ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Polyhedral Computation, Spring 2014
Assignment 1*

February 25, 2014

Problem 1 (Rational Numbers): For any two rational numbers r and s, show that

- $\text{size}(r \times s) \leq \text{size}(r) + \text{size}(s)$
- $\text{size}(r + s) \leq 2\ (\text{size}(r) + \text{size}(s))$

Problem 2 (Euclidean Algorithm):

(a) Explain why the Euclidean algorithm is correct.

(b) Analyze the time complexity of the algorithm.

Problem 3 (Hermite Normal Form): Consider the integral matrix:

$$A = \begin{pmatrix} -4 & 6 & -6 & -6 \\ 6 & -3 & -9 & -3 \\ 4 & -3 & 9 & -3 \end{pmatrix}.$$

(a) Compute the Hermite normal form $[B, 0]$ of the matrix A by hand.

(b) Solve the diophantine equation systems $Ax = b$ and $Ax = b'$ with $x \in \mathbb{Z}^4$ where

$$b = \begin{pmatrix} 0 \\ 12 \\ 18 \end{pmatrix}, \qquad b' = \begin{pmatrix} 4 \\ 6 \\ 3 \end{pmatrix}.$$

If a system has no solution, find a certificate of infeasibility.

*Please hand in your solution to May Szedlák CAB G19.2 (may.szedlak@inf.ethz.ch) no later than **Tuesday, March 11, 2014**.

(c) Compute the transformation matrix T such that $A\,T = [B, 0]$ either by hand or any symbolic mathematics system (such as maple or mathematica).

(d) Find the general solution to any of the feasible systems in (b).

Problem 4 (Lattice Basis): Let $A \in \mathbb{Q}^{m \times n}$ be a rational matrix of full row rank, let B be a basis of the lattice $L(A)$ and let B' be a nonsingular $m \times m$ matrix whose column vectors are points in $L(A)$. Show that

- $|\det(B)| \leq |\det(B')|$.
- B' is a basis of $L(A)$ if and only if $|\det(B)| = |\det(B')|$.

Problem 5 (Fourier-Motzkin Elimination): Consider a system of linear inequalities $Ax \leq b$ where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Apply Fourier-Motzkin Elimination in the i'th variable. Denote the resulting matrix with A', the right-hand side with b'.

1. Prove that $Ax \leq b \Leftrightarrow A'x' \leq b'$ where $x' = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)^T$.
2. Determine whether the following two systems of inequalities admit a feasible solution.

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & -1 \\ -5 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 4 \\ 5 \\ 6 \\ 0 \\ -8 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 4 & 2 \\ -3 & -1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 5 \\ -1 \\ 17 \\ -11 \\ 8 \end{pmatrix}$$