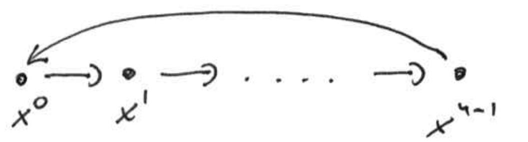
Recap: Derivation infinite and finite time models



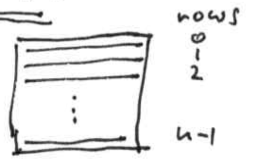
b.c. $x^n = 1 \Rightarrow \mathcal{A} = \mathcal{M} = \mathbb{C}[x]/x^n-1$

$\Phi: \hat{s} \to \sum s_k x^k$

- $\varphi(x)$ = shift matrix = cyclic shift
- $\varphi(h)$ = circulant matrix
- $h(x)s(x) \bmod (x^n - 1)$ circular convolution
- b.c. defines signal extension
- $\mathcal{F} = DFT_n = [\omega_n^{kl}]_{0 \le k,l < n}$
- other b.c.'s = other finite time models are possible
  example: $x^n + 1 \to \tilde{\mathcal{F}} = DFT\text{-}3_n = [$

Real DFT:

DFT_n:



observation: row $0, \frac{n}{2}$ are real
rows $i, n-i$ are complex conjugates

$\Rightarrow$ if $\hat{s}$ is real, then in $\hat{y} = DFT_n \hat{s}$

$y_0, y_{n/2}$ are real

symmetry: y_k, y_{n-k} are conjugates, $1 \le k < \frac{n}{2}$

$RDFT_n$:

row_0
$Re(row_1)$
$Im(row_1)$
$Re(row_2)$
$Im(row_2)$
$\vdots$
$row_{n/2}$

$$\Rightarrow \quad RDFT_n \cdot \hat{s} = \begin{pmatrix} y_0 \\ Re(y_1) \\ Im(y_1) \\ \vdots \\ y_{n/2} \end{pmatrix}$$

$$row_k = (\omega_n^{kl})^T_{0 \le l < n}$$

$$Re(row_k) = \left(\cos \frac{2\pi kl}{n}\right)^T$$

$$Im(row_k) = \left(\sin \frac{2\pi kl}{n}\right)^T$$

Obvious:

$$DFT_n = \begin{pmatrix} 1 & & & & & & & \\ & 1 & i & & & & & \\ & & & 1 & i & & & \\ & & & & \ddots & 1 & i & 1 \\ & & & & & & 1 & -i \\ & & & & & ⋰ & & \\ 0 & 1 & -i & & & & & \end{pmatrix} \cdot RDFT_n \qquad (n \text{ even shown})$$

Summary: if signal $\hat{s}$ is real, use the real transform RDFT (cheaper)

Algebraic view of real DFT

finite <u>complex</u> time model:

$$\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/x^n-1, \quad \Phi: \hat{s} \mapsto \sum s_k x^k,$$

$$\mathcal{F}: \mathbb{C}[x]/x^n-1 \longrightarrow \bigoplus_k \mathbb{C}[x]/x-\omega_n^k \qquad \text{bases: } (1) \text{ in spectral comp.}$$

finite <u>real</u> time model:

$$\mathcal{A} = \mathcal{M} = \mathbb{R}[x]/x^n-1, \quad \Phi: \hat{s} \mapsto \sum s_k x^k$$

$$\mathcal{F}: \mathbb{R}[x]/x^n-1 \longrightarrow \mathbb{R}[x]/x-1 \oplus \bigoplus_{1\le k<\frac{n}{2}} \mathbb{R}[x]/x^2-2c_kx+1 \oplus \underbrace{\mathbb{R}[x]/x+1}_{\text{if } n \text{ even}}$$

$\ulcorner$ $c_k = \cos\frac{2\pi k}{n}$, $x^2-2c_kx+1 = (x-\omega_n^k)(x-\omega_n^{n-k})$ $\lrcorner$

- so we have <u>two-dim.</u> spectral components !
  (since $\mathbb{R}$ is not a "splitting field" for x^n-1)
- filtering is still circular convolution, but of <u>real</u> polynomials

<u>Claim:</u> $b_k = (1, \frac{1}{s_k}x - \frac{c_k}{s_k})$ is a "natural" choice of basis in the spectral components and yields $\mathcal{F} = RDFT_n$

