# On Bichromatic Triangle Game

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**Abstract**

We study a combinatorial game called Bichromatic Triangle Game, defined as follows. Two players $\mathcal{R}$ and $\mathcal{B}$ construct a triangulation on a given planar point set $V$. Starting from no edges, they take turns drawing one straight edge that connects two points in $V$ and does not cross any of the previously drawn edges. Player $\mathcal{R}$ uses color red and player $\mathcal{B}$ uses color blue. The first player who completes one empty monochromatic triangle is the winner. We show that each of the players can force a tie in the Bichromatic Triangle Game when the points of $V$ are in convex position, and also in the case when there is exactly one inner point in the set $V$.

As a consequence of those results, we obtain that the outcome of the Bichromatic Complete Triangulation Game (a modification of the Bichromatic Triangle Game) is also a tie for the same two cases regarding the set $V$.

*Keywords:* Combinatorial games, geometric games, triangulations.

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[1] Grant 2008/06508-8 from *Fundação de Amparo à Pesquisa do Estado de São Paulo*. Partially supported by *Conselho Nacional de Desenvolvimento Científico e Tecnológico* Project 475064/2010-0. Email: `manic.gordana@ufabc.edu.br`

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# 1 Introduction

Games on triangulations belong to the more general area of combinatorial games, which usually involve two players. We consider games with perfect information (i.e., there is no hidden information, in contrast to card games, like poker) and each of the players plays optimally (i.e., both players do their best to win). Any such game that cannot end in a draw has two possible outcomes: *first player's win*, where the first player has a strategy to win no matter how the other player moves throughout the game, and *second player's win*, where we have the same situation with the roles swaped between the players. On top of those two outcomes, combinatorial games with two players may have a third outcome, *a tie*, where both of the players have a strategy to prevent the opponent from winning. For more information on combinatorial game theory we refer the reader to [2], [3] and [4].

Let $V \subseteq \mathbb{R}^2$ be a set of points in the plane with no three collinear points. A *triangulation* of $V$ is a simplicial decomposition of its convex hull whose vertices are precisely the points in $V$. Aichholzer et al. [1] consider several combinatorial games involving the vertices, edges (straight line segments), and faces (triangles) of some triangulation. Their ultimate goal for each game is to characterize who wins the game and design efficient algorithms to compute a winning strategy, or alternatively, show that both players can force a tie and again determine and efficiently compute their defense strategies.

We first give the definition of the Bichromatic Complete Triangulation Game, as introduced in [1]. Two players $\mathcal{R}$ and $\mathcal{B}$ construct a triangulation on a given point set $V$. Starting from no edges, players $\mathcal{R}$ and $\mathcal{B}$ play in turns by drawing one straight edge in each move, with $\mathcal{R}$ making the first move. In each move, the chosen edge is not allowed to cross any of the previously drawn edges. Player $\mathcal{R}$ uses color red and player $\mathcal{B}$ uses color blue. A triangle formed by the already drawn edges is said to be *empty* if it contains no points from $V$ in its interior. Each time a player completes one or more empty monochromatic triangles, the player wins the corresponding number of points and it is again his turn (he has an extra move). Once the triangulation is complete, the game stops and the player who owns more points is the winner.

Bichromatic Triangle Game, also introduced in [1], starts as Bichromatic Complete Triangulation Game, but has a different winning condition. Namely, the player who completes an empty monochromatic triangle first is the winner. If the triangulation is complete and no player has won to that point, the game is a draw.

Aichholzer et al. [1] posed an open problem, to determine the outcomes of the Bichromatic Triangle Game and the Bichromatic Complete Triangulation Game, possibly depending on the configuration of points in $V$. Besides those two games, they analyze a whole family of games on triangulations and manage to obtain game outcomes only for some special configurations of points, suggesting that the whole family of problems in full generality (in terms of point configurations) is quite hard, and for now out of reach. On top of that, they conjecture that for many of the games even determining the outcome may be NP-hard for general triangulations, that is, when there is no predetermined condition for the points of $V$ apart from not having any three collinear points. Therefore, they focus their attention to special classes of triangulations, e.g., when points from $V$ are in convex position, and obtain positive results.

