Minors in Random and Expanding Hypergraphs*

Uli Wagner
Institut für Theoretische Informatik, ETH Zürich
Universitätstrasse 6
CH-8092 Zürich Switzerland
uli@inf.ethz.ch

ABSTRACT

We introduce a new notion of minors for simplicial complexes (hypergraphs), so-called *homological minors*. Our motivation is to propose a general approach to attack certain *extremal problems* for *sparse* simplicial complexes and the corresponding *threshold problems* for random complexes.

In this paper, we focus on threshold problems. The basic model for random complexes is the Linial-Meshulam model $X^k(n,p)$. By definition, such a complex has n vertices, a complete $(k-1)$-dimensional skeleton, and every possible k-dimensional simplex is chosen independently with probability p. We show that for every $k, t \geq 1$, there is a constant $C = C(k,t)$ such that for $p \geq C/n$, the random complex $X^k(n,p)$ asymptotically almost surely contains K_t^k (the complete k-dimensional complex on t vertices) as a homological minor. As corollary, the threshold for (topological) embeddability of $X^k(n,p)$ into $\mathbf{R}^{2k}$ is at $p = \Theta(1/n)$.

The method can be extended to other models of random complexes (for which the lower skeleta are not necessarily complete) and also to more general *Tverberg-type problems*, where instead of continuous maps without *doubly covered image points* (embeddings), we consider maps without q-fold covered image points.

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1. INTRODUCTION

For graphs, there is a rich interplay between combinatorial properties on the one hand and topological properties, such as planarity, on the other hand. For example, for (connected, finite) graphs embedded into the plane $\mathbf{R}^2$ (or the sphere $\mathbf{S}^2$), we have *Euler's relation* $f_0 - f_1 + f_2 = 2$ between the number f_0 of vertices, the number f_1 of edges, and the number f_2 of regions determined by the embedding. This implies the tight upper bound $f_1 \leq 3f_0 - 6$ for the number of edges of a *simple* planar graph with $f_0 \geq 3$ vertices.

Another example are the classical characterizations of planar graphs in terms of *forbidden minors*. Kuratowski [25] proved that a graph is planar iff it contains neither the complete graph K_5 nor the complete bipartite graph $K_{3,3}$ as a *topological minor* (i.e., if it does not contain a *subdivision* of either of these). Wagner [48] showed that the same holds in terms of *deletion-and-contraction minors*, i.e., G is planar iff neither K_5 nor $K_{3,3}$ can be obtained from G by a finite sequence of edge or vertex deletions and edge contractions.

These results form the starting point of the general theory of graph minors, which comprises some of the the deepest results and open problems in combinatorics, notably the Robertson-Seymour *Graph Minor Theorem* [41] (see [28] for a survey) and *Hadwiger's conjecture* [17] (see also [4, 40]).

Minors are also a valuable tool when dealing with extremal problems for *sparse graphs*, due to the following theorem of Mader [30]:

MADER'S THEOREM. *For every integer t there are constants $C(t)$ and $C_{\mathrm{TOP}}(t)$ such that every (simple) graph on n vertices with at least $C(t) \cdot n$ edges (respectively, at least $C_{\mathrm{TOP}}(t) \cdot n$ edges) contains the complete graph K_t as a (topological) minor.*[1]

Consequently, if a given graph property implies a fixed forbidden minor, then graphs with that property have only a linear number of edges. In particular, if one wishes to prove a linear upper bound for the number of edges in planar graphs without appealing to Euler's relation, it is enough to show, first, that K_5 (or some other fixed complete graph, if one is willing to accept a worse constant) is not planar, and second, that planarity is preserved under taking minors (which is easy).

Simplicial Complexes

It is rather natural to wonder how to define minors for higher-dimensional *simplicial complexes* or *hypergraphs*.

We recall that a (finite) *hypergraph* is simply a finite set system, while a (finite, abstract) *simplicial complex* X is a

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[1]Even the dependence of the constants on t is now known quite precisely, $C(t) \approx 0.319t\sqrt{\log t}$ and $C_{\mathrm{TOP}}(t) = \Theta(t^2)$, see [45] and [5, 21], respectively. In the special case $t = 5$, Mader [31] also showed that $3n-5$ edges are already enough to ensure K_5 as a (topological) minor.

finite set system that is closed under taking subsets, i.e., $F \subseteq G \in X$ implies $F \in X$. For our purposes, simplicial complexes are somewhat more natural and convenient, but the difference is more one of viewpoint and terminology than an essential one.[2]

The sets in a simplicial complex X are called *faces* or *simplices* and are classified according to their *dimension* $\dim F := |F| - 1$. The dimension of X is defined as the maximum dimension of any face. The set of i-dimensional faces of X will be denoted by X_i, and the number of i-dimensional faces of X is denoted by $f_i(X) := |X_i|$ or simply by f_i. If $G \subseteq F \in X$ then we say that G is a face of F. The elements of a face F are called the *vertices* of F. Identifying singleton sets with their elements, we also call X_0 the set of vertices of X. Thus, e.g., a 1-dimensional simplicial complex is just a simple graph, and f_0 and f_1 are the numbers of its vertices and edges, respectively. Further basic notions, including *embeddablity*, will be reviewed in Section 2.1.

There are many possible ways of defining minors of simplicial complexes, some of which we will mention in Section 4. Which definition is "the right one" depends on the context and on the aspects of the theory of graph minors that one wishes to generalize. We stress right away that we are not aiming for a characterization of embeddable complexes in terms of finitely many forbidden minors (indeed, recent NP-hardness results [33] regarding embeddability of simplicial complexes are an indication that such a goal might be too ambitious, especially for embeddings of 2-dimensional complexes into $\mathbf{R}^4$). Instead, our principal motivation are topological *extremal problems* for *sparse* simplicial complexes and the corresponding *threshold questions* for random complexes (we think of a k-dimensional complex as sparse if $f_k = O(f_{k-1})$).

Extremal Problems

A typical example of such an extremal problem is the following:

CONJECTURE 1. *If X embeds (topologically) into $\mathbf{R}^{2k}$, denoted $X \hookrightarrow \mathbf{R}^{2k}$, then $f_k(X) \le C_k \cdot f_{k-1}(X)$, where C_k is a constant that depends only on k.*

This is a natural generalization of the fact that the number f_1 of edges of a (simple) planar graph is at most linear in the number $n := f_0$ of vertices. The conjecture seems to be folklore and has appeared various times in the literature[3], see, e.g., [20, 42, 9]. The only known nontrivial upper bound is $f_k = O(n^{k+1-1/3^k})$ for any fixed $k \ge 1$, which is proved using *forbidden subcomplex* arguments, see below.

As in the special case of graphs, it is known that Conjecture 1 in general would have a number of interesting consequences in discrete geometry, including higher-dimensional analogues [9, 10] of the well-known crossing lemma [1, 26] and upper bounds for the number of triangulations of an n-point set in $\mathbf{R}^d$ [9, 11]. The problem is also closely related to the *Upper Bound Theorem* [34, 44] and the *g-Conjecture* for simplicial spheres, via a deep conjecture of Kalai and Sarkaria [20, 42] that connects embeddability and algebraic shifting and would, in particular, imply Conjecture 1 in a very precise form. We also remark that the case of k-dimensional complexes embeddable into $\mathbf{R}^{2k}$ is the critical one, in the sense that Conjecture 1 would imply tight upper bounds for the face numbers of simplicial complexes embeddable into $\mathbf{R}^d$, for any ambient dimension d. These connections and implications will be discussed in more detail in the full version of this paper.

Random and Expanding Complexes

Linial and Meshulam [27] introduced a higher-dimensional analogue of the Erdős-Rényi random graph model $G(n, p)$. By definition, the random k-dimensional complex $X^k(n, p)$ has n vertices, a full $(k-1)$-skeleton (i.e., all subsets of size k or less of the vertices form faces of the complex), and every $(k+1)$-element set of vertices is taken as a k-face independently with probability p, which may be constant or, more generally, a function $p(n)$ depending on n.

This model has received a lot of attention, and the *threshold probabilities* for a number of topological properties of $X^k(n, p)$ have been determined, see [27, 36, 3, 2, 7, 8, 23].

It is known that the sharp threshold probability for planarity of $X^1(n, p) = G(n, p)$ is at $p = 1/n$ (see [29]). For the higher-dimensional embeddability problem, we have the following result.

THEOREM 2. *The threshold for embeddability of $X^k(n, p)$ into $\mathbf{R}^{2k}$ is at $p = \Theta(1/n)$. That is, for every $k \ge 1$ there are constants $C_k > c_k > 0$ depending only on k such that*

$$\lim_{n\to\infty} \Pr[X^k(n,p) \hookrightarrow \mathbf{R}^{2k}] = \begin{cases} 1, & p \le c_k/n, \\ 0, & p \ge C_k/n. \end{cases}$$

In particular, this shows that Conjecture 1 holds for almost all complexes. (For the complex $X^k(n, p)$ we have $f_{k-1} = \binom{n}{k}$, and f_k is strongly concentrated around $p \cdot \binom{n}{k+1}$, so in order to obtain complexes with $f_k = \Theta(f_{k-1})$, the parametrization $p = C/n$ is the right one.)

