ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

# Automation in Dense Linear Algebra

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# Content

Motivation

Building a new algorithm

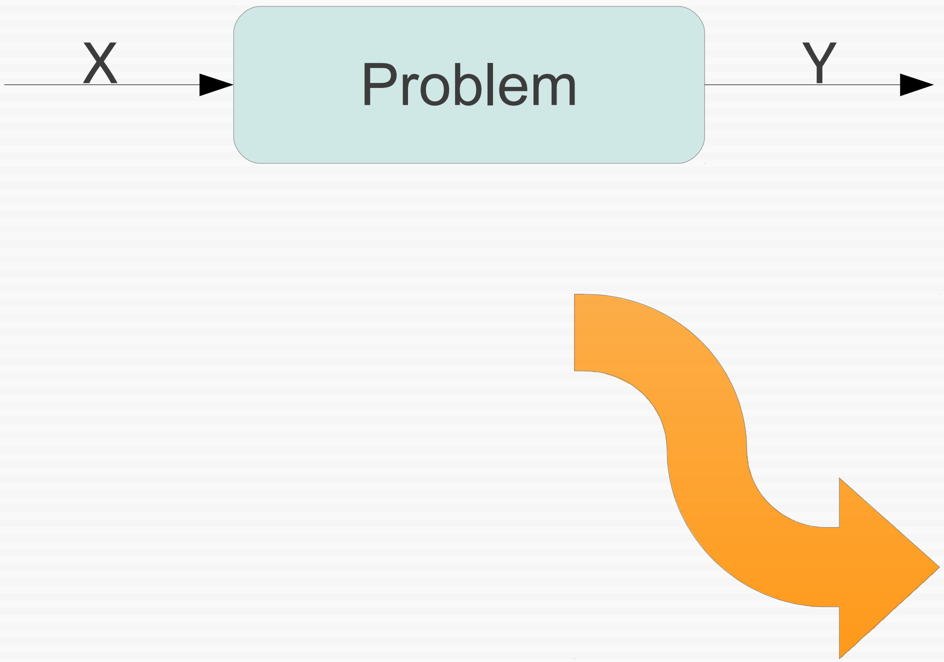
Prototype

Conclusion

# Motivation

- Best algorithm for a dense linear algebra problem?
  - LU
  - Cholesky
  - Eigenvalues
  - SVD
  - ...
- Highly used for:
  - Interpolation of functions
  - Solving systems of equations
  - Optimization

# Idea: Automatically build it!




```
function [A] = choleskyL1( A , nb )

[ ATL, ATR, ...
  ABL, ABR ] = FLA_Part_2x2( A,0,0,'FLA_TL');

%% Loop Invariant
%% ATL=choleskyL[ATL]
%% ABL'=0
%% ABL=ABL
%% ABR=ABR

while( size(ATL,1) ~= size(A,1) | size(ATL,2) ~= size(A,2) )
   b = min( nb, min( size(ABR,1), size(ABR,2) ));

   [ A00, A01, A02, ...
     A10, A11, A12, ...
     A20, A21, A22 ] = FLA_Repart_2x2_to_3x3(ATL, ATR,...
                                             ABL, ABR,...
                                             b, b, 'FLA_BR');

   %* ***************************************************** *%
   A10 = A10 . inv(A00)';
   A11 = choleskyL(A11 - A10 . A10');
   %* ***************************************************** *%
   [ ATL, ATR, ...
     ABL, ABR ] = FLA_Cont_with_3x3_to_2x2(A00, A01, A02, ...
                                           A10, A11, A12, ...
                                           A20, A21, A22, ...
                                           'FLA_TL');
end;
A = ATL;
return;
```