In the present paper we make a step forward in resolving Bichromatic Triangle Game and Bichromatic Complete Triangulation Game, giving the outcomes of the two games for a few special configurations of point configurations. In particular, for the Bichromatic Triangle Game, we show that $\mathcal{B}$ can force a tie when the points in $V$ are in convex position, and also when there is exactly one inner point in $V$ (we say that $v \in V$ is an *inner point* of $V$ if $v$ belongs to the interior of the convex hull of $V$). A simple strategy stealing argument ensures that $\mathcal{B}$ cannot win the game, so the outcome has to be a draw. As a consequence of the results we obtained for the Bichromatic Triangle Game, we get that the outcome of the Bichromatic Complete Triangulation Game is also a draw if the points of $V$ are in convex position, and in the case when there is exactly one inner point in the set $V$.

### *1.1 Notation*

Let $V \subseteq \mathbb{R}^2$ be a set of points in the plane. We say that the points of $V$ are in *convex position* if they are the vertices of some convex polygon. A point $v \in V$ is referred to as *inner*, if it is strictly inside the convex hull of $V$. For $u, v \in V$, denote by $uv$ the line segment with endpoints $u$ and $v$, which we will sometimes call an *edge*. We denote the set of all such line segments with $\binom{V}{2}$.

A *configuration* in a Bichromatic Triangle Game is the triple $(V, E_{\mathcal{R}}, E_{\mathcal{B}})$, where $V \subseteq \mathbb{R}^2$, and $E_{\mathcal{R}}, E_{\mathcal{B}} \subseteq \binom{V}{2}$ are two disjoint sets of edges, drawn by $\mathcal{R}$ and $\mathcal{B}$, respectively, during the course of a (possibly unfinished) game. So, no two edges in $E_{\mathcal{R}} \cup E_{\mathcal{B}}$ are allowed to cross. A *free edge* with respect to a configuration $(V, E_{\mathcal{R}}, E_{\mathcal{B}})$ is an edge in $\binom{V}{2} \setminus (E_{\mathcal{R}} \cup E_{\mathcal{B}})$ that does not cross any of the edges in $E_{\mathcal{R}} \cup E_{\mathcal{B}}$. Given a configuration $\mathscr{C} = (V, E_{\mathcal{R}}, E_{\mathcal{B}})$ and a set of points $W \subseteq V$, we define the *induced configuration* $\mathscr{C}[W] = (W, E'_{\mathcal{R}}, E'_{\mathcal{B}})$, where $E'_{\mathcal{R}} = \binom{W}{2} \cap E_{\mathcal{R}}$ and $E'_{\mathcal{B}} = \binom{W}{2} \cap E_{\mathcal{B}}$. Two induced configurations are said to be *independent* if they share precisely two points $v_1$ and $v_2$, and they both contain the edge $v_1v_2$ which is taken by the same player in both configurations.

Given points $x, y, z \in \mathbb{R}^2$, with the sequence $(x, y, z)$ being in clockwise order and non-collinear, we denote by $\widehat{xyz}$ the open set of points $w \in \mathbb{R}^2$ such that the sequences $(x, y, w)$ and $(w, y, z)$ are both in clockwise order and both non-collinear. Let $V$ be a finite planar set of points in convex position, denoted $v_1, \ldots, v_n$ in clockwise order. For $x = v_i, y = v_j$, by $\overrightarrow{\widehat{xy}}$ we denote the set of points $\{v_i, v_{i+1}, \ldots, v_j\}$, where index addition is taken modulo $n$. Moreover, for $x, y \in V$, if $x = v_i$ and $y = v_{i+1}$, we say that $x$ and $y$ are *consecutive* in $V$.

## 2 Our Results

**Lemma 2.1** *Suppose $\mathscr{C} = (V, E_{\mathcal{R}}, E_{\mathcal{B}})$ is a configuration of the Bichromatic Triangle Game where $V$ is a set of points in convex position and each of the drawn edges, i.e., edges in $E_{\mathcal{R}} \cup E_{\mathcal{B}}$, are between consecutive points in $V$. If $|E_{\mathcal{R}}| < 2$, then $\mathcal{B}$ can force a tie in $\mathscr{C}$ regardless of which player is next to make a move.*

**Proof.** The proof is by induction on $|V|$. If $|V| \leq 3$, the statement trivially holds. Hence, suppose $|V| = n > 3$ and suppose the statement is true for $|V| < n$. We denote by $r$ the cardinality of $E_{\mathcal{R}}$, so $r$ is either 0 or 1.