It seems very likely that like planarity, the general embeddability property has a *sharp threshold*, i.e., that there exists $C = C(k)$ such that for any $\varepsilon > 0$, the probability $\Pr[X^k(n, p) \hookrightarrow \mathbf{R}^{2k}]$ tends to 1 if $p \le (1-\varepsilon)C/n$ and to 0 if $p \ge (1+\varepsilon)C/n$. Computer experiments conducted by G. Pundak (personal communication) suggest that $C(2)$ is around 4.37. Our current proof does not yield such a sharp threshold (even though many of the estimates could be improved, at the expense of making the proof more complicated).

On the other hand, our proof only uses a certain *quasirandomness* or *expansion* property of $X^k(n, p)$, see Section 5, and thus applies to a broader class of complexes, which do not necessarily have a complete $(k-1)$-skeleton, including certain other models of random complexes; this will be discussed in the full version of this paper. The relevant notion of expansion was first defined by Gromov [15]. The same notion arose independently in the work of Linial, Meshulam, and Wallach [27, 36] and in the work of Newman and Rabinovich [39].

The fact that $X^k(n, p) \hookrightarrow \mathbf{R}^{2k}$ asymptotically almost surely for $p \le c_k/n$ follows fairly easily from a recent result of Aronshtam, Linial, Łuczak, and Meshulam (personal communication) concerning collapsibility of random complexes.

[2]Essentially, a *pure k-dimensional* simplicial complex (all maximal faces of dimension k) is the same thing as a $(k+1)$-uniform hypergraph (all hyperedges of cardinality $k+1$).

[3]Sometimes, the conclusion is weakened to $f_k = O(n^k)$, where $n := f_0$ is the number of vertices (we trivially have $f_{k-1} \le \binom{n}{k}$ since a $(k-1)$-simplex has k vertices).

This part of the proof is omitted from this extented abstract and will be given in the full version. Here, we focus on the nonembeddability part, which is the main contribution.

The Structure of this Paper

The remainder of this paper is structured as follows. In Section 2, we review some basic facts concerning simplicial complexes and homology. In Section 3, we describe our general approach to extremal and threshold problems. At the heart of it lies a conjectural higher-dimensional analogue of Mader's theorem. Here, we prove the corresponding threshold result for random complexes (Theorem 7). In Section 4, we give the precise definition of *homological minors* and prove that, in a suitable technical sense, complexes with a nonembeddable homological minor are nonembeddable (Theorem 5). In Section 5, we discuss higher-dimensional face expansion of simplicial complexes in the sense of Gromov and note that $X^k(n, C/n)$ has (a coarse version of) this property. Finally, in Section 6, we prove that expanding complexes contain large complete minors (Proposition 12), which proves Theorems 7 and 2.

2. PRELIMINARIES

We summarize some basic definitions and facts concerning simplicial complexes, homology, and obstructions in order to fix terminology and notation and to provide the necessary background for the definition of homological minors. For further background, see, e.g. [37, 32, 22].

2.1 Simplicial Complexes and Embeddings

Formally, there are two different ways of viewing a simplicial complex, either as an *abstract simplicial complex*, as defined in the introduction, or as a *geometric simplicial complex*, i.e., as a finite collection X of closed geometric simplices in some Euclidean space $\mathbf{R}^m$ such that if $\sigma \in X$ and τ is a face of σ then also $\tau \in X$ and such that any two simplices in X intersect in a common face (which may be empty). Here, by definition, a geometric simplex σ is the convex hull of some affinely independent set of points, called the *vertices* of the simplex, and a face of σ is the convex hull of some subset (possibly empty) of its vertices. Every geometric simplicial complex X defines an *underlying topological space* or *polyhedron* $|X| := \bigcup_{\sigma \in X} \sigma \subset \mathbf{R}^m$, namely the union of all the geometric simplices in X, with the topology inherited as a subspace of the ambient Euclidean space $\mathbf{R}^m$.

Two simplicial complexes are isomorphic if there is a face-preserving bijection between their vertex sets. For any two isomorphic geometric simplicial complexes, there is an obvious homeomorphism between their underlying spaces that is linear on each face. There is a standard way of going back and forth between the abstract and the geometric viewpoint (see, e.g. [32]), and an abstract simplicial complex can be viewed as a purely combinatorial description of a geometric simplicial complex up to isomorphism, by specifying which subsets of vertices form vertex sets of faces.

A *subcomplex* of X is a subset $Y \subset X$ that is itself a simplicial complex. The *i-skeleton* of X is the simplicial complex $X_i = X_{-1} \cup X_0 \cup \ldots \cup X_i$ that consists of all faces of X of dimension at most i. A geometric simplicial complex X' is called a *subdivision* of a geometric simplicial complex X if $|X| = |X'|$ and every simplex of X' is contained in some simplex of X.

A (topological) *embedding* of a simplicial complex X into $\mathbf{R}^d$ is a map $f \colon |X| \to \mathbf{R}^d$ that is a homeomorphism of $|X|$ with $f(|X|)$. Since we only consider finite simplicial complexes, this is equivalent to requiring that f be injective. If such an embedding exists, we say that X embeds into $\mathbf{R}^d$, denoted $X \hookrightarrow \mathbf{R}^d$. For further background on embeddings of simplicial complexes, see the survey [43]. For an introduction especially geared towards combinatorialists and computer scientists and and with an emphasis on algorithmic aspects of embeddability, see also [33].

2.2 Homology and Cohomology

We review the definition of (simplicial) homology and cohomology of a finite simplicial complex X (see, e.g., [37] for a thorough introduction). For simplicity, we restrict ourselves to the case of coefficients in the field $\mathbf{F}_2$ with two elements.

Let X be a finite simplicial complex. For integer i, denote by $C^i(X) = C^i(X;\mathbf{F}_2)$ the vector space $\mathbf{F}_2^{X_i}$ of functions from the set of i-faces of X to the field $\mathbf{F}_2$ (thus, $C^i(X) = 0$ unless $-1 \leq i \leq \dim X$). The elements of this vector space are called *i-dimensional cochains* of X. Since we are working over $\mathbf{F}_2$ we can also think of an i-cochain as (the characteristic vector of) a subset of X_i.

Moreover, let $C_i(X) = C_i(X;\mathbf{F}_2)$ be the vector space over $\mathbf{F}_2$ generated by X_i. In other words, the elements of $C_i(X)$, called *i-chains*, are formal linear combinations of i-faces of X. Since the sets X_i are finite, $C_i(X)$ is again (non-canonically) isomorphicto $\mathbf{F}_2^{X_i}$, so it might seem like exaggerated formalism to distinguish between $C_i(X)$ and $C^i(X)$, but it is sometimes convenient to maintain this distinction.

In fact, the space $C^i(X)$ is the dual vector space of $C_i(X)$, and we have a natural bilinear map $\langle , \rangle \colon C^i(X) \times C_i(X) \to \mathbf{F}_2$. If we identify both spaces with $\mathbf{F}_2^{X_i}$, this map corresponds to the standard inner product on $\mathbf{F}_2^{X_i}$.

We have a linear map $\partial = \partial_i : C_i(X) \to C_{i-1}(X)$, called the *boundary map*, given on basis elements $F \in X_i$ by $\partial F = \sum_{G \subseteq F, \dim G = i-1} G$. In other words, with respect to the standard bases of $C_i(X)$ and $C_{i-1}(X)$, the boundary map is given by the incidence (or inclusion) matrix between i-faces and $(i-1)$-faces.

Its dual map $\partial_i^* : C_{i-1}(X) \to C_i(X)$ is called the *coboundary map*. Again, with respect to the standard bases, this map is given by (the transpose of) the inclusion matrix. We often drop the indices and just write ∂ or ∂^*. Thus, if $S \subseteq X_i$ is a subset of i-faces and we view it as an i-chain then the boundary ∂S corresponds to the the set of all $(i-1)$-faces contained in an odd number of i-faces in S. Conversely, if we think of S as an i-dimensional cochain then its coboundary consists of all $(i+1)$-faces that contain an odd number of faces in S.

The crucial property of the boundary map is that $\partial_{i-1} \circ \partial_i = 0$ (and consequently also $\partial_i^* \circ \partial_{i-1}^* = 0$), which is easily verified. Equivalently, we have the following relation between the kernels and images of these maps:

$$\begin{aligned} B_i = B_i(X;\mathbf{F}_2) := \operatorname{im} \partial_{i+1} \quad &\subseteq \quad Z_i = Z_i(X;\mathbf{F}_2) := \ker \partial_i, \\ B^i = B^i(X;\mathbf{F}_2) := \operatorname{im} \partial_i^* \quad &\subseteq \quad Z^i = Z^i(X;\mathbf{F}_2) := \ker \partial_{i+1}^*. \end{aligned}$$

The elements of Z_i, B_i, Z^i, and B^i are called *i-cycles*, *i-boundaries*, *i-cocycles*, and *i-coboundaries*, respectively. By the above inclusion relations, we can form the quotient vector spaces $H_i(X) := \frac{Z_i(X)}{B_i(X)}$ and $H^i(X) := \frac{Z^i(X)}{B^i(X)}$, which are called the i-th *homology* and the i-th cohomology group of

X, respectively.[4] Thus, every i-cycle $\zeta \in Z_i$ determines a *homology class* $[\zeta] = \zeta + B_i \in H_i$, and likewise every cocycle $\alpha \in Z^i$ determines a *cohomology class* $[\alpha] \in H^i$.