<u>Frequ. response:</u> $\varphi_k(h(x))$ at frequ. k is a 2×2 matrix, $1\le k<\frac{n}{2}$

$$x \cdot 1 = c_k \cdot 1 + s_k\left(\frac{1}{s_k}x - \frac{c_k}{s_k}\right)$$

$$x \cdot \left(\frac{1}{s_k}x - \frac{c_k}{s_k}\right) = \dots = -s_k\cdot 1 + c_k\left(\frac{1}{s_k}x - \frac{c_k}{s_k}\right)$$

$$\Rightarrow \varphi_k(x) = \begin{pmatrix} c_k & -s_k \\ s_k & c_k \end{pmatrix} = R_{\frac{2\pi k}{n}} \quad \text{(rotation)}$$

$$\Rightarrow \varphi_k(h(x)) = h\left(R_{\frac{2\pi k}{n}}\right)$$

$$\varphi_0(h(x)) = h(1)$$

$$\varphi_{n/2}(h(x)) = h(-1)$$

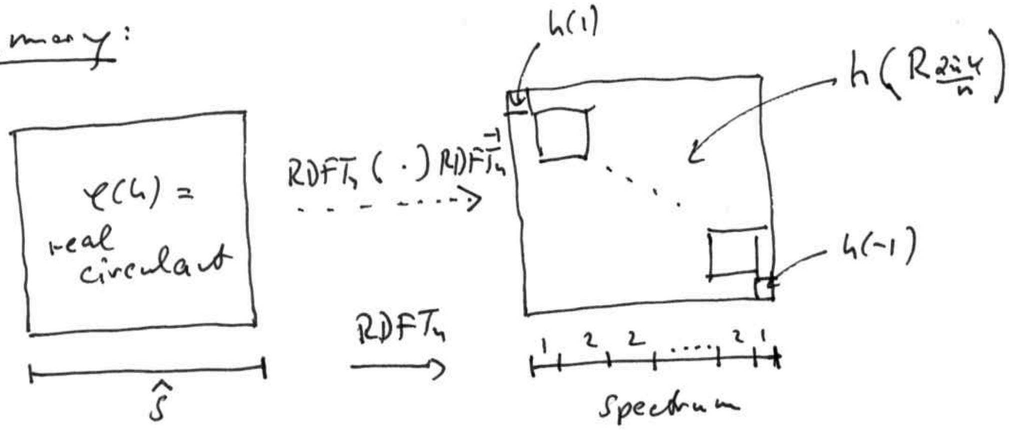
$\mathcal{F}$: remember: $\underbrace{h(x)}_{\text{filt}}\,\underbrace{s(x)}_{\text{sig}} \bmod p(x) \iff \varphi(h)\cdot\hat{s}$

so: $$x^\ell = x^\ell \cdot 1 \bmod (x^2 - c_k x + 1)$$

$$\iff \varphi_k(x^\ell)\cdot\begin{pmatrix}1\\0\end{pmatrix} = \varphi_k(x)^\ell\cdot\begin{pmatrix}1\\0\end{pmatrix} = R^\ell_{\frac{2\pi k}{n}}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\frac{2\pi k\ell}{n}\\ \sin\frac{2\pi k\ell}{n}\end{pmatrix}$$

$$\Rightarrow \mathcal{F} = RDFT_n$$

Summary:

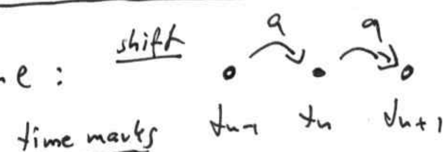


RDFT "block-diagonalizes" real circulant matrices.

- The algebraic theory enables signal models over non-splitting field.
- Other choices of bases in the ~~two~~ two-dim. spectral components are possible. For example, requiring $\mathcal{F}^{-1} = \mathcal{F}$ yields $\mathcal{F} = DHT_n$ (discrete Hartley transform)

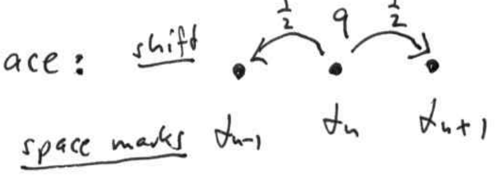
Discrete 1-D Space Models

time:



$q \diamond t_n = t_{n+1}$ + $t_0 = 1$ $\Rightarrow \begin{cases} q = x \\ t_n = x^n \end{cases}$