**Case 1.** If $\mathcal{B}$ is next to play he can choose to draw any free edge that does not connect two consecutive points from $V$, dividing $\mathscr{C}$ into two independent configurations $\mathscr{C}_1$ and $\mathscr{C}_2$. By the induction hypothesis $\mathcal{B}$ can force a tie both in $\mathscr{C}_1$ and in $\mathscr{C}_2$. Hence, $\mathcal{B}$ can force a tie in $\mathscr{C}$.

**Case 2.** If $\mathcal{R}$ is next to play, let $uv$ be the edge that he draws, and let $\overline{\mathscr{C}}$ be the resulting configuration, i.e., $\overline{\mathscr{C}} = (V, E_{\mathcal{R}} \cup \{uv\}, E_{\mathcal{B}})$. This move divided $\mathscr{C}$ into two independent configurations $\mathscr{C}_1 = \overline{\mathscr{C}}[\overrightarrow{\widehat{uv}}]$ and $\mathscr{C}_2 = \overline{\mathscr{C}}[\overrightarrow{\widehat{vu}}]$, with $uv$ belonging to both configurations. Clearly, the remainder of the game goes on independently in these two configurations. In order for $\mathcal{B}$ to force a tie in $\mathscr{C}$, he must force a tie as second player in both $\mathscr{C}_1$ and $\mathscr{C}_2$.

If $r = 1$, we may assume, without loss of generality, that $\mathscr{C}_1$ has two red edges, and that $\mathscr{C}_2$ has one red edge (including the edge $uv$ in both cases). Note that a tie can be easily forced when the configuration $\mathscr{C}_1$ has at most three points (by playing arbitrarily), so from now on we assume that it has at least four points. Now $\mathcal{B}$ picks a free edge $xy$ in $\mathscr{C}_1$ with $\mathscr{C}_1[\overrightarrow{xy}]$ and $\mathscr{C}_1[\overrightarrow{yx}]$ each containing a single red edge. The existence of such an edge is guaranteed by the properties of $\mathscr{C}_1$, i.e., it has at least four points, its vertices are in convex position, and its drawn edges are between consecutive points (note that the two red edges may have a common vertex). After $\mathcal{B}$ draws $xy$ we obtain a configuration $\mathscr{C}_1'$ which is divided into two independent configurations $\mathscr{C}_1'[\overrightarrow{xy}]$ and $\mathscr{C}_1'[\overrightarrow{yx}]$. By the induction hypothesis, $\mathcal{B}$ can force a tie in both $\mathscr{C}_1'[\overrightarrow{xy}]$ and $\mathscr{C}_1'[\overrightarrow{yx}]$ and hence $\mathcal{B}$ can force a tie in $\mathscr{C}_1$. As for $\mathscr{C}_2$, it is either trivial, i.e., a single edge, or $\mathcal{B}$ can force a tie in it by the induction hypothesis.

If, however, $r = 0$ and $u$ and $v$ are not consecutive in $V$, then both $\mathscr{C}_1$ and $\mathscr{C}_2$ have one red edge, and by the induction hypothesis $\mathcal{B}$ can force a tie in $\mathscr{C}_1$, and in $\mathscr{C}_2$.

Finally, if $r = 0$ and $u$ and $v$ are consecutive in $V$, then we set $\mathscr{C} = \overline{\mathscr{C}}$ and apply the argument from Case 1. □

The previous lemma helps us to determine the outcome of the game for points in convex position. Note that the win of $\mathcal{B}$ (as the second player) is ruled out by a simple strategy stealing argument.

**Theorem 2.2** *Player $\mathcal{B}$ can force a tie in the Bichromatic Triangle Game when the points in $V$ are in convex position.*

**Proof.** The first edge that $\mathcal{R}$ draws divides the initial configuration $\mathscr{C}$ into two independent configurations (one possibly trivial). By Lemma 2.1, $\mathcal{B}$ can force a tie in both of these configurations, and hence, $\mathcal{B}$ can force a tie in $\mathscr{C}$.□

**Theorem 2.3** *Player $\mathcal{B}$ can force a tie in the Bichromatic Triangle Game when the set of points $V$ has exactly one inner point.*

**Proof.** First we give an outline of the proof. Roughly speaking, we analyze every possible move of $\mathcal{R}$ and determine what the response of player $\mathcal{B}$ should be. Every time player $\mathcal{B}$ draws an edge in response to a move of $\mathcal{R}$, we divide the problem of forcing a tie into independent subgames. This process may be repeated more than once until each subgame falls into one of the following three types, in which player $\mathcal{B}$ can force a tie.