If X and Y are simplicial complexes, then a *chain map* between them is a sequence of linear maps $\varphi_i : C_i(X) \to C_i(Y)$ with the additional property that these commute with the respective boundary maps in X and Y, i.e., $\varphi_i \circ \partial_i^X = \partial_i^Y \circ \varphi_{i-1}$. As before, we will often drop indices and write $\varphi(F)$ instead of $\varphi_i(F)$. Thus, the chain map represents every i-face F of X by a formal linear combination $\varphi(F)$ of i-faces in Y such that $\partial\varphi(F)$ equals the sum of the collections $\varphi(G)$, where G ranges over all $(i-1)$-faces in $\partial(F)$.

A chain map φ induces linear mappings $\varphi_* : H_i(X) \to H_i(Y)$ and $\varphi^* : H^i(Y) \to H^i(X)$ in (co)homology by defining $\varphi_*([\zeta]) := [\varphi_i(\zeta)]$ and $f^*([\alpha]) := [\varphi^i(\alpha)]$, where $\varphi^i : C^i(Y) \to C^i(X)$ is the transpose of φ_i.

A simplicial map f between complexes X and Y defines a chain map $f_\sharp$ that maps an i-face $F \in X_i$ to $f(F)$ if the latter is also of dimension i, and to zero otherwise. However, not all chain maps are of this form. One necessary condition is that a chain map induced by a simplicial map necessarily maps every vertex of X to a unique vertex of Y. Consequently, it maps the unique basis element $\emptyset$ of $C_{-1}(X)$ to the corresponding basis element $\emptyset$ of $C_{-1}(Y)$ (and not to zero). More generally, by a technique called *simplicial approximation*, one can show that an arbitrary continuous map $f : |X| \to |Y|$ between simplicial complexes induces homomorphisms (linear maps) $f_* : H_i(X) \to H_i(Y)$ and $f^* : H^i(Y) \to H^i(X)$ in (co)homology (note that the direction of the map is reversed in cohomology). Furthermore, homotopic maps induce identical homomorphisms in (co)homology.

On the level of chain maps, a *chain homotopy* P between two chain maps φ, ψ from X to Y is a family of linear maps (not chain maps) $P_i : C_i(X) \to C_{i+1}(Y)$ such that $\psi_i - \varphi_i = \partial_{i+1}^Y P_i + P_{i-1}\partial_i^X$ for all i. Chain homotopic chain maps induce identical maps in (co)homology.

One fact we will need below as a black box is that the direct sum $H^*(X) = \bigoplus_i H^i(X)$ can be turned into a graded ring, called the *cohomology ring*. That is, one can define a bilinear multiplication $H^i(X) \times H^j(X) \to H^{i+j}(X)$ called the *cup product*, and the homomorphisms induced by continuous maps also respect this product.

Moreover, we will need that the everything said so far also applies to complexes more general than simplicial complexes. Specifically, we will need *polytopal* complexes, whose i-faces are i-dimensional convex polytopes that meet in common faces. The boundary matrix is still given by the incidences between i-faces and $(i-1)$-faces and and all other definitions carry over verbatim.

2.3 Deleted Products and Obstructions

Let X be a simplicial complex and $|X|$ its underlying topological space. The (twofold) *topological deleted product* of X is the space $|X|^2_{\mathrm{del}} := (|X| \times |X|) \setminus \{(x,x) : x \in |X|\}$, i.e., the twofold cartesian product with the "diagonal" removed.

The (twofold) *combinatorial deleted product* of X is the polytopal cell complex $X^2_{\mathrm{del}} := \{\sigma \times \tau : \sigma, \tau \in K, \sigma \cap \tau = \emptyset\}$. Thus, the cells of the deleted product are cartesian products of vertex-disjoint simplices of X. The combinatorial deleted product is a subspace of the topological one. The latter deformation retracts onto the former, so the two spaces are homotopy equivalent.

The topological deleted product comes with an obvious action of the group $\mathbf{Z}_2$ that simply exchanges the order of coordinates, $(x,y) \mapsto (y,x)$. This is inherited by the combinatorial deleted product. For the latter, the action maps cells to cells, namely $\sigma \times \tau \mapsto \tau \times \sigma$ (some care has to be taken regarding orientations, which we ignore here). The action is *free*, i.e., it does not have any fixed-points. The homotopy equivalence between the topological and the combinatorial deleted product can also be chosen to be equivariant. Henceforth, we will not distinguish between the two deleted products and simply write X^2_{del}.

An embedding $f \colon X \hookrightarrow \mathbf{R}^d$ induces a continuous map $\tilde{f} \colon X^2_{\mathrm{del}} \to S^{d-1}$ by setting $\tilde{f}(x,y) := \frac{f(x)-f(y)}{\|f(x)-f(y)\|}$. Moreover, this map is $\mathbf{Z}_2$-*equivariant*, i.e., $\tilde{f}(y,x) = -\tilde{f}(x,y)$ for all $(x,y) \in X^2_\Delta$. Thus, the existence of an equivariant map from X^2_{del} to S^{d-1} is a necessary condition for embeddability of X into $\mathbf{R}^d$. A celebrated theorem by Haefliger and Weber [18, 49] asserts that for $\dim X \le (2d-3)/3$ (the so-called *metastable range*), the existence of an equivariant map from X^2_{del} to $\mathbf{S}^{d-1}$ is also a sufficient condition for $X \hookrightarrow \mathbf{R}^d$ (outside the metastable range, this fails). We refer to [43] for a modern overview, proof sketch, and extensions.

The van Kampen obstruction yields a necessary algebraic conditions for the existence of equivariant maps. Several equivalent definitions are known. Here, we choose a rather abstract one that allows us to derive the necessary facts quickly (for more details see, e.g.[22, 35]). It is important to remark, however, that ultimately, the van Kampen obstruction boils down to a very concrete system of linear equations given by the simplicial complex, see, e.g. [33] for an elementary exposition.

It is a basic fact that there is a (cellular) $\mathbf{Z}_2$-equivariant map from X^2_{del} into the infinite-dimensional sphere $\mathbf{S}^\infty$. Moreover, this map is unique up to $\mathbf{Z}_2$-equivariant homotopy. In fact, both the map and the homotopy are easy to construct inductively on successively higher skeleta of X^2_{del}, using that $\mathbf{S}^\infty$ is contractible, i.e., that all its homotopy groups vanish. This $\mathbf{Z}_2$-map induces a unique map (up to homotopy) between the quotient spaces $X^2_{\mathrm{del}}/\mathbf{Z}_2 \to \mathbf{R}P^\infty$, and hence a unique homomorphism in cohomology $H^*(\mathbf{R}P^\infty; \mathbf{F}_2) \to H^*(X^2_{\mathrm{del}}/\mathbf{Z}_2; \mathbf{F}_2)$.

The $\mathbf{Z}_2$-cohomology ring of $\mathbf{R}P^\infty$ is known to be isomorphic to the polynomial ring $\mathbf{F}_2[x]$. In particular, in each dimension d, the element x^d is the unique nonzero element of $H^d(\mathbf{R}P^\infty; \mathbf{F}_2)$. The image of this element under the above homomorphism $H^d(\mathbf{R}P^\infty; \mathbf{F}_2) \to H^d(X^2_{\mathrm{del}}/\mathbf{Z}_2; \mathbf{F}_2)$ is called the van Kampen obstruction (modulo 2) for embeddability of X into $\mathbf{R}^d$ and denoted by $\mathfrak{o}^d_{\mathbf{F}_2}(X)$.

If X embeds into $\mathbf{R}^d$ then, as noted above, we get an equivariant map from X^2_{del} to $\mathbf{S}^{d-1}$. Composing this with the inclusion $\mathbf{S}^{d-1} \hookrightarrow \mathbf{S}^\infty$, we get a particular representative of the unique (up to homotopy) equivariant map from X^2_{del} into $\mathbf{S}^\infty$. Thus on the level of cohomology, the induced map $H^*(\mathbf{R}P^\infty; \mathbf{F}_2) \to H^*(X^2_{\mathrm{del}}/\mathbf{Z}_2; \mathbf{Z}_2)$ can be written as a composition

$$H^*(\mathbf{R}P^\infty; \mathbf{F}_2) \to H^*(\mathbf{R}P^{d-1}; \mathbf{F}_2) \to H^*(X^2_{\mathrm{del}}/\mathbf{Z}_2; \mathbf{Z}_2).$$

It is known that $H^*(\mathbf{R}P^{d-1}; \mathbf{F}_2)$ is isomorphic to the quotient $\mathbf{F}_2[x]/(x^d)$ and that the the map on the left is just the quotient map $\mathbf{F}_2[x] \to \mathbf{F}_2[x]/(x^d)$, so the image of x^d is zero.

