Bimatrix Games

The bimatrix game here is a non-cooperative two person game. Denote by M and N the finite sets of strategies of Player I and Player II ($M \cap N = \emptyset$).

The game is given by two matrices A and B in $\mathbb{R}^{M \times N}$ where a_{ij} (b_{ij}, respectively) represents the payoff to player I (II) when player I takes the strategy $i \in M$ and player the strategy $j \in N$. The payoffs can be considered as profit or loss for the player. We consider it to be profit in this note. The game is called zerosum if $A + B = \mathbf{0}$.

The famous prisoner's dilemma is a bimatrix game with payoff matrices

$$A^1 = \begin{bmatrix} -6 & -1 \\ -10 & -2 \end{bmatrix}, \qquad B^1 = \begin{bmatrix} -6 & -10 \\ -1 & -2 \end{bmatrix},$$

where both players have two strategies, $M = N = \{\text{confess}, \text{silent}\}$. The payoff matrix represents the negative of the number of years in prison (the larger, the better). An American law gives a favorable pardon to the confessor if the other did not confess. This makes the prisoner's decision complex.

Another classical example of bimatrix game is the battle of sexes, where a couple of a man and a woman have to decide where to go in the evening. The choices are either a soccer or a ballet, i.e. $M = N = \{\text{soccer}, \text{ballet}\}$. Even though the man strongly prefers to see a soccer, he still prefers to go to see a ballet if he has to go alone to see a soccer game. Symmetrically for the woman. Thus the payoff matrices are something like

$$A^2 = \begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}, \qquad B^2 = \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix}.$$

What would be a rational decision for each of them, if they have to decide independently?

Both of the games above admit a strategy pair that is stable in the following ("Nash") sense. For example, for the prisoner's dilemma, if both choose to confess, then there is no incentive for anyone to change his mind. As long as the other one does not change his mind, it is optimal for a player to maintain his decision. In the battle of sexes, if both decide to go to the soccer game (or both decide to go to the ballet), then nobody will be better off by changing his/her decision as long as the opponent does not change her/his decision.

However, there are games that admit no "pure" strategy pair that is stable. The following game is such an example.

$$A^3 = \begin{bmatrix} 1 & -3 \\ -1 & -2 \end{bmatrix}, \qquad B^3 = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix}.$$

One can interpret the game above by setting the first player to be a software vendor and the second player to be a user. The user can either comply with the software license or cheat, i.e. $N = \{\text{comply}, \text{cheat}\}$. The vendor can either inspect the usage of the software with high cost or not, $M = \{\text{not-inspect}, \text{inspect}\}$.

A mixed strategy for player I is a probabilistic decision vector $x \in \mathbb{R}^M$ such that $x \geq \mathbf{0}$ and $\mathbf{1}^T x = 1$. We define the same for player II. Denote by X and Y the set of mixed strategies for player I and II. A pure strategy is a mixed strategy that has only one nonzero component (i.e. 0/1).

Nash introduced the notion of equilibria for the N-person nonzero sum games and showed the existence of an equilibrium by using Brouwer's fixed point theorem.

A pair of mixed strategies $(\overline{x}, \overline{y}) \in X \times Y$ is said to be a Nash equilibrium if

$$\overline{x}^T A\overline{y} \geq x^T A\overline{y} \text{ for all } x \in X, \tag{1}$$

$$\overline{x}^T B\overline{y} \geq \overline{x}^T By \text{ for all } y \in Y. \tag{2}$$

In other words, a Nash equilibrium is a pair of strategies that does not motivate any one of the players to change his/her strategy as long as the other stay with his/her strategy. One can easily verify the following simple characterization of Nash equilibria:

Proposition 1 *A pair of mixed strategies $(\overline{x}, \overline{y})$ is a Nash equilibrium if and only if for any $r \in M$ and $s \in N$,*

$$\overline{x}_r > 0 \textit{ implies } (A\overline{y})_r = \max_{i \in M}(A\overline{y})_i, \textit{ and}$$
$$\overline{y}_s > 0 \textit{ implies } (\overline{x}^T B)_s = \max_{j \in N}(\overline{x}^T B)_j.$$

The well-known theorem of Nash [2] on the n-person games specialized to the two-person case says:

Theorem 2 *Every bimatrix game admits a Nash equilibrium.*

One can verify that for the third game, $(\overline{x}, \overline{y}) = (((3/4, 1/4)^T, (1/3, 2/3)^T)$ is Nash equilibrium. To see this, we evaluate $A^3\overline{y}$ and $(\overline{x}^T B^3)$: $A^3\overline{y} = (-5/3, -5/3)^T$ and $(\overline{x}^T B^3) = (5/4, 5/4)$. By Proposition 1, this pair is a Nash equilibrium.

Lemke-Howson [1] gave an elegant pivot algorithm to compute a Nash equilibrium. The polynomial computability of a Nash equilibrium is still open, see the article [5] by von Stengel. The special case of zero-sum game can be solved in polynomial time, since it can be reduced to linear programming. Note that Savani and von Stengel [3] have recently shown that the Lemke-Howson algorithm is not a polynomial-time algorithm.

How many Nash equilibria can a "nondegenerate" bimatrix have? (Of course, it can have infinitely many e.g. when all entries of A and B are equal. Nondegenerate means that the matrices are sufficiently generic.) Can one generate all efficiently? The first question was studied in [4] and a construction using dual cyclic polytopes showed that there are games with asymptotically more than $2.414^n/\sqrt{n}$ equilibria.

Here again, the polyhedral computation can help in generating all Nash equilibria via vertex enumeration algorithms. In particular, there are two polytopes associated with a bimatrix game. To make this work, we must assume that both A and B are (strictly) positive matrices. It is easy to transform the game (by adding large positive constants to the payoffs) to satisfy this without changing the Nash equilibria. Let

$$P_1 = \{x \in \mathbb{R}^M \mid x \geq \mathbf{0}, B^T x \leq \mathbf{1}\},$$
$$P_2 = \{y \in \mathbb{R}^N \mid Ay \leq \mathbf{1}, y \geq \mathbf{0}\}.$$

It can be shown by Proposition 1 that the equilibria are in one-to-one correspondance with the certain pairs of (normalized) nonzero extreme points of P_1 and P_2, see [5]). if the both inequality systems are nondegenerate.

There are many interesting open questions to investigate, such as the efficient generation of Nash equilibria, in Bimatrix Game Theory.

Exercise.

Compute all Nash equilibrium pairs of extreme points for the game:

$$A = \begin{bmatrix} 6 & 3 \\ 1 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & 3 \\ 1 & 6 \end{bmatrix}.$$

What about for the game?

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 2 \\ 3 & 3 & 2 \\ 3 & 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}.$$

Is there any equilibrium better than the others? Is it pure?
(Here is a web site that helps you verify your answer: http://banach.lse.ac.uk/form.html.)

References

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[2] J.F. Nash. Non-cooperative games. Annals of Mathematics, 54:286–295, 1951.

[3] R. Savani and B. von Stengel. Exponentially many steps for finding a Nash equilibrium in a bimatrix game. In Proc. 45th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2004), pages 258–267, 2004.

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[5] B. von Stengel. Computing equilibria for two-person games. In R. J. Aumann and S. Hart, editors, Handbook of game theory, volume 3, pages 1947–1987. North-Holland, Amsterdam, 2002. available from http://www.maths.lse.ac.uk/Personal/stengel/bvs-publ.html.