General radix, recursive Cooley-Tukey FFT

assume $n = km$

$$DFT_{km} = (\underset{\uparrow \text{radix}}{DFT_k} \otimes I_m)\, \underbrace{T_m^n}_{\text{diagonal matrix}}\, (I_k \otimes DFT_m)\, L_k^n$$

3 key structures: $I_k \otimes A_m$, $A_k \otimes I_m$, L_k^n

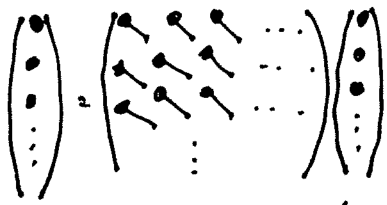
1.) $y = (I_k \otimes A_m)\, x$

$$\begin{pmatrix} | \end{pmatrix} = \begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix} \begin{pmatrix} | \end{pmatrix}$$

k A's at stride 1

for $i = 0 : k-1$
  $y[im : 1 : im+m-1]$
$= A \cdot x[$ " $]$

2.) $y = (A_k \otimes I_m)\, x$

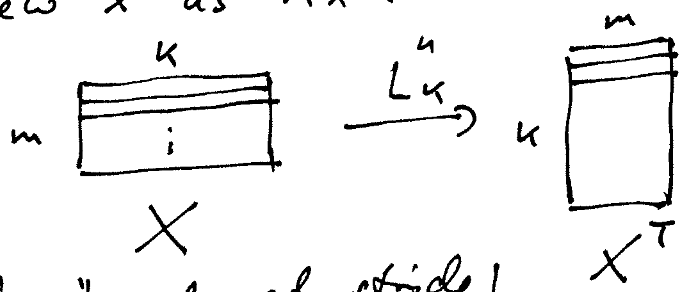


m A's at stride m

for $i = 0 : m-1$
  $y[i : m : i+(k-1)m]$
$= A \cdot x[$ " $]$

3.) $y = L_k^n x$: different ways of viewing it

a.) view x as $m \times k$ matrix:



transposition!

b.) L_k^n "reads at stride 1 and writes at stride m"

stride 1 -> m
is the same as
stride k -> 1

c.) L_k^n performs permutation $im + j \rightarrow jk + i$ $0 \le i < k$, $0 \le j < m$

FFT again:

$$DFT_{km} = \underbrace{(DFT_k \otimes I_m)}_{\text{stride } m \to m}\, T_m^n\, \underbrace{(I_k \otimes DFT_m)}_{\text{stride } 1 \to 1}\, \underbrace{L_k^n}_{\text{stride } 1 \to m}$$

stride 1 -> m
is the same as
stride k -> 1

this is the "decimation-in-time" version

Decimation in frequency: transpose!

Use: - DFT is symmetric

- $(L^n_k)^T = L^n_m$

- $(A \otimes B)^T = A^T \otimes B^T$

Gives:

$$DFT_{km} = L^n_m (I_k \otimes DFT_m) T^n_m (DFT_k \otimes I_m)$$

Cost analysis: (was in exam for radix 2), assume n=2

Measure: (complex adds, complex mults)

Cost: independent of radix $(n \log_2(n), \frac{1}{2} n \log_2(n))$

complex add = 2 real adds

" mult ≤ 4 real mults
2 real adds

$\Rightarrow$ real cost $\leq 2n\log_2(n) + 3n\log_2(n) = 5n\log_2(n)$

<u>Iterative radix-2 FFT</u>

$$DFT_{2^t} = \underbrace{R_{2^t}}_{\substack{\text{bit-reversal}\\\text{permutation}}} \prod_{i=1}^{t} \underbrace{(I_{2^{t-i}} \otimes T^{2^i}_{2^{i-1}})}_{\substack{\text{diagonal}\\\text{matrix}}} \underbrace{(I_{2^{t-i}} \otimes DFT_2 \otimes I_{2^{i-1}})}_{\substack{2^{t-1}\ DFT_2\text{'s}\\\text{at varying strides}}}$$

Most people consider this "the FFT"