Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Polyhedral Computation, Spring 2014
Assignment 3*

March 25 , 2014

Problem 1 (Lower bound on the Size of a Hilbert Basis): In the proof of Theorem 4.1, assume that $t = n$ and the rational cone is generated by n linearly independent vectors $C = \text{cone}(\{a_1, a_2, \ldots, a_n\})$. Derive a tight lower bound of k in terms of n and the absolute value of the determinant $\det([a_1, a_2, \ldots, a_n])$. Note that k is the number of lattice points in the zonotope Z and $k \geq 2^n$, because the zonotope Z is combinatorially a cube.

Problem 2 (Dual of a Zonotope): Suppose we are given some vectors $v_1, v_2, \ldots, v_k$ in $\mathbb{R}^n$. Let I_i denote the line segment $[\mathbf{0}, v_i]$, that is, $I_i := \{x : x = \lambda_i v_i \text{ and } 0 \leq \lambda_i \leq 1\}$. The Minkowski sum $Z := I_1 + I_2 + \cdots + I_k$ over all line segments is called a zonotope.

1. Prove that every extreme point of Z has the form $\sum_{i=1}^{k} \lambda_i v_i$ where $\lambda_i \in \{0, 1\}$ for all $i = 1, \ldots, k$.

2. Give an example of a zonotope for which not every point of the form $\sum_{i=1}^{k} \lambda_i v_i$ where $\lambda_i \in \{0, 1\}$ for all $i = 1, \ldots, k$ is an extreme point.

3. Find an H-representation of a dual of the zonotope Z.

Problem 3 (Euler's Relation): Let P be any 3-polytope. The vector $f(P) = (f_{-1}, f_0, f_1, f_2, f_3)$ where f_i denotes the number of k-dimensional faces is the f-vector of P. Euler's relation says that for every 3-polytope, we have $f_0 - f_1 + f_2 = 2$.

1. Draw four different 3-polytopes with at most nine edges.

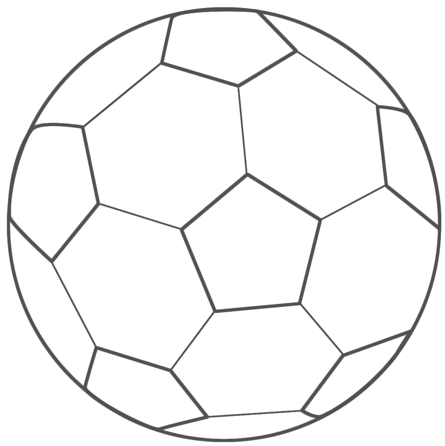
2. Prove that those are the only ones by using Euler's relation.

---

*Please hand in your solution to May Szedlák CAB G19.2 (may.szedlak@inf.ethz.ch) no later than **Tuesday, April 08, 2014**.

Problem 4 (Soccer Ball): Answer the following questions on the graphs of 3-polytopes.

1. Let any simple 3-polytope with n vertices be given. Determine the number of edges and the number of facets as function of n by using Euler's relation.
2. The soccer ball, depicted below, consists of pentagonal and hexagonal facets. Derive their numbers.



3. Determine the maximum number of edges and the maximum number of facets a 3-polytope with n vertices can have as a function of n. Does the soccer ball, or its dual, attain the maximum?