[4]Thus, for the case $i = 0$ we work with what is sometimes called *reduced* (co)homology.

But then also the image of x^d under the composed map is zero, i.e., $\mathfrak{o}^d_{\mathbf{F}_2}(X) = 0 \in H^*(X^2_{\mathrm{del}}/\mathbf{Z}_2; \mathbf{Z}_2)$.

A crucial observation that will be important later is that the existence and uniqueness up to homotopy of a cellular $\mathbf{Z}_2$-map can be mimicked on the level of chain maps, by using that all homology groups of $\mathbf{S}^\infty$ vanish.

LEMMA 3. *Let $a \in \mathbf{F}_2$ be arbitrary, and fix a cellular decomposition of $\mathbf{S}^\infty$ that is compatible with the $\mathbf{Z}_2$-action. Then there is an $\mathbf{Z}_2$-equivariant chain map from X^2_{del} to $\mathbf{S}^\infty$ that maps $\emptyset \in C_{-1}(X^2_{del})$ to $a \cdot \emptyset \in C_{-1}(\mathbf{S}^\infty)$, and any two such chain maps are $\mathbf{Z}_2$-equivariantly chain homotopic.*

The proof is a very simple and rather standard inductive argument and is omitted from this extended abstract. Since the van Kampen obstruction is the pullback of a cohomology class, it only depends on the chain homotopy class of the map used for the pullback. Thus, we obtain the following lemma (again, the (simple) details are omitted from this extended abstract).

LEMMA 4. *If ψ is an equivariant chain map from K^2_{del} to X^2_{del} such that $\psi(\emptyset) = 1 \cdot \emptyset$, and if $\mathfrak{o}^d_{\mathbf{F}_2}(K) \neq 0$ for some d then also $\mathfrak{o}^d_{\mathbf{F}_2}(X) \neq 0$.*

2.4 Small Nonembeddable Complexes

Every k-dimensional complex embeds into $\mathbf{R}^{2k+1}$. On the other hand, there are k-dimensional complexes that do not embed into $\mathbf{R}^{2k}$. The basic examples are K^k_t, the *complete* k-dimensional simplicial complex on t vertices, and $K^k_{t,\ldots,t}$, the *complete multipartite* complex on $k+1$ classes $V_0, \ldots, V_k$ of t vertices each. By definition, the faces of K^k_t are all subsets of vertices of cardinality at most $k+1$ (in other words, K^k_t is the k-dimensional skeleton of the simplex on t vertices), and the faces of $K^k_{t,\ldots,t}$ are precisely the *rainbow* sets $F \subseteq V_0 \cup \ldots \cup V_k$, where F is rainbow if $|F \cap V_i| \leq 1$ for all i.

It is a classical result of van Kampen [47] and Flores [13] that the complexes K^k_{2k+3} and $K^k_{3,\ldots,3}$ do not embed into $\mathbf{R}^{2k}$. This is proved by showing that the van Kampen obstructions of these complexes are nonzero.

These complexes are the direct generalizations of the nonplanar graphs K_5 and $K_{3,3}$. It is also known that these complexes are *minimally nonembeddable*, i.e., if we remove from their underlying topological space an arbitrarily small open neighborhood of any point then the resulting space becomes embeddable. However, in higher dimensions, there are not just these two but in fact infinitely many minimally nonembeddable complexes [50, 46].

3. THE FORBIDDEN MINOR APPROACH

The fact that the complete multipartite complex $K^k_{3,\ldots,3}$ does not embed into $\mathbf{R}^{2k}$ immediately leads to a nontrivial upper bound $f_k = O(n^{k+1-1/3^k})$ for embeddable complexes, by the following Turán-type result.

THEOREM (ERDŐS [12]). *If a simplicial complex X on n vertices does not contain $K^k_{t,\ldots,t}$ as a* subcomplex *then $f_k(X) = O(n^{k+1-1/t^k})$.*

We remark that in higher dimensions, this extremal result is not known to be tight (see, e.g., the discussion in [16]). For $k \geq 2$, a probabilistic construction with alterations yields a slightly smaller exponent of $k + 1 - \frac{(k+1)t-k}{t^{k+1}-1}$ for the lower bound. Even so, forbidden subhypergraph arguments work well only for fairly dense complexes and are too weak to obtain tight bounds in our context.

A natural approach is to use *forbidden minors* instead. The challenge is to find a suitable notion of minors that, on the one hand, preserves sufficient topological information and, on the other hand, is sufficiently flexible so as to deal with very sparse complexes.

In Section 4, we will briefly discuss some other notions of minors and the difficulties that arise when trying to carry out the above aproach. The notion of *homological minors*, denoted by $\preccurlyeq_H$, is designed so as to circumvent these difficulties. As a first step, we will show that homological minors satisfy the first requirement.

THEOREM 5. *If $K \preccurlyeq_H X$ and $\mathfrak{o}^{2k}_{\mathbf{F}_2}(K) \neq 0$ then $\mathfrak{o}^{2k}_{\mathbf{F}_2}(X) \neq 0$.*

In other words: if there is a good reason for the nonembeddability of the minor K (nonvanishing of its van Kampen obstruction modulo 2) then X is nonembeddable for the the same reason.[5]

Next, we propose the following generalization of Mader's theorem.[6]

CONJECTURE 6. *For every $k, t \geq 1$, there is a constant $C = C(k,t)$ with the following property: If X is a simplicial complex with $f_k(X) \geq C \cdot f_{k-1}(X)$ then $K^k_t \preccurlyeq_H X$.*

We have not yet been able to verify Conjecture 2 in full generality, but we can show that it holds for almost all complexes.

THEOREM 7. *For every k, t, there exists a constant $C = C(k,t)$ such that $\Pr[K^k_t \preccurlyeq_H X^k(n,p)] \to 1$ as $n \to \infty$ for $p \geq C/n$.*

The proof actually shows that with high probability, $X = X^k(n, C/n)$ contains a K^k_t-minor even after moderate *alterations*, i.e., if we remove εn^k many k-faces from X. This is of particular interest since random complexes $X^2(n,p)$ with small alterations (and $p \approx 1/\sqrt{n}$) show that an analogue of Conjecture 6 fails for topological minors of simplicial complexes, see Section 4.

Moreover, as remarked above, the proof only uses certain *quasirandomness* or *expansion* properties of X and hence applies to a much broader class of complexes, which do not necessarily have a complete $(k-1)$-skeleton. We consider this as positive evidence that Conjecture 6, the notion of homological minors, and the overall approach proposed here are reasonable and promising.

Remark 1. In sparse (constant average degree) expanding graphs on n vertices, one can find not just constant-size complete minors, but much larger ones of size $\sqrt{n/\log n}$, see e.g. [24]. The crucial fact that makes this possible is that such a graph has very small diameter $O(\log n)$ and that this

[5] There is an analogous result for the integer van Kampen obstruction, which characerizes embeddability if $k \neq 2$.

[6] The case $t = 2k+3$ would be sufficient for the embeddability problem, but this special case seems to pose the same difficulties as the general one, so the restriction appears distracting rather than helpful. Moreover, there are other applications of homological minors that require larger complete minors. We also remark that up to a change in the constant, it does not matter whether one excludes complete or complete multipartite minors.

remains the case even if a moderate number of vertices are deleted. In higher dimensions, the picture is more complicated, and the higher-dimensional analogue of the diameter in an expanding complex need not be logarithmic, see [39].

4. MINORS OF SIMPLICIAL COMPLEXES

4.1 Topological Minors

From a topological point of view, the most natural and straightforward choice to define minors of simplicial complexes may be to generalize topological minors. Already for this, there are several possibilities in higher dimensions. For the most restrictive definition, we say that a simplicial complex K is a *subdivision minor* of X, denoted $K \preccurlyeq_{\mathrm{SD}} X$, if X contains a subcomplex that is isomorphic to a subdivision of K. For the most general one, we say K is a *topological minor* of X if $|X|$ contains a subspace (not necessarily a subcomplex) homeomorphic to $|K|$. These notions obviously preserve embeddability: if K topologically embeds into X and X into $\mathbf{R}^d$ then K embeds into $\mathbf{R}^d$.

However, even the most permissive notion of topological minors fails when it comes to the existence of large complete minors in 2-complexes. One can show that a 2-dimensional complex X contains a subspace homeomorphic to $\mathbf{S}^2$ iff it contains a subcomplex homeomorphic to $\mathbf{S}^2$, in fact, a subdivision of K_4^2 (the boundary of a tetrahedron). Brown, Erdős and Sós [6] showed that there are 2-dimensional complexes on n vertices with as many as $\Omega(n^{5/2})$ triangles that do not contain a subcomplex homeomorphic to the 2-sphere $\mathbf{S}^2$, i.e., that do not contain K_4^2 as a topological minor.[7] Thus, the analogue of Conjecture 2 fails for topological minors.

