought of as the *second moment* of the distribution of $j$-facets. Another connection between $j$-facets and $\Box(P)$ is mentioned in Section 5.

The aim of this note is to prove the following lower bound:

**Theorem 1.**

$$\Box^* \geq (53 + 5\sqrt{13})/216 > 0.3288 \tag{5}$$

Let us digress here for a moment to briefly discuss the lower bound part of the findings of Aichholzer, Aurenhammer, and Krasser, $\Box^* > 0.3115$. This bound is based upon the fact that $\overline{\mathrm{cr}}(K_{11}) = 102$, which they prove by a clever computer enumeration of all "abstract combinatorial configurations" (more precisely, uniform oriented matroids of rank 3) of up to 11 points. It then follows by monotonicity that $\Box^* \geq 102/\binom{11}{4} > 0.3090$. Slight further improvements result from the facts that $\overline{\mathrm{cr}}(K_n)$ and $\binom{n}{4}$ agree in parity for odd $n$ (see [4]) and that $\overline{\mathrm{cr}}(K_n)$ is integer. For example, $\overline{\mathrm{cr}}(K_{14}) \geq \lceil 102\binom{14}{4}/\binom{11}{4} \rceil = \lceil 309.4 \rceil = 310$ yields $\Box(14) > 0.3096$, which in turn implies $\overline{\mathrm{cr}}(K_{15}) \geq 423$, hence $\Box(15) > 0.3098$, and so forth. Earlier Brodsky, Durocher, and Gethner [3] proved "by hand" that $\overline{\mathrm{cr}}(K_{10}) = 62$, which in the fashion outlined above yields $\Box^* > 0.3001$.

Unfortunately, determining $\overline{\mathrm{cr}}(K_n)$ even for small $n$ can be a tedious business (the Brodsky et al. paper is thirty pages long), and enumerating all order types of 12 or more points appears to be a computationally hopeless task, so this road does not seem to lead much farther.

## 2 Preparations

We describe a different approach towards a lower bound for $\Box^*$. The idea is to exploit a certain trade-off, to be made precise later on, between the "global" number of all crossings on the one hand and the "local" number of crossings involving a given point on the other hand.

For $p \in P$, we define

$$\Box(p, P \setminus p) = \frac{|\{T \in \binom{P \setminus p}{3} : T \cup p \text{ in convex position}\}|}{\binom{n-1}{3}}. \tag{6}$$

As a first step, observe that for a set $P$ of $n$ points in general position, we can express $\Box(P)$ as the average

$$\Box(P) = \frac{1}{n} \sum_{p \in P} \Box(p, P \setminus p) \tag{7}$$

of the corresponding "local" quantities.

We now introduce the key ingredient of our proof of Theorem 1, which combines "global" and "local" considerations:

**Definition 1.** Let $S$ be a set of $n+1$ points in general position in the Euclidean plane, and let $v$ be a vertex of $\mathrm{conv}(S)$. We define

$$\Lambda(v, S \setminus v) := \max\left\{\Box(S \setminus v), \Box(v, S \setminus v)\right\}, \tag{8}$$

and $\Lambda(n) := \min_S \max_v \Lambda(v, S \setminus v)$, where the minimum is taken over all sets $S$ of $|S| = n+1$ points in general position and the maximum over all vertices $v$ of $\mathrm{conv}(S)$.

**Lemma 1.**

$$\Box^* = \liminf_{n \to \infty} \Lambda(n). \tag{9}$$

*Proof.* Denote the right-hand side of (9) by $c$. On the one hand, for every $S$ and $v$, $\Lambda(v, S \setminus v) \geq \Box(S \setminus v)$, hence $\Lambda(n) \geq \Box(n)$ and therefore, $c \geq \Box^*$.

On the other hand, fix $\varepsilon > 0$, and choose $n_0 \in \mathbf{N}$ such that $\Lambda(n) \geq c - \varepsilon$ for all $n \geq n_0$.