The first type of subgame has a configuration with the set of points in convex position, with all drawn edges being between consecutive points and with at most one red edge. To show that $\mathcal{B}$ can force a tie in such a subgame, we apply Lemma 2.1. The second type of subgame has a configuration on four points, possibly with some edges already drawn. As such subgames are of finite size, they are easy to analyze – checking through all possible configurations and possibilities we get that player $\mathcal{B}$ can force a tie in each of them. Finally, the third configuration type is more complex, and we are going to analyze it separately.

Throughout the game, it is part of $\mathcal{B}$'s defense strategy to make his move in the subgame in which player $\mathcal{R}$ played his last move. Exceptionally, if player $\mathcal{R}$ draws the last free edge in a subgame, $\mathcal{B}$ will respond in another subgame which still has free edges (if there is no more free edges, the game is finished). Clearly, this extra move does not harm player $\mathcal{B}$, so ignoring those moves of his does not affect our analysis.

Let $x$ be the unique inner point of $V$. Initially, there are no edges drawn and it is $\mathcal{R}$'s turn, so we have the configuration $(V, \emptyset, \emptyset)$. Regarding the first move of $\mathcal{R}$, we have two possibilities:

(i) *$\mathcal{R}$ draws an edge $ux$, for some $u \neq x$.*

(ii) *$\mathcal{R}$ draws an edge $uv$, for $u, v \neq x$.*

**Case (i).** Let $r$ be the line passing through $x$ that is perpendicular to the segment $ux$, and let $s$ be the line containing the segment $ux$, see Figure 1. Note that $r$ divides the plane into two half-planes. There must exist at least one point $v$ in the half-plane not containing $u$, otherwise $x$ would not be an inner point of $V$. Among all possible choices, we pick $v$ as close as possible to the line $s$. Now $\mathcal{B}$ draws $xv$ and we are left with subgames $(\overset{\frown}{\overrightarrow{uv}} \cup \{x\}, \{ux\}, \{xv\})$ and $(\overset{\frown}{\overrightarrow{vu}} \cup \{x\}, \{ux\}, \{xv\})$. One of these subgames is convex, so $\mathcal{B}$ can force a tie by Lemma 2.1. The other one will be handled in the following claim. Note that $uv$ is a free edge that belongs to both subgames. However, the color of $uv$ does not influence the outcome of the non-convex subgame. Hence, if edge $uv$ is ever played during the game we consider it to be a move in the convex subgame.

**Claim 2.4** *(Special subgame) Player $\mathcal{B}$ can force a tie in the configuration $(W, \{ux\}, \{xv\})$, where $x$ is the inner point of $W$, $\overset{\frown}{\overrightarrow{uv}} = \{u, v\}$ (i.e., $u$ and $v$ are consecutive in $W$), and no configuration points are contained in $\widehat{vux}$.*

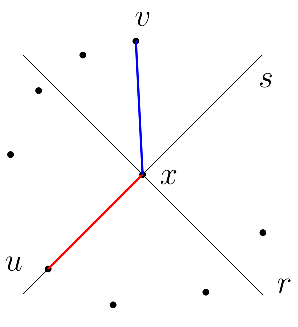


Fig. 1.

**Proof of the claim.** We look at the points of $W - x$ in clockwise order, and let $w$ be the first point after $v$. When player $\mathcal{R}$ makes another move in the configuration $(W, \{ux\}, \{xv\})$, if player $\mathcal{B}$ can draw $uw$, he does so and we are left with a trivial subgame on the points $u, v, x, w$, and one or more convex subgames (depending on the move of player $\mathcal{R}$), which can be taken care of by Lemma 2.1, see Figure 2a.