4.2 Contractions and Nevo's Minors

Another very natural attempt is to extend the combinatorial definition of minors and to allow arbitrary deletions and arbitrary contractions of faces of a simplicial complex. Here, the most naive notion of a contraction would be to identify two vertices along a common edge. In the course of such a contraction of u with v, for every face $F = G \cup \{v\}$ in X with $u \notin G$, we remove F and replace it by the face $\mathbf{F'} = G \cup \{u\}$. If F' is already present in X, we just retain one copy (that is, we do not keep "multiple faces", as in edge contractions in simple graphs).

Unfortunately, simple examples show that the first desired property fails, i.e., we can have $K \preccurlyeq_{\mathrm{DC}} X$, and X embeds into $\mathbf{R}^d$, but $\mathfrak{o}^d_{\mathbf{F}_2}(K) \neq 0$. For instance, consider the complex Y on the left-hand side of Figure 1, which is homeomorphic to a triangle with a small open hole punched in its center. By contracting the edge uv, we obtain the complex Y' on

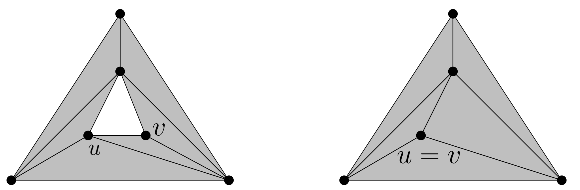

Figure 1: Closing a hole by contracting an edge.

[7]A simple alternative proof of this is due to Linial (personal communication) and uses a random complex $X^2(n, cn^{-1/2})$ with small alterations

the right, which is a subdivided triangle. Now consider the complete 2-complex K_7^2 on seven vertices. If we replace one of the triangles of K_7^2 with the complex Y, we obtain a complex X that is homeomorphic to K_7^2 with a small hole in one of the triangles. It is known (and easy to see) that X embeds into $\mathbf{R}^4$. However, if we contract the edge uv in X, we obtain a complex X' that is homeomorphic to K_7^2, hence $\mathfrak{o}^4_{\mathbf{F}_2}(X') \neq 0$ and X' does not embed into $\mathbf{R}^4$. Thus, a notion of minors based on arbitrary contractions is unsuitable for embeddability questions.

One way around this difficulty was suggested by Nevo [38], who introduced a notion of *admissible* contractions and a corresponding notion of minors. The idea is that a contraction as in the example closes a "higher-dimensional hole" in the complex because the edge uv is incident to a *missing triangle* (a triangle whose boundary belongs to the complex but the triangle itself doesn't). Identifying the two endpoints of an edge uv is an *admissible contraction* only if the edge is not contained in any missing triangle (in higher dimensions, one also has to exclude missing faces of other dimensions, see [38] for the details). Nevo showed that the corresponding notion of deletion-and-admissible-contraction minors does preserve embeddability information, i.e., if $K \preccurlyeq_{\mathrm{Nevo}} X$ and if $\mathfrak{o}^d_{\mathbf{F}_2}(K) \neq 0$ then $\mathfrak{o}^d_{\mathbf{F}_2}(X) \neq 0$.

However, it seems difficult to prove a variant of Conjecture 2 for this notion of minors. To illustrate the difficulty, consider the random complex $X^2(n, C/n)$. Then the probability that a given edge uv is incident to a certain number t of triangles equals $\binom{n-2}{t} p^t (1-p)^{n-2-t} \approx C^t e^{-C}/t!$ for small t and $n \to \infty$. Thus, the triangle-degree of a given edge is roughly Poisson distributed. Since the 1-skeleton is complete, it follows that with high probability, every edge is indicent to many missing triangles, and it is not clear how to delete edges (and their incident triangles) in a controlled manner so as to obtain edges that are admissible to contract.

4.3 Homological Minors

In order to circumvent the difficulties pointed out above, we propose the notion of homological minors.[8]

To motivate the definition, let us rephrase definition of subdivision minors slightly differently.We have $K \preccurlyeq_{\mathrm{sd}} X$ iff, for every face $F \in K$, there is a subcomplex $Y_F \subseteq X$ that is isomorphic to a subdivision of F and such that $Y_F \cap Y_G = Y_{F \cap G}$ for all $F, G \in K$. To define homological minors $K \preccurlyeq_H X$, we relax this in two ways: First, we drop the requirement that an i-face $F \in K$ be represented by a subcomplex $Y_F \subseteq X$ that is a combintorial i-dimensional *ball* (isomorphic to a subdivided i-simplex). Instead, we allow more general i-dimensional chains. Second, we relax the conditions on how the Y_F may intersect.

Definition 1. Let K and X be simplicial complexes. K is a *homological minor* of X, denoted $K \preccurlyeq_H X$, if

0. there is a *chain map* φ from K to X such that

[8]We mention that there is a line of research that studies generalizations of minors to matroids and related structures, see, for instance [14] or [19]. These notions are somewhat related to the notion of homological minors, since the starting point for both is the linear algebra setting of cycles and cocycles (for a graph, simplicial complex, or matroid). However, matroid minors seem to ignore the interactions beweeen boundary maps of various dimension and thus seem to have no direct bearing on embeddability and related questions.

1. $\varphi(\emptyset) = 1 \cdot \emptyset$; equivalently, for every vertex v on K, its image $\varphi(v)$ is set of vertices of *odd* cardinality.

2. φ preserves *disjoint vertex supports*: if F and G are vertex-disjoint simplices, then every simplex in $\varphi(F)$ is vertex-disjoint from all simplices occurring in $\varphi(G)$.

We remark that apart from the disjointness Condition 2, the definition boils down to linear algebra (which is the reason why chains are easier to handle than disks): Condition 0 (being a chain map) amounts to a homogeneous system of linear equations, and Condition 1 to one inhomogeneous linear equation. As an immediate consequence of the definition, we obtain the following:

OBSERVATION 8. *If φ is a chain map witnessing $K \preccurlyeq_H X$ then by Property 2, φ induces a equivariant chain map $\widetilde{\varphi}$ between the* deleted products *K^2_{del} and X^2_{del}, defined by "tensoring", i.e., $\widetilde{\varphi}(F \times G) = \sum_{A,B} A \times B$, where A ranges over the simplices in $\varphi(F)$ and B ranges over the simplices in $\varphi(G)$. Moreover, by Property 1, $\widetilde{\varphi}(\emptyset) = 1 \cdot \emptyset$, i.e., $\widetilde{\varphi}$ maps the empty face of K^2_{del} to that of X^2_{del}.*

Theorem 5 is a direct consequence of the preceding observation and Lemma 4.

Remark 2. In this extended abstract, we restrict ourselves to coefficients in the field $\mathbf{F}_2$, but the definitions carry over verbatim to arbitrary coefficient rings, with finite fields being most convenient to work with. In particular, the methods extend to Tverberg-type questions (where we are interested in p-fold covered image points instead of twofold ones), for which $\mathbf{F}_p$ is the appropriate choice of coefficients. One can also define a notion of minors based directly on the existence of chain maps between deleted products as in Lemma 4. These extensions and modifications will be discussed in the full version of this paper.

Remark 3. One can show that a minor in Nevo's sense is also a homological minor (the argument is implicit in [38]). However, homological minors are strictly more general. For instance, one can show that an arbitrary homology k-cycle X (considered as a simplicial complex) contains K^k_{k+2} as a homological minor but $K^k_{k+2} \preccurlyeq_{\text{Nevo}} X$ if and only if X is a piecewise linear k-sphere. A particularly simple example that shows that the two notions differ already for graphs (for which Nevo's minors are just usual graph minors) was pointed our by Nevo (personal communication): The "claw" $K^1_{1,3}$ contains K^1_3 (the boundary of a triangle) as a homological minor but not as a graph minor.

5. HIGHER-DIMENSIONAL EXPANSION

For graphs of arbitrary density, *edge expansion* can be defined as follows.

Definition 2. Let $G = (V, E)$ be a graph, and let $\varepsilon > 0$. We say that G is ε-edge expanding if for every $S \subseteq V$,

$$\frac{|E(S, V \setminus S)|}{|E|} \geq \varepsilon \cdot \frac{\min\{|S|, |V \setminus S|\}}{|V|}, \qquad (1)$$

where $E(S, V \setminus S)$ is the set of edges across the cut $(S, V \setminus S)$, i.e., with one endpoint in S and the other one in $V \setminus S$.

(For graphs with bounded degrees, (1) is easily seen to be equivalent, up to a change in the constant ε, to the more ususal condition that $|E(S, V \setminus S)| \geq \varepsilon|S|$ whenever $|S| \leq |V|/2$.) In order to generalize this to higher dimensions, we rephrase everything in terms of cochains.

