18-799F Algebraic Signal Processing Theory
Spring 2007
Assignment 3
Due Date: Feb. 14th 2:30pm (at the beginning of class)

1. *(26 pts)* Which of the following are vector spaces? Show or disprove. For each vector space specify its basis and dimension, if you can.
   (a) $\mathbb{F}_n[x] = \{\sum_{i=0}^{n} a_i x^i \mid a_i \in \mathbb{F}\}$ (polynomials of degree at most n) as $\mathbb{F}$-vector space.
   (b) $\mathrm{GL}_n(\mathbb{R})$ as $\mathbb{R}$-vector space.
   (c) $\mathbb{F}(x)$ as $\mathbb{F}$-vector space.
   (d) $\mathbb{C}$ as $\mathbb{R}$-vector space.
   (e) $\mathbb{R}$ as $\mathbb{C}$-vector space.
   (f) $\mathbb{Q} + \mathbb{Q}\sqrt{2}$ as $\mathbb{Q}$-vector space. Is it also a ring? A field?
   (g) Any ideal in $\mathbb{R}[x]$ as $\mathbb{R}$-vector space.
   (h) $\mathbb{R}[x]/(p(x)\mathbb{R}[x])$ as $\mathbb{R}$-vector space.

2. *(56 pts)* Prove which of the following are linear mappings. For each linear mapping do the following
   - specify the kernel, its dimension, and a basis
   - specify the image and its dimension
   - state whether it is injective, surjective, bijective
   - apply the homomorphism theorem

   (a)
$$\begin{aligned} \phi: \quad \mathbb{R}^3 &\to \mathbb{R} \\ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} &\mapsto \alpha_1 + \alpha_2 + \alpha_3 \end{aligned}$$

   (b)
$$\begin{aligned} \phi: \quad \mathbb{R}^3 &\to \mathbb{R} \\ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} &\mapsto \alpha_1 \end{aligned}$$

   (c)
$$\begin{aligned} \phi: \quad \mathbb{C} &\to \mathbb{R} \\ z &\mapsto |z| \end{aligned}$$

   Reminder: for any $z \in \mathbb{C}: \ z = |z| e^{\sqrt{-1} \arg z}$.

   (d)
$$\begin{aligned} \phi: \quad \mathbb{C} &\to \mathbb{R} \\ z &\mapsto \arg z \end{aligned}$$

   (e)
$$\begin{aligned} \phi: \quad \mathbb{F}[x] &\to \mathbb{F}[x] \\ q(x) &\mapsto x q(x) \end{aligned}$$

   (f)
$$\begin{aligned} \phi: \quad \mathbb{Z} &\to \mathbb{Z} \\ x &\mapsto 2x \end{aligned}$$

(g)

$$\begin{array}{rrcl} \phi: & \mathbb{F}[x] & \rightarrow & \mathbb{F}[x] \\ & q(x) & \mapsto & q'(x) \text{ (the derivative)} \end{array}$$

(h) (Note: $\mathbb{F}_n[x]$ was defined in problem 1(a))

$$\begin{array}{rrcl} \phi: & \mathbb{F}_n[x] & \rightarrow & \mathbb{F}_n[x] \\ & q(x) & \mapsto & q'(x) \text{ (the derivative)} \end{array}$$

3. *(18 pts)* (Note: $\mathbb{F}_n[x]$ was defined in problem 1(a))

   (a) Find a vector space U, such that $\mathbb{F}[x] = \mathbb{F}_n[x] \oplus U$ and show this holds.

   (b) Give a basis and the dimension of U.

   (c) Give a basis and the dimension of $\mathbb{F}[x]/\mathbb{F}_n[x]$.

4. **Extra credit problem** *(20 pts)* Let $\mathbb{F}$ be a field. Then $\mathbb{F}[x]$ is a Euclidean ring with the usual division with rest. A polynomial $p(x) \in \mathbb{F}[x]$ is called irreducible over $\mathbb{F}$ is it does not have a nontrivial factorization $p(x) = q(x)r(x)$, $\deg(q), \deg(r) > 0$. Assume $p(x) \in \mathbb{F}[x]$ is irreducible.

   (a) Show that $\mathbb{F}' = \mathbb{F}[x]/p(x)$ is a field. (Note: it is called an extension field of $\mathbb{F}$).

   (b) Show that $\mathbb{F}'$ is an $\mathbb{F}$ vector space of dimension $\deg(p)$. We write $[\mathbb{F}' : \mathbb{F}] = \deg(p)$; it is the degree of the field extension.

   (c) Give an example of a field extension $\mathbb{F}'$ of degree 2 for the fields $\mathbb{F} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$.