Otherwise, $\mathcal{R}$ has made a move that prevents $\mathcal{B}$ from playing $uw$. Hence, $\mathcal{R}$ has drawn either $uw$ or $vz$ or $xz$, for some $z \in \overset{\frown}{wu}$, with $z \neq u, w$. If $\mathcal{R}$ drew $uw$, then $\mathcal{B}$ draws $xw$, splitting the problem into a trivial subgame on the points $u, v, x, w$, and a convex subgame, see Figure 2b. If, however, $\mathcal{R}$ drew $vz$, then $\mathcal{B}$ draws $uz$, and produces two convex subgames (one of them possibly trivial - a single edge), and a subgame on the points $u, v, x, z$, see Figure 2c. Finally, if $\mathcal{R}$ drew $xz$, then again $\mathcal{B}$ draws $uz$ and we are left with two subgames $(\overset{\frown}{zu}, \emptyset, \{uz\})$ and $(\overset{\frown}{vz} \cup \{x\}, \{xz\}, \{xv\})$, see Figure 2d. While the first subgame is always convex (or trivial - a single edge), the second subgame may or may not be convex. However, if $(\overset{\frown}{vz} \cup \{x\}, \{xz\}, \{xv\})$ is not convex, then it is a smaller instance of this special subgame, which can be handled by induction. Hence, the claim is proved. △

**Case (ii).** We assume, without loss of generality, that $x$ is in the interior of $\overset{\frown}{uv}$, and we proceed as follows.

Suppose first that there exists an edge $yz$ leaving $x$ on one side and $uv$ on the other side. In this case, we assume that $y$ and $z$ are chosen so that $x$ is as close as possible to $yz$. (We may choose $z$ and $y$ according to the following simple algorithm. Starting from $v$, consider the points of $V - x$ in counterclockwise order and pick $y$ as the last point that is on the same side as $v$ with respect to the line $ux$. Now choose $z$ as the last point in clockwise order, starting from $u$, that is on the same side as $u$ with respect to $yx$. Note that $y$ may

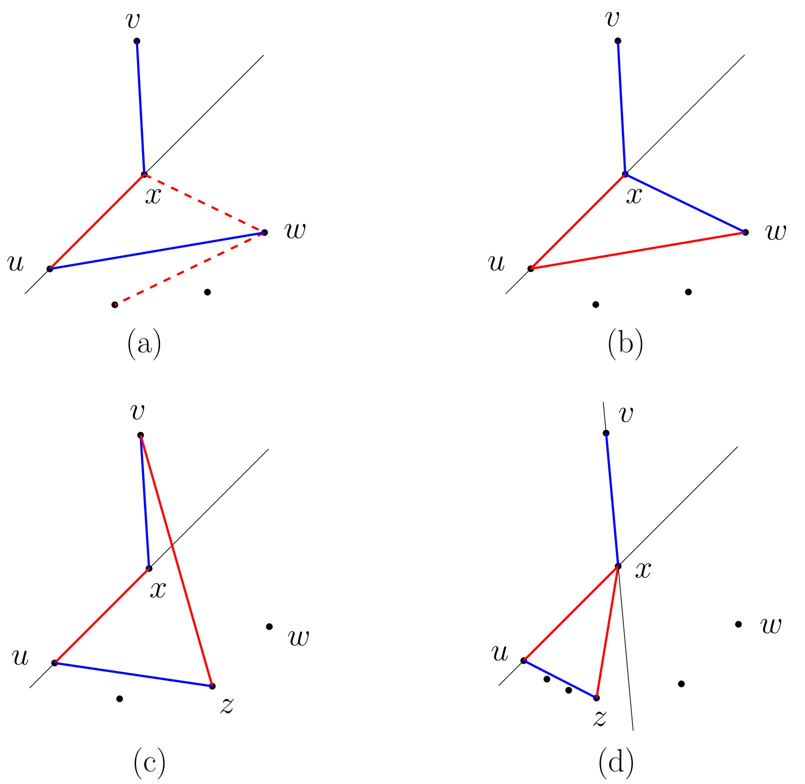


Fig. 2. (a) The dashed red edges are examples of possible moves of $\mathcal{R}$ that do not prevent $\mathcal{B}$ from playing $uw$. (d) If one of the subgames is not convex, then it is a smaller instance of the special subgame, which can be handled by induction.

be equal to $v$ and $z$ can be equal to $u$, but not both simultaneously.) Then $\mathcal{B}$ draws $yz$, and the problem is divided into three smaller subgames $(\overset{\curvearrowright}{yv} \cup \overset{\curvearrowright}{uz}, \{uv\}, \{yz\})$, $(\overset{\curvearrowright}{vu}, \{uv\}, \emptyset)$ and $(\overset{\curvearrowright}{zy} \cup \{x\}, \emptyset, \{yz\})$, see Figure 3. Note that the first two subgames are convex (some of them possibly degenerate). For the third subgame we show that $\mathcal{B}$ can force a tie in the following paragraph.