http://www.aices.rwth-aachen.de:8080/~pauldj/pubs/CC-2009.pdf

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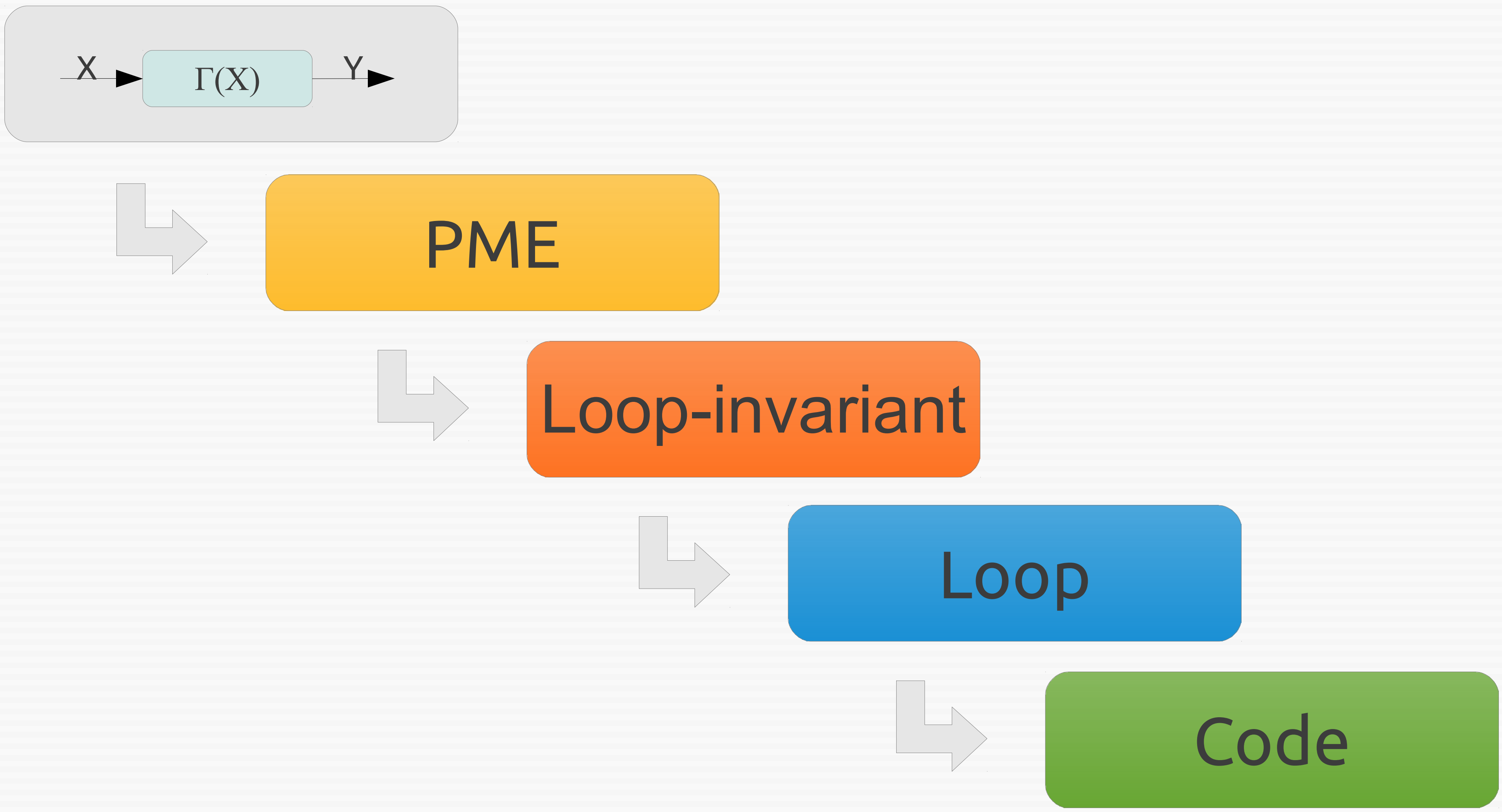
# Steps needed