*Claim A.* Suppose $|S| = n+1 > n_0$. Then there exists a point $p \in S$ such that $\Box(p, S \setminus p) \geq (c-\varepsilon)\binom{n-1}{3}/\binom{n}{3}$. To see this, let $v$ be a vertex of $\mathrm{conv}(S)$ such that $\Lambda(v, S \setminus v) \geq c - \varepsilon$. If $\Box(v, S \setminus v) \geq (c-\varepsilon)$, then $v$ is the point we are looking for. Otherwise, $\Box(S \setminus v) \geq (c-\varepsilon)$, by definition of $\Lambda(v, S \backslash v)$. Thus, by (7), there is some $p \in S \setminus v$ for which

$$\square(p, S \setminus p)\binom{n}{3}/\binom{n-1}{3} \geq \square(p, S \setminus \{v, p\}) \\ \geq \square(S \setminus v) \geq (c - \varepsilon),$$

which proves Claim A.

*Claim B.* For all $n \geq n_0$,

$$\square(n) \geq (c - \varepsilon)\binom{n-1-n_0}{4}/\binom{n}{4}$$

We proceed by induction on $n$. The claim is clearly true for $n = n_0$. Moreover, if $n > n_0$, let $P$ be a set of $n$ points achieving $\square(P) = \square(n)$. By Claim A, there is some point $p \in P$ with $\square(p, P \setminus p) \geq (c - \varepsilon)\binom{n-2}{3}/\binom{n-1}{3}$. By induction, $\square(P \setminus p) \geq (c - \varepsilon)\binom{n-2-n_0}{4}/\binom{n-1}{4}$. Together with the quadrilaterals in which $p$ participates, this yields

$$\binom{n}{4}\square(P) \geq (c - \varepsilon)\left(\binom{n-2}{3} + \binom{n-2-n_0}{4}\right) \\ \geq (c - \varepsilon)\binom{n-1-n_0}{4}$$

quadrilaterals in $P$, which proves Claim B.

Moreover, since $\lim_{n\to\infty}\binom{n-1-n_0}{4}/\binom{n}{4} = 1$, it follows that $\square^* \geq c - \varepsilon$, and because this holds for all $\varepsilon > 0$, the proof is complete. $\square$

## 3 Staircases of Encounters

In view of Lemma 1, our goal is to estimate $\Lambda(n)$. In this section, we introduce the necessary tools. Let $S$ be a set of $n+1$ points in general position in the plane. Fix a vertex $v$ of the convex hull of $S$, and set $P = S \setminus v$.

Consider a point $p \in P$, and let $\ell$ be the oriented line through $p$ and $v$. Let $L$ be the set of points of $P$ that lie to the left of $\ell$, and $R$ the set of those that lie to the right.

For $k := |L|$, the grid

$$\beta(p) := \beta(v, P, p) := \{0 \dots k-1\} \times \{0 \dots n-2-k\}$$

will be referred to as the *box* of $p$. We "fill" this box, i.e. we define a subset $\lambda(p) \subseteq \beta(p)$, in the following fashion: Enumerate the points in $L$ in the order $q_0, q_1, \dots q_{k-1}$ in which they are first encountered when we rotate $\ell$ clockwise, and set

$$(10) \quad \lambda(p) := \lambda(v, P, p) \\ := \{(a, b) \in \beta(v, P, p) : b < |H^-(q_a, p) \cap R|\}$$

(see Figure 1). We call $\lambda(p)$ the *staircase of encounters* of $p$.