Note that there can be no configuration points in $\widehat{yzx}$ or in $\widehat{zyx}$. Consider the points of $V - x$ in clockwise order, let $b$ be the point immediately after $z$, and let $a$ be the point preceding $y$. After $\mathcal{R}$ has played in the configuration $(\overset{\curvearrowright}{zy} \cup \{x\}, \emptyset, \{yz\})$, if $\mathcal{B}$ can draw either $za$ or $yb$, then $\mathcal{B}$ plays such a move, and again, we are left with a trivial subgame on the points $a, x, y, z$, or $b, x, y, z$, and one or two convex subgames (depending on the move of player $\mathcal{R}$), see Figure 4a, so $\mathcal{B}$ can force a tie on them by Lemma 2.1. Otherwise, $\mathcal{R}$ has just made a move that prevents $\mathcal{B}$ from playing $za$ and $yb$, which means that $\mathcal{R}$

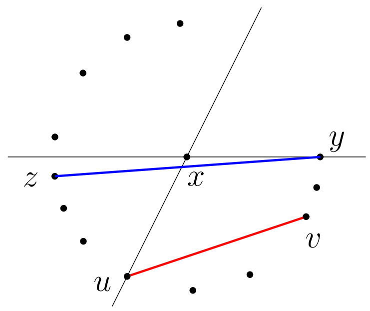


Fig. 3.

drew either $za$ or $yb$ or $xw$, for some $w \in \overset{\curvearrowright}{zy}$, with $w \neq z, y$. If $\mathcal{R}$ has drawn $za$ (or $yb$), then $\mathcal{B}$ plays $xa$ (respectively $xb$), splitting the problem into a trivial subgame on the points $a, x, y, z$ ($b, x, y, z$ respectively), and a convex subgame, see Figure 4b. Assume now that $\mathcal{R}$ has drawn $xw$ for some $w \in \overset{\curvearrowright}{zy}$, with $w \neq z, y$. Then $\mathcal{B}$ draws $zx$, forcing $\mathcal{R}$ to play $xy$, and finally $\mathcal{B}$ plays $yw$ (see Figure 4c). At this point we obtain two more convex subgames, namely $(\overset{\curvearrowright}{zw} \cup \{x\}, \{xw\}, \{zx\})$ and $(\overset{\curvearrowright}{wy}, \emptyset, \{wy\})$ (one of them may be trivial).

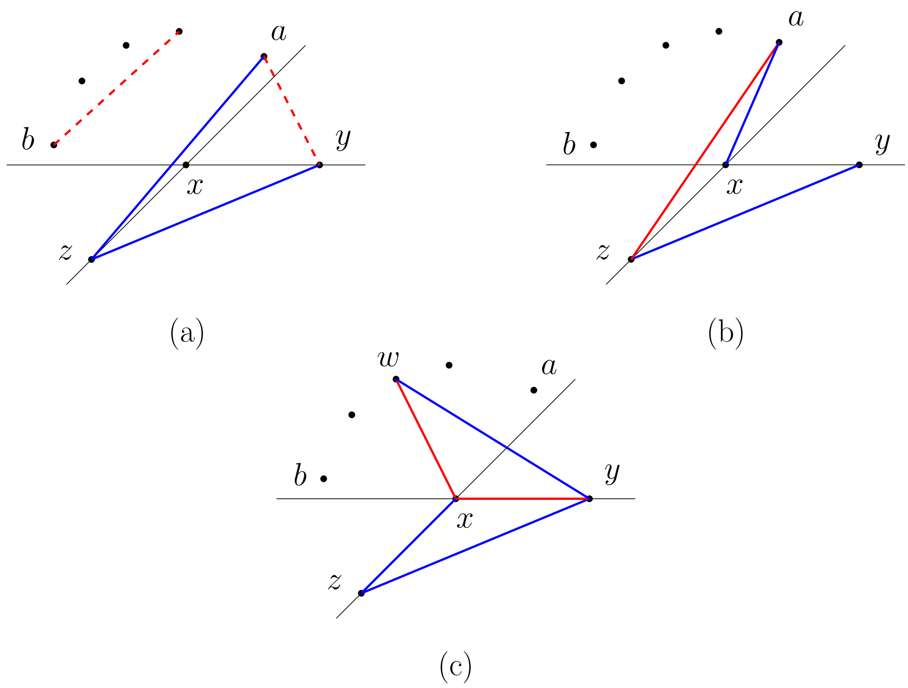


Fig. 4. (a) The dashed red edges are examples of possible moves of $\mathcal{R}$ that do not prevent $\mathcal{B}$ from playing $za$.