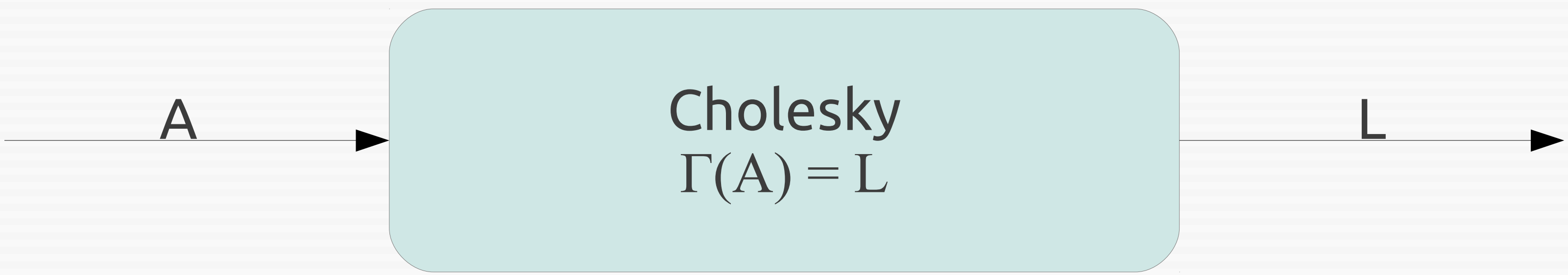


# Layout of the algorithm

```
function Γ(A, B, ...)
    // Some arrangements of the input

    while G do
        // Stepwise computation of Γ
    end while
end function
```

# Example: Cholesky factorization




- How do we compute L, so that $A = LL^T$?

# Problem to PME

$$A = L L^T$$
$$\Gamma(A) = L$$

$$\underbrace{\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)}_{A} = \underbrace{\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)}_{L} \underbrace{\left(\begin{array}{c|c} L_{TL}^T & L_{BL}^T \\ \hline 0 & L_{BR}^T \end{array}\right)}_{L^T}$$

$$= \left(\begin{array}{c|c} L_{TL}L_{TL}^T & L_{TL}L_{BL}^T \\ \hline L_{BL}L_{TL}^T & L_{BL}L_{BL}^T + L_{BR}L_{BR}^T \end{array}\right)$$

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$$A_{TL} = L_{TL}L_{TL}^T \Rightarrow L_{TL} = \Gamma(A_{TL})$$

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$$= \left(\begin{array}{c|c} L_{TL}L_{TL}^T & L_{TL}L_{BL}^T \\ \hline L_{BL}L_{TL}^T & L_{BL}L_{BL}^T + L_{BR}L_{BR}^T \end{array}\right)$$

$$A_{TL} = L_{TL}L_{TL}^T \Rightarrow L_{TL} = \Gamma(A_{TL})$$

$$A_{BL} = L_{BL}L_{TL}^T \Rightarrow L_{BL} = A_{BL}L_{TL}^{-T}$$

# Problem to PME

$A = L L^T$
$\Gamma(A) = L$

$$\underbrace{\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)}_{A} = \underbrace{\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)}_{L} \underbrace{\left(\begin{array}{c|c} L_{TL}^T & L_{BL}^T \\ \hline 0 & L_{BR}^T \end{array}\right)}_{L^T}$$

$$= \left(\begin{array}{c|c} L_{TL}L_{TL}^T & L_{TL}L_{BL}^T \\ \hline L_{BL}L_{TL}^T & L_{BL}L_{BL}^T + L_{BR}L_{BR}^T \end{array}\right)$$

$$A_{TL} = L_{TL}L_{TL}^T \Rightarrow L_{TL} = \Gamma(A_{TL})$$

$$A_{BL} = L_{BL}L_{TL}^T \Rightarrow L_{BL} = A_{BL}L_{TL}^{-T}$$

$$A_{BR} - L_{BL}L_{BL}^T = L_{BR}L_{BR}^T \Rightarrow L_{BR} = \Gamma(A_{BR} - L_{BL}L_{BL}^T)$$