Since we are working over $\mathbf{F}_2$, there is a 1-1 correspondence between subsets $S \subseteq V$ and 0-cochains (i.e., functions $\alpha\colon V \to \mathbf{F}_2$), by identifying S with its characteristic function $\mathbf{1}_S$, and the set $E(S, V \setminus S)$ of edges corresponds to the 1-cochain $\partial^*\alpha$. The constant 0-cochains $\mathbf{0}$ and $\mathbf{1}$ are precisely the coboundaries of the two possible (-1)-dimensional cochains, and they correspond to the trivial cuts with $S = \emptyset$ and $S = V$, respectively. Adding the constant 0-dimensional cochain $\mathbf{1}$ to a 0-cochain $\alpha = \mathbf{1}_S$ is the same as exchanging the two sides S and $V \setminus S$ of the corresponding cut.

In general, let X be a simplicial complex. We equip the vector space $\alpha \in C^i(X) \cong \mathbf{F}_2^{X_i}$ with the *Hamming norm*, i.e., we define $|\alpha|$ to be the number of 1's that appear in the vector α, or equivalently, the number of i-faces in that are mapped to 1 by α. Then we normalize by the number of all i-faces and define $\|\alpha\| := \frac{|\alpha|}{f_i(X)}$.

In the case of 0-cochains, we defined expansion by bounding the norm $\|\partial^*\alpha\|$ of the coboundary from below in terms of $\min\{\|\alpha\|, \|\mathbf{1} + \alpha\|\}$. In general, the right measure is the *normalized distance* of α from the space B^i of coboundaries (the trivial kernel of ∂^*). That is, we define

$$\|[\alpha]\| := \min\{\|\alpha + \partial^*\beta\| : \beta \in C_{i-1}\}.$$

We remark that the definitions of $\|\alpha\|$ and of $\|[\alpha]\|$ depend on the ambient complex (if also $\alpha \in C_i(Y)$ for some $Y \subseteq X$, then the norms may be different with respect to Y).

Now we are ready to define higher-dimensional expansion, which we refer to as *face expansion* or *cohomological expansion*. For our proof we will also need a *coarse* version of it that only applies to large cochains.

Definition 3. Let $\varepsilon > 0$. We say that a finite simplicial complex X has *i-dimensional face expansion ε* or that it is *ε-expanding in dimension i* if

$$\|\partial^*\alpha\| \geq \varepsilon \cdot \|[\alpha]\| \qquad (2)$$

holds for all $\alpha \in C^{i-1}(X)$. If we only require (2) for all $\alpha \in C^{i-1}(X)$ with $\|\alpha\| \geq \delta$ for some δ then we say that X is *coarsely* (ε, δ)-expanding in dimension i. We also call ε the *expansion factor* and δ the *coarseness*.

Note that in the definition, we shifted the notation from i-chains to $(i-1)$-chains compared to the preceding discussion so that expansion in dimension i captures properties of the i-dimensional faces of X. The basic observation in this context (see [15, 27, 36]) is that the complete complex K^k_n is a face expander in all dimensions:

PROPOSITION 9. *The complete complex K^k_n has i-dimensional face expansion 1 for all $i \in \{0, 1, \ldots, k\}$.*

Well-known Chernoff-type concentration bounds easily imply the following:

LEMMA 10. *Let $X = X^k(n, C/n)$, where $C > 0$ is a sufficiently large constant. Then, X is 1-expanding in every dimension $i \leq k - 1$, and asymptotically almost surely X is coarsely expanding in dimension k with expansion facor $1/2$ and coarseness $\Omega(1/C)$.*

This is as an analogue of the fact that the random graph $G(n, C/n)$ has a giant component of size $\Omega(n)$ and that this component is edge expanding. The (straightforward) proof of this lemma is omitted here.

6. PARTITIONS AND COLORFUL HOMOLOGICAL MINORS AND COMINORS

Let K and X be simplicial complexes of dimension k. Consider a graded linear map $\varphi : C_*(K) \to C_*(X)$, i.e., a family of linear maps $\varphi_i : C_i(K) \to C_i(X)$. We can identify each φ_i with a matrix of dimension $f_i(X) \times f_i(K)$ over $\mathbf{F}_2$, and by writing the entries of the matrices in a row (in some specified order), we can view φ as a vector over $\mathbf{F}^2$ of length $\sum_i f_i(X) f_i(K)$. The entries of this vector are indexed by pairs of faces $(A, F) \in K \times X$ of equal dimension $\dim A = \dim F$, and the entry of φ in position (A, F) equals the coefficient with which F appears in the chain $\varphi(A)$.

Condition 0 in the definition of homological minors, the property of being a chain map, simply amounts to a number of homogeneous linear conditions on the entries of φ, with the coefficients of the linear equations given by the boundary matrices of K and X.

Condition 1, $\varphi(\emptyset) = \emptyset$ is also a linear condition, albeit an imhomogeneous one: it simply says that a fixed entry of φ (w.l.o.g. the first entry) equals 1. Only Condition 2, disjointness of vertex supports, is not linear. However, we can enforce it as follows.

6.1 Partitions and Colorful Chain Maps

Suppose that K has t vertices, w.l.o.g. $K_0 = \{1, \ldots, t\} =: [t]$. Fix a partition of the vertices of X into t parts or *color classes*, in other words, a map $\mathcal{P} : X_0 \to [t]$. We restrict our attention to *colorful* simplices of X, i.e., faces $F \in X$ that contain at most one vertex of each color.

For a given $A \in K$, let us say that a simplex $F \in X$ is *A-colored* if F contains precisely one vertex of each color $i \in A$ (such a simplex is necessarily of the same dimension as A), and let $X[A]$ denote the set of A-colored simplices in X (strictly speaking, $X[A]$ depends on the partition $\mathcal{P}$, but we suppress this from the notation). Let us say that a graded map $\varphi : C_*(K) \to C_*(X)$ is *color-faithful* if, for every $A \in K$, the chain $\varphi(A)$ is supported on the A-colored simplices $X[A]$, i.e., every simplex $F \in X$ that appears with a nonzero coefficient in the chain $\varphi(A)$ is A-colored. Such a φ trivially preserves vertex-disjoinness, since for vertex-disjoint $A, B \in K$, any $F \in X[A]$ and $G \in X[B]$ are vertex-disjoint as well.

From now on, we only consider color-faithful graded linear maps $\varphi : C_*(K) \to C_*(X)$. As before, we can identify such a map with a vector over $\mathbf{F}_2$, whose entries are indexed by pairs (A, F), where $A \in K$ and F is now required to be A-colored. Thus, the length of φ equals $\sum_{A \in K} |X[A]|$. Condition 0 of being a chain map means that φ is a solution to a homogeneous system of linear equations,

$$M\varphi = 0. \qquad (3)$$

Here, M is called the *enhanced boundary matrix of* X and defined as follows. The columns of M are indexed by pairs $(A, F) \in K \times X$ with $F \in X[A]$ as above, and the rows of M are indexed by pairs (G, i), where G is a colorful face of X *of dimension at most* $(k-1)$, G is B-colored for some $B \in K$, and $i \in [t] \setminus B$ is a color that does not appear in G. The entry of M in row (G, i) and column (A, F) equals 1 if either $A = B \cup \{i\}$ and $G \subseteq F$ (i.e., if G is a facet of F and i is the color of the unique vertex of F missing from G) or if $G = F$. Otherwise, the entry of M at that position is zero.

For a fixed K and a *random partition* of the vertices of X, the existence of a nonzero solution to (3) is almost trivial if $f_k(X) \geq C \cdot f_{k-1}(X)$ for a sufficiently large constant C, by counting variables and constraints. However, we are looking for a solution φ of (3) that also satisfies one additional *inhomogeneous condition* $\varphi(\emptyset) = \emptyset$, i.e., for a vector $\varphi \in \ker M$ whose entry in the first position equals 1. Because of this inhomogeneity, the simple dimension counting argument breaks down.

6.2 Cominors

When does there exist a solution $\varphi \in \ker M$ with first coordinate equal to 1? It does *not* exist if and only if the row vector $(1, 0, \ldots, 0)$ lies in the row space of M i.e., if there is a row vector α such that $\alpha M = (1, 0, \ldots, 0)$. We call any such row vector α a *cominor*.

Let us take a closer look at what it means. The entries of α are indexed by the same pairs (G, i) as the rows of M, i.e., G is a colorful face of X of dimension $\dim(G) < k$, and $i \in [t]$ is a color that does not appear in G. The conditions for α to be a cominor are as follows: first of all, $\sum_{i=1}^{t} \alpha(\emptyset, i) = 1$ (all calculations are modulo 2). Secondly, for every nonempty colorful face F of X,

$$\sum_{i \in [t] \setminus \mathcal{P}} \alpha(F, i) \; + \sum_{G \subset F, |G| = |F|-1} \alpha(G, \mathcal{P}(F \setminus G)) = 0$$

if $\dim F < k$ and

$$\sum_{G \subset F, |G| = |F|-1} \alpha(G, \mathcal{P}(F \setminus G) = 0$$

if $\dim F = k$. Equivalently, we can view α as a collection $\{\alpha_A\}$ of $\dim(A)$-dimensional cochains on the subcomplexes[9] $X[A]$, one for each simplex $A \in K$, by setting $\alpha_A(F) := \alpha(F, i)$, where F is face of X that is B-colored for the unique facet $B \subset A$ with $\{i\} = A \setminus B$, i.e., i is the unique color of A that does not appear in F.

