18-799 Algorithms and Computation in Signal Processing
Spring 2005
Assignment 1
Due Date: Jan. 25th 1:30pm (before or in class)

1. *(9 pts)* Show that the following identities hold by determining the explicit constants c and n_0 that are a part of the definition of O.
   (a) $n + 1 = O(n)$
   (b) $n^3 + an^2 + bn + c = O(n^3)$
   (c) $n^5 = O(n^{\log_2 n})$
2. *(16 pts)*
   (i) In the first class, you learned that $\Theta(log_a n) = \Theta(log_b n)$ for $a, b > 1$. Does $\Theta(a^n) = \Theta(b^n)$ hold? Justify your answer.
   (ii) Prove or disprove: $2^{2n} = O(2^n)$.
   (iii) Show that for $k > 0,\ \alpha > 1$: $n^k = O(\alpha^n)$ (i.e., polynomial functions grow slower than exponential functions).
   (iv) Find a function $f(n)$ such that $f(n) = O(1)$, $f(n) > 0$ for all n, and $f(n) \neq \Theta(1)$. Justify the answer.
3. *(21 pts)* Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \leq 2$. Make your bounds as tight as possible. Justify your answers.
   (a) $T(n) = 2T(n/2) + n^3$.
   (b) $T(n) = T(9n/10) + n$.
   (c) $T(n) = 16T(n/4) + n^2$.
   (d) $T(n) = 7T(n/3) + n^2$.
   (e) $T(n) = 7T(n/2) + n^2$.
   (f) $T(n) = 2T(n/4) + \sqrt{n}$.
   (g) $T(n) = 4T(n/2) + n^2 \log n$.
4. *(24 pts)* Compute the exact (arithmetic) cost

   $$C(n) = (\text{number of adds}, \text{number of mults})$$

   of the Karatsuba algorithm, recursively applied, for the multiplication of the polynomials:

   $$h(x) = h_{n-1}x^{n-1} + \ldots + h_0, \quad p(x) = p_{n-1}x^{n-1} + \ldots + p_0,$$

   assuming $n = 2^k$.
   Extension to 4 (Extra Credit Problem, *20 pts*): Now compute the exact (arithmetic) cost

   $$C(m, n) = (\text{number of adds}, \text{number of mults})$$

   in the more general case

   $$h(x) = h_{m-1}x^{m-1} + \ldots + h_0, \quad p(x) = p_{n-1}x^{n-1} + \ldots + p_0,$$

   assuming $n = 2^k,\ m = 2^\ell,\ m \leq n$.
5. *(30 pts)* Solve the recurrence $f_0 = 1,\ f_1 = 1, \quad f_n = f_{n-1} + 2f_{n-2}$, using the method of generating functions.