# Problem to PME

$$A_{TL} = L_{TL}L_{TL}^T \Rightarrow L_{TL} = \Gamma(A_{TL})$$

$$A_{BL} = L_{BL}L_{TL}^T \Rightarrow L_{BL} = A_{BL}L_{TL}^{-T}$$

$$A_{BR} - L_{BL}L_{BL}^T = L_{BR}L_{BR}^T \Rightarrow L_{BR} = \Gamma(A_{BR} - L_{BL}L_{BL}^T)$$

$$\text{PME: } \left\{ \left( \begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array} \right) = \left( \begin{array}{c|c} \Gamma(A_{TL}) & 0 \\ \hline A_{BL}L_{TL}^{-T} & \Gamma(A_{BR} - L_{BL}L_{BL}^T) \end{array} \right) \right\}$$

# Loop-invariant

- Holds before, at the begin and after the loop

$\{P_{inv}\}$
**while** $G$ **do**
  $\{P_{inv}\}$
  $[\dots]$
  $\{P_{inv}\}$
**end while**
$\{P_{inv} \wedge \neg G\}$

$c = a;\ i = 0$
$\{c = a + i\}$
**while** $i \neq b$ **do**
  $\{c = a + i\}$
  $c = c + 1$
  $i = i + 1$
  $\{c = a + i\}$
**end while**
$\{c = a + i \wedge \neg(i \neq b)\}$
$//\ c = a + b$

# Loop-invariant

- Holds before, at the begin and after the loop

$$\{P_{inv}\}$$
**while** $G$ **do**
$$\{P_{inv}\}$$
$$[\ldots]$$
$$\{P_{inv}\}$$
**end while**
$$\{P_{inv} \wedge \neg G\}$$

$$A \overset{!}{=} LL^T$$

$$c = a;\ i = 0$$
$$\{c = a + i\}$$
**while** $i \neq b$ **do**
$$\{c = a + i\}$$
$$c = c + 1$$
$$i = i + 1$$
$$\{c = a + i\}$$
**end while**
$$\{c = a + i \wedge \neg(i \neq b)\}$$
$$//\ c = a + b$$

# Choosing a Loop-invariant

$$\text{PME: } \left\{ \left( \begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array} \right) = \left( \begin{array}{c|c} \Gamma(A_{TL}) & 0 \\ \hline A_{BL} L_{TL}^{-T} & \Gamma(A_{BR} - L_{BL} L_{BL}^T) \end{array} \right) \right\}$$

- Any subset of the PME
- Some blocks can be 0x0-matrices
- Has to respect the dependencies

| # | Loop-invariants for Cholesky Factorization |
|---|---|
| 1 | $\left( \begin{array}{c\|c} L_{TL} = \Gamma(A_{TL}) & * \\ \hline L_{BL} = 0 & L_{BR} = 0 \end{array} \right)$ |
| 2 | $\left( \begin{array}{c\|c} L_{TL} = \Gamma(A_{TL}) & * \\ \hline L_{BL} = A_{BL} L_{TL}^{-T} & L_{BR} = 0 \end{array} \right)$ |
| 3 | $\left( \begin{array}{c\|c} L_{TL} = \Gamma(A_{TL}) & * \\ \hline L_{BL} = A_{BL} L_{TL}^{-T} & L_{BR} = A_{BR} - L_{BL} L_{BL}^T \end{array} \right)$ |

| | |
|---|---|
| 4 | $\left( \begin{array}{c\|c} L_{TL} = 0 & * \\ \hline L_{BL} = A_{BL} L_{TL}^{-T} & L_{BR} = 0 \end{array} \right)$ |

http://www.aices.rwth-aachen.de:8080/~pauldj/pubs/CC-2009.pdf

# Choosing a Loop-invariant

$$\{P_{inv}\}: \left\{ \left( \begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & L_{TR} = 0 \\ \hline L_{BL} = 0 & L_{BR} = 0 \end{array} \right) \right\}$$