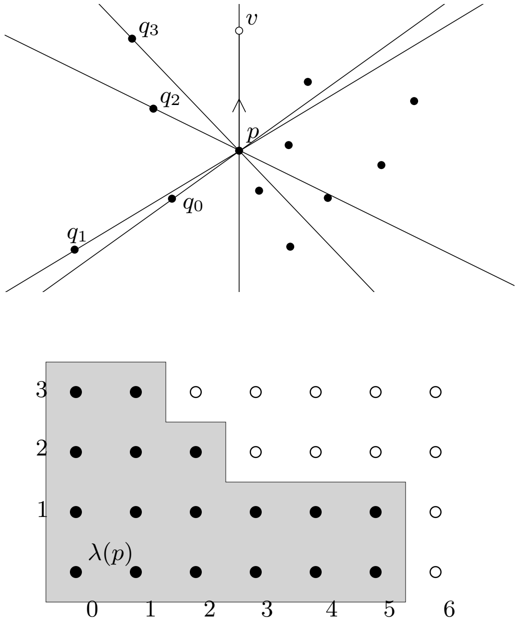

Figure 1: Staircase of Encounters

We proceed to relate these staircases of encounters to the object of our investigation, $\Lambda(v, P)$. For $p \in P$ and $0 \leq i \leq n-2$, let $f_i(p)$ be the number of entries of $\lambda(p)$ on the $i$th diagonal, i.e.

$$(11) \quad f_i(p) := f_i(v, P, p) \\ := |\{(a, b) \in \lambda(v, P, p) : a + b = i\}|.$$

The upcoming Lemma 2 and Corollary 1 express $\Lambda(S, v)$ in terms of the numbers $f_i(p)$. The first describes the connection of the $f_i(p)$'s with $\square(v, P)$ and is immediate from the definitions.

**Lemma 2.** *The probability that a random triple $T \subset P$ forms a convex quadrilateral with $v$ is*

$$\square(v,P) = 1 - \frac{\sum_{p\in P}\sum_i f_i(p)}{\binom{n}{4}}. \tag{12}$$

In order to relate the $f_i(p)$'s to $\square(P)$, we have to work a little. For $p \in P$, let $\triangle(p, P\setminus p)$ be the probability that a random triple $T \subseteq P \setminus p$ *embraces* $p$, i.e., contains $p$ in its convex hull. Observe that

$$\square(P) = 1 - \frac{4}{n-3}\sum_{p\in P}\triangle(p, P\setminus p). \tag{13}$$

**Lemma 3.** *For $p \in P$, we have*

$$\triangle(p, P\setminus p) = \frac{\sum_i (n-3-2i) f_i(p)}{\binom{n-1}{3}} \tag{14}$$

*Proof.* Both sides of (14) depend only on the circular ordering of the rays that emanate from $p$ through the points in $P \setminus p = L \cup R$. Thus, by sliding the points along these rays if necessary, we may assume that the all points in $L$, respectively $R$, lie on two lines $\ell_L$, respectively $\ell_R$, that are parallel to $\ell$ and at a distance $\varepsilon$ to the left, respectively right, of $\ell$ (see Figure 2).

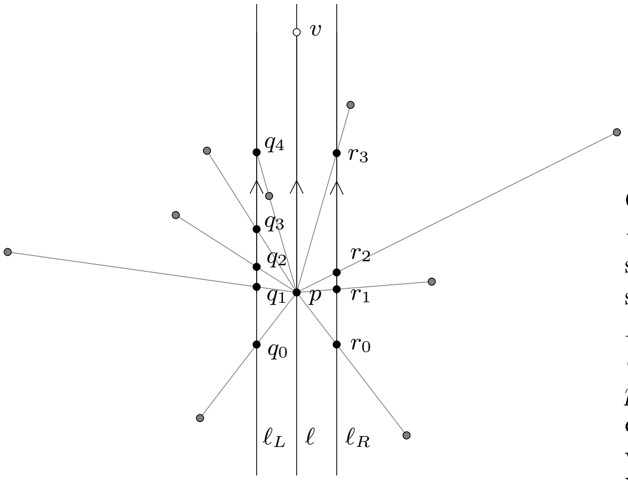

Figure 2: Numberings of $L$ and $R$.