If, on the other hand, there exists no edge $yz$ which has $x$ on one side and $uv$ on the other side, there can be no points in $\widehat{vux}$ or $\widehat{uvx}$. Consider the points of $V - x$ in clockwise order and let $y$ be the point preceding $v$. Now $\mathcal{B}$ draws $uy$, and we are left with a trivial subgame on the points $u, v, x, y$, and two convex subgames $(\overrightarrow{\widehat{uy}}, \emptyset, \{uy\})$ and $(\overrightarrow{\widehat{vu}}, \{uv\}, \emptyset)$ (one of them may be degenerate), see Figure 5 below, resolved by Lemma 2.1.

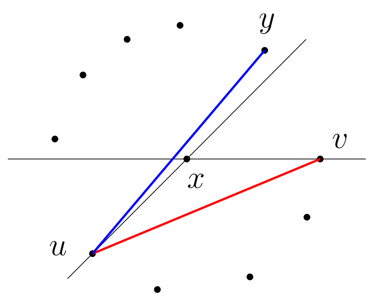


Fig. 5.

This completes the proof. □

Now we show that player $\mathcal{R}$ can force a tie in the Bichromatic Triangle Game when the points in $V$ are in convex position, and when $V$ has exactly one inner point.

**Lemma 2.5** *Suppose $\mathscr{C} = (V, E_{\mathcal{R}}, E_{\mathcal{B}})$ is a configuration of the Bichromatic Triangle Game where $V$ is a set of points in convex position and all drawn edges, i.e., edges in $E_{\mathcal{R}} \cup E_{\mathcal{B}}$, are between consecutive points in $V$. If $|E_{\mathcal{B}}| < 2$, then $\mathcal{R}$ can force a tie in $\mathscr{C}$ regardless of who is next to play.*

**Proof.** The proof is symmetric to the proof of Lemma 2.1 – this time $\mathcal{R}$ plays the role of $\mathcal{B}$, and vice versa. □

**Theorem 2.6** *Player $\mathcal{R}$ can force a tie in the Bichromatic Triangle Game when the points in $V$ are in convex position.*

**Proof.** Player $\mathcal{R}$ is the first to make a move, thus he can draw any edge between two consecutive points in $V$. Now, the statement follows immediately from Lemma 2.5. □

**Theorem 2.7** *Player $\mathcal{R}$ can force a tie in the Bichromatic Triangle Game when the set of points $V$ has exactly one inner point.*

**Proof.** First, player $\mathcal{R}$ draws any edge $yz$ with the property that there are no points in $\widehat{yzx}$ or $\widehat{zyx}$, and such that $x$ is in the interior of $\overset{\curvearrowright}{zy}$. Such an edge may be obtained as follows. Start with any edge $yz$, with $y, z \neq x$, such that $x$ is in the interior of $\overset{\curvearrowright}{zy}$. If $yz$ has the desired property, we are done. Otherwise, set $u = z$ and $v = y$ and use the same simple algorithm as in Case (ii) of the proof of Theorem 2.3 to find the desired edge $yz$.

By drawing $yz$, player $\mathcal{R}$ splits the problem into two smaller subgames, namely $(\overset{\curvearrowright}{yz}, \{yz\}, \emptyset)$ and $(\overset{\curvearrowright}{zy} \cup \{x\}, \{yz\}, \emptyset)$. Player $\mathcal{R}$ can force a tie in the first subgame by Lemma 2.5. To show that he can also force a tie in the second subgame, we proceed analogously as in the second paragraph of Case (ii) of the proof of Theorem 2.3, but this time $\mathcal{R}$ plays the role of $\mathcal{B}$, and vice versa. We also use Lemma 2.5 instead of Lemma 2.1. □

## 3 Summary and Remarks

We consider the Bichromatic Triangle and the Bichromatic Complete Triangulation Games and show that either player can force a tie in these games when the given points are in convex position or when there is exactly one inner point. Natural open questions that arise are to design polynomial algorithms to compute winning strategies or to determine which player can force a tie in these games in the general case, i.e., when there is no predetermined conditions for the given set of points, or else to show that the problem of characterizing the outcome of any of these games in the general case is NP-hard.

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