# Choosing a Loop-invariant

$$\{P_{inv}\}: \left\{ \left( \begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & L_{TR} = 0 \\ \hline L_{BL} = 0 & L_{BR} = 0 \end{array} \right) \right\}$$

- At the beginning:

$$\{P_{inv}\}: \left\{ \left( \begin{array}{c|c} L_{TL} = 0 \times 0 & \\ \hline & L_{BR} = L = 0 \end{array} \right) \right\}$$

# Choosing a Loop-invariant

$$\{P_{inv}\}: \left\{ \left( \begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & L_{TR} = 0 \\ \hline L_{BL} = 0 & L_{BR} = 0 \end{array} \right) \right\}$$

- At the beginning:

$$\{P_{inv}\}: \left\{ \left( \begin{array}{c|c} L_{TL} = 0 \times 0 & \\ \hline & L_{BR} = L = 0 \end{array} \right) \right\}$$

- At the end:

$$\{P_{inv}\}: \left\{ \left( \begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) \Rightarrow L = \Gamma(A) & \\ \hline & L_{BR} = 0 \times 0 \end{array} \right) \right\}$$

# Ensuring progress

$$\left(\begin{array}{c|c} A_{TL} & * \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|cc} A_{00} & * & * \\ \hline A_{10} & A_{11} & * \\ A_{20} & A_{21} & A_{22} \end{array}\right)$$

$$\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|cc} L_{00} & 0 & 0 \\ \hline L_{10} & L_{11} & 0 \\ L_{20} & L_{21} & L_{22} \end{array}\right)$$

$$\rightsquigarrow$$

$$\left(\begin{array}{cc|c} A_{00} & * & * \\ A_{10} & A_{11} & * \\ \hline A_{20} & A_{21} & A_{22} \end{array}\right) \rightarrow \left(\begin{array}{c|c} A_{TL} & * \\ \hline A_{BL} & A_{BR} \end{array}\right)$$

$$\left(\begin{array}{cc|c} L_{00} & 0 & 0 \\ L_{10} & L_{11} & 0 \\ \hline L_{20} & L_{21} & L_{22} \end{array}\right) \rightarrow \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)$$

# Construction of the loop

$\{P_{inv}\}$

**while** $\neg$ SAMESIZE$(L, L_{TL})$ **do**

$\{P_{inv}\}$ → $\left\{\left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & L_{TR} = 0 \\ \hline L_{BL} = 0 & L_{BR} = 0 \end{array}\right)\right\}$

// Repartition of A and L → $\left(\begin{array}{c|c} A_{TL} & * \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|cc} A_{00} & * & * \\ \hline A_{10} & A_{11} & * \\ A_{20} & A_{21} & A_{22} \end{array}\right)$, $\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|cc} L_{00} & 0 & 0 \\ \hline L_{10} & L_{11} & 0 \\ L_{20} & L_{21} & L_{22} \end{array}\right)$

$$\left\{\left(\begin{array}{c|cc} L_{00} = \Gamma(A_{00}) & L_{01} = 0 & L_{02} = 0 \\ \hline L_{10} = 0 & L_{11} = 0 & L_{12} = 0 \\ L_{20} = 0 & L_{21} = 0 & L_{22} = 0 \end{array}\right)\right\}$$

$L_{10} = A_{10} L_{00}^{-T}$

$L_{11} =$ CHOLESKY$(A_{11} - L_{10} L_{10}^T)$

$$\left\{\left(\begin{array}{cc|c} L_{00} = \Gamma(A_{00}) & L_{01} = 0 & L_{02} = 0 \\ L_{10} = A_{10} L_{00}^{-T} & L_{11} = \Gamma(A_{11} - L_{10} L_{10}^T) & L_{12} = 0 \\ \hline L_{20} = 0 & L_{21} = 0 & L_{22} = 0 \end{array}\right)\right\}$$