Thus, in the case $K = K_t^k$, for each $i \in [t]$, we have a (-1)-dimensional cochain α_i on the 0-dimensional complexes $X[i]$. For each pair $\{i, j\} \in \binom{[t]}{2}$, we have a 0-cochain on $X[\{i, j\}]$, etc. The condition of being a cominor translates to the requirement that $\sum_{i=1}^{t} \alpha_{\{i\}}(\emptyset) = 1$, and for $A \in K_t^k$ and $F \in X[A]$,

$$\partial^* \alpha_A(F) = \sum_{i \in [t] \setminus A} \alpha_{A \cup i}(F) \qquad (4)$$

if $0 \leq \dim A < k$ and

$$\partial^* \alpha_A = 0 \qquad (5)$$

if $\dim A = k$. Thus, in the latter case, α_A is a *cocycle* on $X[A]$ (note that α_A need not be a cocycle when considered as a chain in X).

In order to show the existence of a K_t^k minor, we will assume there is a cominor and derive a contradiction. For this, we use face expansion.

[9] Here, we abuse notation and think of $X[A]$ as the subcomplex consisting of all A-colored simplices and their faces.

6.3 Minors in Expanding Complexes

We are now ready to prove Theorem 7. The first step is the following lemma.

LEMMA 11. *Let X be a k-dimensional simplicial complex with a given t-partition of the vertices, and assume that α is a K_t^k-cominor with respect to that partition. Then we may modify α, while preserving the cominor conditions, and achieve that for every $A \in K_t^k$, the cochain α_A is* cohomologically minimal, *i.e.* $\|\alpha_A\| = \|[\alpha_A]\|$, *in the subcomplex* $X[A]$.

PROOF. We prove this top-down. Let $i \le k$ and $C \in \binom{[t]}{i+1}$. Then the component α_C of the cominor is an $(i-1)$-dimensional cochain on $X[C]$. Let β be an arbitrary $(i-1)$-dimensional coboundary on $X[C]$. It suffices to show that we can modify α to obtain a new cominor α' such that $\alpha'_C = \alpha_C + \beta$ and that $\alpha'_A = \alpha_A$ for all $A \subseteq [t]$ with $|A| \ge i+1$ and $A \ne C$. (If we can prove this then we can make all α_A cohomologically minimal in a top-down fashion, since our next modification never interferes with any of the previous ones in the same or higher dimensions.)

Let γ be an $(i-2)$-dimensional cochain on $X[C]$ with $\partial^*\gamma_C = \beta_C$. For every "facet" $B \in \binom{C}{i}$ of C, define $\alpha'_B := \alpha_B + \gamma|_{X[B]}$ (each $X[B]$ is a subcomplex of $X[C]$, so the restriction of γ is defined). Moreover, define $\alpha'_C := \alpha_C + \beta$, and set $\alpha'_A := \alpha_A$ for every $A \subseteq [t]$ that is not equal to C or a facet of C. We claim that the resulting α' is still a cominor.

Consider a subset $A \subseteq [t]$. If $A = C$ then $\partial^*\alpha'_C = \partial^*(\alpha_C + \beta) = \partial^*\alpha_C$, and $\alpha'_S = \alpha_S$ for all proper supersets $S \supset C$. Therefore, whichever of the two cominor relations (4) and (5) applies, it is preserved.

Next, suppose that A is a facet of C and let $F \in X[A]$. We have $\partial^*\alpha'_A(F) = \partial^*\alpha_A(F) + \beta(F)$. On the other hand, there is a unique $i \in [t] \setminus A$ such that $A \cup i = C$. For this i, we have $\alpha'_{A\cup i}(F) = \alpha'_C(F) = \alpha_C(F) + \beta(F)$. For all other $j \in [t] \setminus A$, $A \cup j$ is neither equal to C nor to a facet of C, hence $\alpha'_{A\cup j}(F) = \alpha_{A\cup j}(F)$ for all these other j. Summing up, we see that relation (4) is preserved since we simply add $\beta(F)$ once on either side of the equation.

If $A \subseteq C$ and $|A| = i-1$ then there are precisely two superfaces $B \subset A$ of size $|B| = i$ that are facets of C. On each of these two facets, α_B is modified by adding γ. On all other B, α_B is unchanged. Thus, in the cominor relation (4) for A, we add $\beta(F)$ exactly twice to the right-hand side and make no changes on the left-hand side, so the relation is preserved.

For all other A, neither α_A nor the cochains $\alpha_{A\cup i}$ are affected. This shows that the modified α' is a cominor, which completes the proof of the lemma. □

PROPOSITION 12. *Suppose that X is a simplicial complex on n vertices, and suppose that we have a partition $X_0 = V_1 \cup \ldots V_t$ of the vertices of X into t classes (colors) such that the following properties hold:*

1. *The partition is an* equipartition *of the complex (of the vertices as well as of the higher-dimensional faces). That is, we assume that all $|V_i|$ are equal to $\frac{1+o(1)}{t}n$, and that for each dimension $i < t$ and each $A \in \binom{[t]}{i+1}$, we have $f_i(X[A]) = (1+o(1))\left(\frac{i+1}{t}\right)^{i+1} \cdot f_i(X)$.*

   (This is the behavior we get with high probability when we t-color the vertices uniformly at random.)
2. *For $1 \le i \le k$ and $A \in \binom{[t]}{i+1}$, the complex $X[A]$ is coarsely face expanding in dimension i with face expansion at least $\varepsilon_i > 0$ and coarseness $\delta_i < \frac{i!\varepsilon_1\varepsilon_2\cdots\varepsilon_{i-1}}{(i+1)^i t^i}$.*

Then there is no K_t^k-cominor for this partition, i.e., X has a colorful homological minor.

PROOF. Assume that α is a cominor for the partition. By the preceding lemma, we may assume that every component α_A is cohomologically minimal.

Since $\sum_i \alpha_i(\emptyset) = 1$, we must have $\alpha_i(\emptyset) = 1$ for some i, say $\alpha_1(\emptyset) = 1$. For each $v \in V_1$, we have $\partial^*\alpha_1(v) = 1 = \sum_{i\in[t]\setminus\{1\}} \alpha_{\{1,i\}}(v)$. It follows that there is some i such that for at least $1/t$ of the vertices in V_1, we have $\alpha_{1i}(v) = 1$, say $i = 2$. It follows that $|\alpha_{12}| \ge |V_1|/t = |V_1 \cup V_2|/2t$, hence $\|\alpha_{12}\| = \|[\alpha_{12}]\| \ge \frac{1}{2t}$ in $X[12]$. Here, we use that α_{12} is cohomologically minimal and that $|V_1| = |V_2|$.

We assume that $\delta_0 < 1/2t$ and that $X[12]$ is coarsely expanding. Thus, $\|\partial^*\alpha_{12}\| \ge \epsilon_0/2t$ in $X[12]$. Thus, $\partial^*\alpha_{12}$ is supported by a fraction of at least $\varepsilon_1/2t$ of the edges of $X[12]$. By the cominor relation, there must be an index j such that α_{12j} is supported by a fraction of at least $\frac{\varepsilon_1}{2t^2}$ of the edges in $X[12]$, say $j = 3$.

By the equipartition property and since α_{123} is minimal, we get $\|[\alpha_{123}]\| \ge \frac{\varepsilon_1}{2t^2} \cdot \frac{4}{9}(1+o(1))$ in $X[123]$. If $\delta_2 < \frac{2\varepsilon_1}{9t^2}$ then coarse expansion of $X[123]$ implies that $\|\partial^*\alpha_{123}\| \ge \frac{2\varepsilon_1\varepsilon_2}{9t^2}$ in $X[123]$.

Inductively, we get, for every $i \le k$, a set $A \in \binom{t}{i+1}$ such that

$$\|[\alpha_A]\| \ge \frac{i!\varepsilon_1\varepsilon_2\cdots\varepsilon_{i-1}}{(i+1)^i t^i} > \delta_i$$

and

$$\|\partial^*\alpha_A\| \ge \frac{i!\varepsilon_1\varepsilon_2\cdots\varepsilon_i}{(i+1)^i t^i}$$

in $X[A]$. For $i = k$, this is a contradiction, since in this case, we should have $\partial^*\alpha_A = 0$. □

PROOF OF THEOREM 7. Let $X = X^k[n, C/n]$, where C is a constant to be determined. Fix any partition (coloring) of the vertices of X into t parts of equal size, say $[n] = V_1 \cup \ldots \cup V_t$ with $V_i = \{\frac{(i-1)n}{t}+1, \ldots, \frac{in}{t}\}$, $1 \le i \le t$.

Observe that for this coloring each face of K_t^k, the complex $X[A]$ is isomorphic to $X^k(n', C'/n')$ where $n' = \frac{k+1}{t}n$ and $C' = \frac{k+1}{t}C$. It follows easily that asymptotically almost surely the coloring yields an equipartition of X in the sense of Proposition 12.