Consider the points in $L$ and $R$ in the order in which they appear along the lines $\ell_L$ and $\ell_R$, respectively, $L = \{q_0, \dots, q_{k-1}\}$ and $R = \{r_0, \dots, r_{n-k-1}\}$. (For the points in $L$, this agrees with the ordering in the definition of $\lambda(p)$.) Then each pair $(a,b) \in B(p)$ corresponds to the pair $(q_a, r_b) \in L \times R$. We have $v \in H^+(q_a, r_b)$, $a = |H^-(q_a, r_b) \cap L|$, and $b = |H^-(q_a, r_b) \cap R|$. Moreover, $(a,b) \in \lambda(p)$ iff $p \in H^+(q_a, r_b)$.

Now, consider a triple $T = \{q, r, s\} \in \binom{P\setminus p}{3}$ such that $p \in \operatorname{conv}(T \cup v)$; call such a triple a *candidate triple*. There is a unique edge of $\operatorname{conv}(T)$, say $qr$, such that $q \in L$, $r \in R$, and $p, v, s \in H^+(q,r)$; call $qr$ the *lower edge* of $T$. On the other hand, given $q \in L$ and $r \in R$, we can take any point $s \in (P \setminus p) \cap H^+(q,r)$, and only those, to form a candidate triple $T = \{q,r,s\}$ with lower edge $qr$. Thus, if $|H^-(q,r) \cap (P \setminus p)| = i$, then $qr$ forms the lower edge of $|P \setminus \{p,q,r\}| - i = n-3-i$ candidate triples. Hence, the sum $\sum_i (n-3-i) f_i(p)$ counts the number of candidate triples.

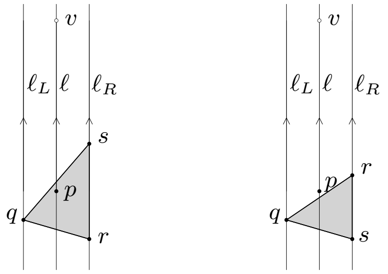

Figure 3: A triple embracing $p$, and a wanna-be.

On the other hand, consider a candidate triple $T = \{q, r, s\}$ whose convex hull does not contain $p$; call such $T$ a *wanna-be*. Then there is a unique edge, say $qr$, of $\operatorname{conv}(T)$ such that $q \in L$, $r \in R$, $p, v \in H^+(q,r)$, and $s \in H^-(q,r)$. Let us call $qr$ the *upper edge* of $T$. Conversely, given $q \in L$ and $r \in R$ with $p \in H^+(q,r)$, there are $|H^-(q,r) \cap (P \setminus p)|$ many choices for a point $s$ such that $\{q,r,s\}$ is a wanna-be with upper edge $qr$. Hence, the number of wanna-bes is $\sum_i i f_i(p)$. The number of triples $T \in \binom{P\setminus p}{3}$ that embrace $p$ equals the number of candidate triples minus the number of wanna-bes, i.e., $\sum_i (n-3-$

$2i)f_i(p)$, as we set out to prove. $\square$

By (13), Lemma 3 immediately yields the following:

**Corollary 1.**

$$\square(P) = 1 - \frac{\sum_{p\in P}\sum_{i=0}^{n-3}(n-3-2i)f_i(p)}{\binom{n}{4}}. \tag{15}$$

Hence, combining (12) and (15), we obtain

$$\Lambda(v,P) = 1 - \underbrace{\min\{\Gamma_1(v,P),\Gamma_2(v,P)\}}_{=:\,\Gamma(v,P)}, \tag{16}$$

where

$$\Gamma_1(v,P) := \frac{\sum_{i=0}^{n-3}(n-3-2i)\sum_{p\in P}f_i(p)}{\binom{n}{4}}$$

and

$$\Gamma_2(v,P) := \frac{\sum_{i=0}^{n-3}\sum_{p\in P}f_i(p)}{\binom{n}{3}}.$$

# 4 A Lower Bound for General Staircases

A *staircase* is a set $\lambda \subseteq \mathbf{N}_0 \times \mathbf{N}_0$ of pairs of nonnegative integers such that $(a,b)\in\lambda$ and $0\le a'\le a$ and $0\le b'\le b$ imply $(a',b')\in\lambda$. Let us look back at what we did so far: In order to analyze $\Lambda(v,P)$, we associated a certain staircase $\lambda(p)=\lambda(v,P,p)$ with every point $p$. Then we counted the number of entries on the $i$th diagonal of each of these staircases, and, in (16), expressed $\Lambda(v,P)$ in terms of the resulting numbers $f_i(p)$.