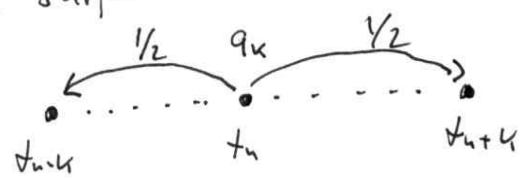
space:



$$q \diamond t_n = \frac{1}{2}\left(t_{n-1} + t_{n+1}\right)$$

- simplest symmetric shift
- $\frac{1}{2}$ for convenience
- yields visualization:

k-fold shift:



- $q_k \neq q^k$!
- $q_k = T_k(q)$

(Chebyshev polynomials of the first kind)

linear extension:

$$\mathcal{M} = \left\{ \sum s_n t_n \right\}$$

$$\mathcal{A} = \left\{ \sum_{k \geq 0} h_k T_k(q) \right\}$$

realization: $q = x$, $\diamond = \cdot$, $t_0 = 1$, $t_n = C_n(x)$

$$x \cdot C_n = \frac{1}{2}\left(C_{n-1} + C_{n+1}\right) \iff C_{n+1} = 2x\, C_n - C_{n-1}$$

$C_0 = 1$

$C_1 = ax + b$ degree of freedom

<u>Definition:</u> Any sequence of polynomials $(C_n)_{n \in \mathbb{Z}}$ that satisfies $C_{n+1} = 2x\, C_n - C_{n-1}$ and $C_0 = 1$ and $\deg(C_1) = 1$ is called a sequence of "Chebyshev polynomials."

Notes:

- $\deg(C_n) = n$, $n \geq 0$
- the C_{-n}, $n \geq 0$, are again polynomials (since the recurrence can be run in the negative direction via
$$C_{n-1} = 2x\, C_n - C_{n+1} \text{)}$$
of $\deg(C_{-n}) \leq n$.
$$\Rightarrow C_{-n} = \sum_{k \geq 0}^{n} \beta_k C_k$$
yields a left signal extension

So, the infinite space models will be for right-sided signals only:

$$\mathcal{M} = \{ \sum_{n \geq 0} s_n C_n \}, \quad \mathcal{A} = \{ \sum_{k \geq 0} h_k T_k \}$$

Many choices for C: choose the ones that have a monomial signal extension.

Lemma: The (left) signal extension associated with C is monomial precisely in the following 4 cases

$$\underbrace{\begin{array}{l} C_0 = 1, \ C_1 = x \\ C_0 = 1, \ C_1 = 2x \\ C_0 = 1, \ C_1 = 2x-1 \\ C_0 = 1, \ C_1 = 2x+1 \end{array}}_{\text{initial conditions}} \quad \text{or} \quad \underbrace{\begin{array}{l} C = T \\ C = U \\ C = V \\ C = W \end{array}}_{\substack{\text{name:} \\ \text{Cheb. poly. of the} \\ \text{1st/2nd/3rd/4th kind}}} \quad \text{or} \quad \underbrace{\begin{array}{l} C_{-1} = C_1 \\ C_{-1} = 0 \\ C_{-1} = C_0 \\ C_{-1} = -C_0 \end{array}}_{\text{left b.c.}}$$

$$\text{or} \quad \underbrace{\begin{array}{l} C_{-n} = C_n \\ C_{-n} = -C_{n-2} \\ C_{-n} = C_{n-1} \\ C_{-n} = -C_{n-1} \end{array}}_{\text{signal extension}}$$

$$\boxed{\text{left b.c.} \Rightarrow \text{left signal extension}}$$

Convergence: $\hat{s} \in \ell_2(\mathbb{N})$, $\hat{h} \in \ell_1(\mathbb{N})$ works

Summary: We define 4 infinite 1-D space models:

$$\mathcal{A} = \{ \sum_{k \geq 0} h_k T_k(x) \mid \hat{h} \in \ell^1(\mathbb{N}) \}, \quad \mathcal{M} = \{ \sum_{n \geq 0} s_n C_n \mid \hat{s} \in \ell_2(\mathbb{N}) \}, \quad C \in \{T, U, V, W\}$$

$$\Phi: \ell_2(\mathbb{N}) \to \mathcal{M}$$

$$\hat{s} \mapsto \sum_{n \geq 0} s_n C_n, \quad C \in \{T, U, V, W\}$$

is called the T-/U-/V-/W-transform.