// Recombination of A and L

$\{P_{inv}\}$

**end while**

$\{P_{inv} \ \wedge \ $ SAMESIZE$(L, L_{TL})\}$

# Construction of the loop

$\{P_{inv}\}$
**while** $\neg$ SAMESIZE$(L,\, L_{TL})$ **do**
$\{P_{inv}\}$

$$\left\{\left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & L_{TR} = 0 \\ \hline L_{BL} = 0 & L_{BR} = 0 \end{array}\right)\right\}$$

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$$\left(\begin{array}{cc|c} A_{00} & * & * \\ A_{10} & A_{11} & * \\ \hline A_{20} & A_{21} & A_{22} \end{array}\right) \rightarrow \left(\begin{array}{c|c} A_{TL} & * \\ \hline A_{BL} & A_{BR} \end{array}\right)$$

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// Recombination of A and L
$\{P_{inv}\}$
**end while**
$\{P_{inv} \;\wedge\; \text{SAMESIZE}(L,\, L_{TL})\}$

PME:

$$\left\{\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) = \left(\begin{array}{c|c} \Gamma(A_{TL}) & 0 \\ \hline A_{BL} L_{TL}^{-T} & \Gamma(A_{BR} - L_{BL} L_{BL}^{T}) \end{array}\right)\right\}$$

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$\{P_{inv}\}$

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// Repartition of A and L

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// Recombination of A and L

$\{P_{inv}\}$

**end while**

$\{P_{inv} \ \wedge\ $ SAMESIZE$(L,\ L_{TL})\}$

PME:

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# Recap

1. Building the PME
2. Choosing the Loop-Invariant
3. Construction of the loop
   - Repartioning
   - Computation of one step

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1. Building the PME ✓
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# Content

# Prototype system

- Takes loop-invariant, returns loop-algorithm
- Generates worksheet, Matlab- or C-code

- More than 300 algorithms for the Level-3 BLAS library
- Found 50 algorithms for the triangular coupled Sylvester equation (3 previously known)

# Prototype system

```
function [A] = choleskyL1( A , nb )

[ ATL, ATR, ...
  ABL, ABR ] = FLA_Part_2x2( A,0,0,'FLA_TL');

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%% ABL'=0
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while( size(ATL,1) ~= size(A,1) | size(ATL,2) ~= size(A,2) )
    b = min( nb, min( size(ABR,1), size(ABR,2) ));
    [ A00, A01, A02, ...
      A10, A11, A12, ...
      A20, A21, A22 ] = FLA_Repart_2x2_to_3x3(ATL, ATR,...
                                              ABL, ABR,...
                                              b, b, 'FLA_BR');

    %* ******************************************************** *%
    A10 = A10 . inv(A00)';
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    [ ATL, ATR, ...
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                                            A10, A11, A12, ...
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end;
return ATL;
```

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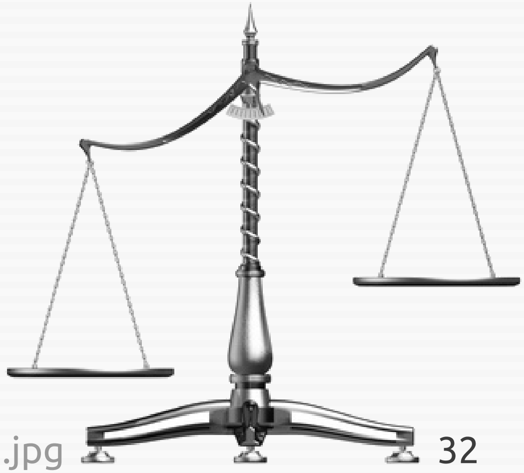
# Conclusion

+ Problem description ($A = LL^T$) sufficient for automatic algorithm generation
+ Possible to generate proof of correctness side by side with generation of algorithm
+ Performance: Familiy of algorithms => autotune these
- Numerical stability is not ensured. Proof for every algorithm needed.



http://kaichido.com/wp-content/uploads/2010/10/Fotolia_733842_XS-Balance-scale1.jpg

# Thank you!

Are there any questions?