Let us now forget about the geometric context. For a staircase $\lambda$ and integer $i$, let

$$f_i(\lambda) := \{(a,b)\in\lambda : a+b=i\}.$$

For $1\le k\le n-3$, let $\beta_k := \{0\dots k\}\times\{0\dots n-3-k\}$, and consider a sequence $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_{n-3})$ of staircases $\lambda_k\subseteq\beta_k$. Taking (16) as a starting point, we define

$$\Gamma_1(\boldsymbol{\lambda}) := \frac{\sum_i(n-3-2i)\sum_k f_i(\lambda_k)}{\binom{n}{4}}, \tag{17}$$

$$\Gamma_2(\boldsymbol{\lambda}) := \frac{\sum_i\sum_k f_i(\lambda_k)}{\binom{n}{3}}, \tag{18}$$

and

$$\Gamma(\boldsymbol{\lambda}) := \min\{\Gamma_1(\boldsymbol{\lambda}),\Gamma_2(\boldsymbol{\lambda})\}. \tag{19}$$

(Observe that $\binom{n}{3}\Gamma_2(\boldsymbol{\lambda}) = \sum_k|\lambda_k|$.) We proceed to prove an upper bound for $\Gamma(\boldsymbol{\lambda})$, which, by (16) and by Lemma 1, yields a lower bound for $\square^*$.

Observe that there is a certain trade-off between $\Gamma_1$ and $\Gamma_2$: On the one hand, $\Gamma_2$ is maximized if $\lambda_k=\beta_k$ for all $k$ ("all boxes are full"). On the other hand, it is not hard to see (but we need not worry about that) that $\Gamma_1$ is maximized if $\lambda_k=\{(a,b)\in\beta_k : a+b\le(n-3)/2\}$ ("all boxes are filled up to the middle diagonal"). Roughly speaking, we obtain the upper bound for $\Gamma(\boldsymbol{\lambda})$ by finding the "equilibrium" of $\Gamma_1$ and $\Gamma_2$.

As a first step, we observe that we can restrict our attention to staircases of a special shape. Let us say that $\lambda_k$ results from *filling the box $\beta_k$ up to the $j$th diagonal* if, for all $(a,b)\in\beta_k$,

$$a+b<j \Rightarrow (a,b)\in\lambda_k, \quad\text{and}\quad a+b>j\Rightarrow(a,b)\notin\lambda_k.$$

(Observe that we do not say anything about the elements of $\lambda_k$ *on* the $j$th diagonal.)

**Lemma 4.** *Suppose that $M=\binom{n}{3}\Gamma_2(\boldsymbol{\lambda})=\sum_i\sum_k f_i(\lambda_k)$ is prescribed. Under this constraint, $\Gamma_1(\boldsymbol{\lambda})$ is maximized iff each $\lambda_k$ is obtained by filling $\beta_k$ up to the $j$th diagonal, for a certain $j=j(M)$.*

*Proof.* Let $j$ be maximal with the property that $\sum_k\sum_{i<j}f_i(\beta_k)\le M$. Suppose that $(a,b)\in\beta_k\setminus\lambda_k$ and $(a',b')\in\lambda_{k'}$, for some $k,k'$, such that $a+b<j$ and $a'+b'>j$. Then by removing $(a',b')$ from $\lambda_{k'}$ and by adding $(a,b)$ to $\lambda$, we increase $\Gamma_1$ while leaving $\Gamma_2$ invariant. The remaining elements of the staircases are distributed in an arbitrary fashion on the $j$th diagonals. $\square$

Thus, we may assume that all $\lambda_k$'s are of this kind. The question remains, up to which diagonal the $\beta_k$'s are filled.