Moreover, by Lemma 10 the necessary expansion properties hold asymptotically almost surely provided

$$\frac{k+1}{t}C = C' > \frac{8(k+1)\ln(2)(k+1)^k t^k}{k!}.$$

This proves the theorem. □

Remark 4. There is a strong formal similarity between the pigeonholing argument in the proof of Proposition 12 and the combinatorial part of Gromov's proof of the *Filling-Contraction Inequality* in [15, Section 2.4]. On the other hand, the underlying topological arguments that lead to these combinatorial problems seem to be different, at least at first sight. It would be interesting to understand this connection better.

7. CONCLUDING REMARKS

We have introduced a new notion of minors and used it to prove the asymptotically tight threshold for embeddability of random k-complexes. The same methods can also be used to obtain analogous results for various Tverberg-type problems. This will be discussed in the full version of this paper.

Moreover, we hope that the approach presented here will also be useful for the corresponding extremal problems. This is work in progress.

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8. REFERENCES

[1] M. Ajtai, V. Chvátal, M. Newborn, and E. Szemerédi. Crossing-free subgraphs. *Ann. Discrete Math.*, 12:9–12, 1982.

[2] L. Aronshtam, N. Linial, T. Luczak, and R. Meshulam. Vanishing of the top homology of a random complex. arXiv:1010.1400v1, 2010.

[3] E. Babson, C. Hoffman, and M. Kahle. The fundamental group of random 2-complexes. *J. Amer. Math. Soc.*, 24(1):1–28, 2011.

[4] B. Bollobás, P. A. Catlin, and P. Erdős. Hadwiger's conjecture is true for almost every graph. *European J. Combin.*, 1(3):195–199, 1980.

[5] B. Bollobás and A. Thomason. Proof of a conjecture of Mader, Erdős and Hajnal on topological complete subgraphs. *European J. Combin.*, 19(8):883–887, 1998.

[6] W. G. Brown, P. Erdős, and V. T. Sós. Some extremal problems on r-graphs. In *New directions in the theory of graphs (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich, 1971)*, pages 53–63. Academic Press, New York, 1973.

[7] D. C. Cohen, M. Farber, and T. Kappeler. The homotopical dimension of random 2-complexes. arXiv:1005.3383v1, 2010.

[8] A. Costa, M. Farber, and T. Kappeler. Topology of random 2-complexes. arXiv:1006.4229v2, 2010.

[9] T. K. Dey. On counting triangulations in d dimensions. *Comput. Geom.*, 3(6):315–325, 1993.

[10] T. K. Dey and J. Pach. Extremal problems for geometric hypergraphs. *Discrete Comput. Geom.*, 19(4):473–484, 1998.

[11] T. K. Dey and N. R. Shah. On the number of simplicial complexes in $\mathbf{R}^d$. *Comput. Geom.*, 8(5):267–277, 1997.

[12] P. Erdős. On extremal problems of graphs and generalized graphs. *Israel J. Math.*, 2:183–190, 1964.

[13] A. Flores. Über die Existenz n-dimensionaler Komplexe, die nicht in den $\mathbf{R}^{2n}$ topologisch einbettbar sind. *Ergeb. Math. Kolloqu.*, 5:17–24, 1933.

[14] J. Geelen, B. Gerards, and G. Whittle. Towards a matroid-minor structure theory. In *Combinatorics, complexity, and chance*, volume 34 of *Oxford Lecture Ser. Math. Appl.*, pages 72–82. Oxford Univ. Press, Oxford, 2007.

[15] M. Gromov. Singularities, expanders and topology of maps. Part 2: From combinatorics to topology via algebraic isoperimetry. *Geom. Funct. Anal.*, 20(2):416–526, 2010.

[16] A. Gundert. On the complexity of embeddable simplicial complexes. Diplomarbeit, Freie Universität Berlin, 2009.

[17] H. Hadwiger. Über eine Klassifikation der Streckenkomplexe. *Vierteljschr. Naturforsch. Ges. Zürich*, 88:133–142, 1943.

[18] A. Haefliger. Plongements de variétés dans le domaine stable. *Sém. Bourbaki*, 245, 1982.

[19] T. Kaiser. Minors of simplicial complexes. *Discrete Appl. Math.*, 157(12):2597–2602, 2009.

[20] G. Kalai. The diameter of graphs of convex polytopes and f-vector theory. In *Applied geometry and discrete mathematics*, volume 4 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 387–411. Amer. Math. Soc., Providence, RI, 1991.

[21] J. Komlós and E. Szemerédi. Topological cliques in graphs. II. *Combin. Probab. Comput.*, 5(1):79–90, 1996.

[22] D. Kozlov. *Combinatorial Algebraic Topology*, volume 21 of *Algorithms and Computation in Mathematics*. Springer, Berlin, 2008.

[23] D. Kozlov. The threshold function for vanishing of the top homology group of random d-complexes. *Proc. Amer. Math. Soc.*, 138(12):4517–4527, 2010.

[24] M. Krivelevich and B. Sudakov. Minors in expanding graphs. *Geom. Funct. Anal.*, 19(1):294–331, 2009.

[25] K. Kuratowski. Sur le problème des courbes gauches en topologie. *Fundamenta*, 15:271–283, 1930.

[26] F. T. Leighton. *Complexity Issues in VLSI.* MIT Press, Cambridge, MA, 1983.

[27] N. Linial and R. Meshulam. Homological connectivity of random 2-complexes. *Combinatorica*, 26(4):475–487, 2006.

[28] L. Lovász. Graph minor theory. *Bull. Amer. Math. Soc. (N.S.)*, 43(1):75–86 (electronic), 2006.

[29] T. Łuczak, B. Pittel, and J. C. Wierman. The structure of a random graph at the point of the phase transition. *Trans. Amer. Math. Soc.*, 341(2):721–748, 1994.

[30] W. Mader. Homomorphieeigenschaften und mittlere Kantendichte von Graphen. *Math. Ann.*, 174:265–268, 1967.

[31] W. Mader. $3n-5$ edges do force a subdivision of K_5. *Combinatorica*, 18(4):569–595, 1998.

[32] J. Matoušek. *Using the Borsuk-Ulam theorem.* Springer-Verlag, Berlin, 2003.

[33] J. Matoušek, M. Tancer, and U. Wagner. Hardness of embedding simplicial complexes in $\mathbf{R}^d$. *J. Eur. Math. Soc.*, 13(2):259–295, 2011.

[34] P. McMullen. The maximum numbers of faces of a convex polytope. *Mathematika*, 17:179–184, 1970.

[35] S. A. Melikhov. The van Kampen obstruction and its relatives. *Proc. Steklov Inst. Math.*, 266(1):142–176, 2009.

[36] R. Meshulam and N. Wallach. Homological connectivity of random k-dimensional complexes. *Random Structures Algorithms*, 34(3):408–417, 2009.

[37] J. R. Munkres. *Elements of Algebraic Topology.* Addison-Wesley, Menlo Park, CA, 1984.

[38] E. Nevo. Higher minors and van Kampen's obstruction. *Math. Scand.*, 101:161–176, 2007.

[39] I. Newman and Y. Rabinovich. Finite volume spaces and sparsification. arXiv:1002.3541v3, 2010.

[40] N. Robertson, P. Seymour, and R. Thomas. Hadwiger's conjecture for K_6-free graphs. *Combinatorica*, 13(3):279–361, 1993.

[41] N. Robertson and P. D. Seymour. Graph minors. XX. Wagner's conjecture. *J. Combin. Theory Ser. B*, 92(2):325–357, 2004.

[42] K. S. Sarkaria. Shifting and embeddability of simplicial complexes. Technical report, Max-Planck-Institut für Mathematik, Bonn, MPI/92-51, 1992.

[43] A. B. Skopenkov. Embedding and knotting of manifolds in Euclidean spaces. In *Surveys in contemporary mathematics*, volume 347 of *London Math. Soc. Lecture Note Ser.*, pages 248–342. Cambridge Univ. Press, Cambridge, 2008.

[44] R. P. Stanley. The upper bound conjecture and Cohen-Macaulay rings. *Studies in Appl. Math.*, 54(2):135–142, 1975.

[45] A. Thomason. The extremal function for complete minors. *J. Combin. Theory Ser. B*, 81(2):318–338, 2001.

[46] B. R. Ummel. Imbedding classes and n-minimal complexes. *Proc. Amer. Math. Soc.*, 38:201–206, 1973.

[47] E. R. van Kampen. Komplexe in euklidischen Räumen. *Abh. Math. Sem. Univ. Hamburg*, 9:72–78, 1932.

[48] K. Wagner. Über eine Eigenschaft der ebenen Komplexe. *Math. Ann.*, 114(1):570–590, 1937.

[49] C. Weber. Plongements de polyedres dans le domaine metastable. *Comment. Math. Helv.*, 42:1–27, 1967.

[50] J. Zaks. On minimal complexes. *Pacif. J. Math.*, 28:721–727, 1969.