**Lemma 5.** *Let $j = \lfloor \alpha(n-3) \rfloor$, for $\alpha \in [0,1]$, and suppose that each $\lambda_k$ is obtained by filling $\beta_k$ up to the $j$th diagonal. Then,*

$$\Gamma_1(\boldsymbol{\lambda}) = \underbrace{12\alpha^2(1-2\alpha+\alpha^2)}_{=: F_1(\alpha)} + O(1/n) \tag{20}$$

*and*

$$\Gamma_2(\boldsymbol{\lambda}) = \underbrace{\alpha^2(3-2\alpha)}_{=: F_2(\alpha)} + O(1/n). \tag{21}$$

The proof of Lemma 5 consists of straightforward calculations and is left to the reader.

Having the preceding lemma at our disposal, it is now easy to prove the desired estimate for $\Gamma(\boldsymbol{\lambda})$: For $\boldsymbol{\lambda}$ as in Lemma 5, we have $\Gamma(\boldsymbol{\lambda}) = \min\{F_1(\alpha), F_2(\alpha)\} + o(1)$. Moreover, since we are interested in the limit behaviour as $n \to \infty$, we can ignore the $o(1)$ error term. Thus, since we want to prove an upper bound for $\Gamma$, the question remains which $\alpha$ maximizes $\min\{F_1(\alpha), F_2(\alpha)\}$.

Let us first consider the interval $[\frac{1}{2}, 1]$: Here, $F_1$ is a monotonically decreasing function while $F_2$ is increasing. Moroever, $F_1(1/2) = 3/4 > 1/2 = F_2(1/2)$ and $F_1(1) = 0 < 1 = F_2(1)$, so $\max_{\alpha \in [1/2,1]} \min\{F_1(\alpha), F_2(\alpha)\}$ is attained at some $\alpha$ for which $F_1(\alpha) = F_2(\alpha)$. The roots of $F_1(\alpha) - F_2(\alpha) = 9\alpha^2 - 22\alpha^3 + 12\alpha^4$ are

$$0,\ 0,\ \frac{1}{12}(11+\sqrt{13}),\ \frac{1}{12}(11-\sqrt{13}).$$

Thus, the root we are looking for is $\alpha^* = (11 - \sqrt{13})/12$. Moreover, by considering first and second derivatives at 0, we see that $F_1 \geq F_2$ on the interval $[0, \alpha^*]$. Therefore, since $F_2$ is increasing, $\alpha^*$ maximizes $\min\{F_1, F_2\}$ over the whole interval $[0,1]$, and $F_1(\alpha^*) = F_2(\alpha^*) = (163 - 5\sqrt{13})/216) < 0.6712$. We have proved:

**Theorem 2.** *For every sequence $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{n-3}$ of staircases $\lambda_k \subseteq \{0 \ldots k\} \times \{0 \ldots n-3-k\}$, we have*

$$\Gamma(\boldsymbol{\lambda}) \leq (163 - 5\sqrt{13})/216) + O(1/n). \tag{22}$$

*In particular, for every set $S$ of $n+1$ points in general position and for any vertex $v$ of* $\operatorname{conv}(S)$*,*

$$\begin{aligned} \Lambda(v, S \setminus v) &\geq 1 - (163 - 5\sqrt{13})/216) + O(1/n) \\ &= (53 + 5\sqrt{13})/216 + O(1/n). \end{aligned} \tag{23}$$

By Lemma 1, this also establishes Theorem 1.

# 5 Concluding Remarks

It must be admitted that the key ingredient of the presented proof, $\Lambda(v, P)$, may appear rather mysterious at first sight, so a few remarks about how it came into existence seem in order. This will also allow us to point out how our proof builds upon earlier work, in particular by Welzl [11] and by Wagner and Welzl [10].

Our proof developped out of the attempt to analyze $\square(P)$ in terms of the probabilities $\triangle(p, P \setminus p)$, $p \in P$, which we considered in Lemma 3. For a fixed point $p$, Welzl [11] gave precise upper bounds for the probabilities $\triangle(p, P \setminus p)$. More generally, for every dimension $d$ and every set $P$ of $n$ points in general position in $\mathbf{R}^d$, by considering $j$-facets of a suitable lifting of $P$ to dimension $d+1$, he proved precise upper bounds for the probability that a fixed point $p \in P$ is contained in the convex hull of a random $k$-element subset of $P \setminus p$, $d+1 \leq k \leq n-1$. He also showed that these results are equivalent, through a variant of Gale duality, to the Upper Bound Theorem for convex polytopes. It should also be noted that the proof of Lemma 3 is inspired by Section 4 of [11]. (It also follows from this proof that the vector $(f_i(p) - f_{n-3-i}(p) : 0 \leq i \leq n-3)$ is the *g-vector* of the polytope that results from dualizing $P \setminus p$ with respect to $p$ as origin.)

These results, and the method of proof, were extended from discrete point sets to absolutely continuous probabilty measures $\mu$ by Wagner and Welzl [10]. In the latter, "continuous" set-up, the object of investigation is the probability that a fixed point $p$ lies in the convex hull of $k$ independent $\mu$-random points. Wagner and Welzl also analyzed this probability in terms of the *depth* $\delta_\mu(p)$ of $p$ with respect to the measure $\mu$, which is defined as the infimum of $\mu(H)$ over all halfspaces $H$ containing $p$.

With these results at our disposal, the following seemed a natural approach towards a lower bound for $\square^*$: For every set $P$ of $n$ points and every $i \in \{0 \dots \lfloor n/2 \rfloor - 1\}$, there are at least $2(i+1)$ points in $P$ whose depth does not exceed $i$ (as can be seen by considering only halfspaces with a common normal vector). Combining this simple observation with the bounds for $\triangle(p, P \setminus p)$, however, one only obtains $\square^* \geq 1/4$ (we omit the calculations, as we have omitted the bounds for $\triangle(p, P \setminus p)$ in terms of depth).

Moreover, in a sense, both steps in the above reasoning are best possible: On the one hand it is not hard to see that the bounds for $\triangle(p, P \setminus p)$ in terms of the depth of $p$ are tight, at least in the plane. On the other hand, here is a construction of a set $P_k$ of $3+2k$ points that contains 3 points of depth 0 and 2 points of depth $i$, for $1 \leq i \leq k$: Let $P_0$ be the vertex set of an arbitrary triangle which contains the origin $\boldsymbol{o}$ in its interior. Assume now that we have constructed $P_k$, that $\boldsymbol{o}$ does not lie on any line spanned by two points of $P_k$ and that $\delta_{P_k}(\boldsymbol{o}) = k+1$. Choose any line through $\boldsymbol{o}$ which avoids $P_k$. This line determines two halfplanes, one of which contains exactly $k+1$ points of $P_k$. For a suitably chosen $\varepsilon > 0$, let $p, p'$ be two new points in that halfplane such that $pop'$ is an isosceles triangle of height $\varepsilon^2$ whose base $pp'$ is parallel to the chosen line and of length $\varepsilon$. It is easy to see that for sufficiently small $\varepsilon$, $P_{k+1} := P_k \cup \{p, p'\}$ has again the desired properties.

When looking for means of a finer analysis of the probabilities $\triangle(p, P \setminus p)$, staircases of encounters came up, at first with a *viewpoint* $v$ at infinity. It seems difficult to pin down the dependencies between the staircases of encounters of different points, but luckily, in the hypothetical worst case $\square(P) = 1/4$ the viewpoint $v$ would be involved in an awful lot of crossings, namely $\square(v, P) = 1/2$. To exploit this, the viewpoint is forced into the point set under consideration, and $\square(P)$ and $\square(v, P)$ are considered simultaneously, which leads to the definition of $\Lambda(v, P)$.

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