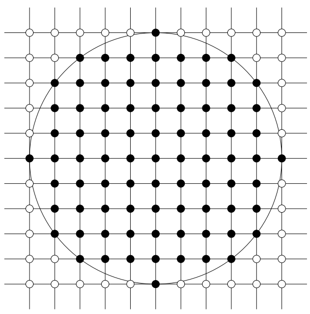
Geometric Selection Problems and Hypergraphs



PhD thesis

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Contents

Introduction

Many extremal problems in areas like combinatorics, graph theory, geometry and number theory can be formulated roughly in the following way: Given a structure and a collection of forbidden configurations. How many elements of the structure can be chosen, such that none of the forbidden configurations occur? Problems of this type will be called *selection problems*. Here are five examples:

1. Given a fixed graph G. How many edges can a graph on n vertices have without containing an isomorphic copy of G?

2. How many points can be chosen from the $[n]^2$-grid, such that there are no 4 points on a common circle and no 3 points on a common line?

3. How many elements can be chosen from the set of squares $\{1^2, 2^2, \dots, n^2\}$, such that all pairwise sums are distinct?

4. Given a set P of n points in $\mathbb{R}^d$. What is the maximum size of a subset $X \subseteq P$ with the property that all mutual distances determined by X are distinct?

5. Given a hypergraph H. Determine the independence number $\alpha(H)$.

Many other selection problems will be considered throughout this thesis. In some cases, selection problems can be solved by a construction giving a solution (a subset that avoids all bad configurations) of maximum size or of size within a constant factor of the maximum. Improving earlier bounds of Erdős and Purdy, we give such a construction for example (2) in Chapter 1.

It is quite typical that selection problems – although easily stated and easy to understand – have shown to be hard in the sense that even the asymptotics of the maximum solutions are unknown and the gaps between lower and upper bounds are considerable. Moreover, the best known lower bounds are based on non-constructive proofs.

It turns out that example (5), the determination of the independence number of a hypergraph, plays a central role. Many selection problems can be formulated in an equivalent way as an independent set problem in a certain hypergraph. A solution for the selection problem is in one-to-one correspondence to an independent set in this hypergraph. Therefore, lower bounds on the independence number of hypergraphs can be used to derive lower bounds for selection problems. We will see that this "hypergraph approach" leads to improvements of known bounds for several extremal problems. This motivates the study of the independence number of hypergraphs, which is the subject of Chapter 2.

The aim of this thesis is: (i) to show that the hypergraph approach is a powerful tool for extremal problems and (ii) to derive new lower bounds for extremal problems using this tool, where we focus on geometric problems.

The application of the hypergraph approach to a given problem requires the study of the underlying structure. This raises questions, which are often interesting on their own. Example (3) leads to number theoretic questions concerning the representation of integers by sums of squares, which we answer in Section 4.1. Example (4) raises questions about the distribution of distances of point sets in $\mathbb{R}^d$, which will be considered in Section 4.3.

Overview

In Chapter 1 we are concerned with point sets in general position. We derive bounds on the maximum size of a subset of the $[n]^2$-grid with no 4 points on a common circle and no 3 points on a common line. Our lower bound, based on a construction, is tight up to a constant factor. Generalizations to higher dimensions are also considered. In Section 1.3 we discuss an application of such point sets to computational geometry. Section 1.4 deals with the problem of how to perturb lattice points in such a way that they will be in general position but certain topological properties are invariant.

In Chapter 2 we study the independence number of graphs and hypergraphs. We give an overview about known lower bounds for the independence number of graphs and uniform hypergraphs. In Section 2.3 we give a new result, which generalizes these bounds to arbitrary hypergraphs. Section 2.6 shows that the lower bounds can be improved if the given graph or hypergraph has a special structure. This chapter forms the basis for the hypergraph approach.

In Chapter 3 we develop and illustrate the hypergraph approach as a tool for selection problems. We give alternative proofs for two known results as examples for the typical use of this approach. In Section 3.2 we infer bounds on sets of lattice points with distinct slopes. In Section 3.3 we study so-called anti-Ramsey type results that form a suitable framework for the problems considered in Chapter 4. In addition to existence results we investigate asymptotic problems concerning threshold functions. Moreover, we consider algorithmic aspects in Section 3.5.

Finally, in Chapter 4 we use the knowledge developed in the previous chapters to infer new results for problems concerning point sets with distinct distances. To this end, we prove results about the representation of integers by sums of squares in Section 4.1. These results will also have implications for example (3) given above as we will show in Section 4.5. Furthermore, we will study the distribution of distances of point sets in Section 4.3. In addition to these existence results we determine threshold functions for point sets with distinct distances.

Notations

- $[n]$ $(n \in \mathbb{N})$: Set of integers $\{0, 1, \dots, n-1\}$.
- $[n]^d$ $(n \in \mathbb{N},\, d \in \mathbb{N})$: Set of lattice points in $\mathbb{R}^d$ with integer coordinates between 0 and $n-1$.
- $\lfloor x \rfloor$, $\lceil x \rceil$ $(x \in \mathbb{R})$: Largest integer not exceeding x and smallest integer greater or equal n, respectively.
- $x^{\underline{r}}$ $(x \in \mathbb{R},\, r \in \mathbb{N}_0)$: Falling factorial defined as $x\,(x-1)\dots(x-r+1)$.
- $[S]^k$ $(S$ a set, $k \in \mathbb{N})$: Set of all subsets of S of size k, also called k-sets of S.
- 2^S $(S$ a set$)$: Set of all subsets of S.
- $G = (V, E)$: Standard notation for graphs with vertex set V and edge set $E \subseteq [V]^2$. All occurring graphs are assumed to be simple (no loops, no multiple edges).
- $H = (V, \mathcal{E})$: Standard notation for hypergraphs with vertex set V and edge set $\mathcal{E} \subseteq 2^V \setminus \{\emptyset\}$.
- $d(v)$: Degree of a vertex v of a graph or uniform hypergraph.
- $\mathbf{d}(v)$: Degree vector of a vertex of an arbitrary hypergraph.
- $\mathbf{e}_k$, $\mathbf{0}$: k-th unit vector, zero vector.
- $\alpha(G)$, $\alpha(H)$: Independence number of a graph G and hypergraph H, respectively.
- K_n : Complete graph on n vertices.
- $r(f)$ $(f$ edge coloring of a complete graph$)$: Rainbow number of f.
- $\mathbf{E}(X)$ $(X$ random variable$)$: Expectation of X.
- $\mathbf{Var}(X)$ $(X$ random variable$)$: Variance of X.
- $f(n) \ll g(n)$: Abbreviation for $f(n) = O(g(n))$.
- $e(y)$ $(y \in \mathbb{R})$: Defined as $e(y) = e^{2\pi i y}$.
- $(a, b) = c$ $(a, b \in \mathbb{Z},\, c \in \mathbb{N})$: The greatest common divisor of a and b is c.
- $\phi(n)$ $(n \in \mathbb{N})$: Eulerian totient function defined as the number of positive integers k not exceeding n that are relative prime to n, i.e. $(n, k) = 1$.

Acknowledgement

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I thank Hanno Lefmann for our fruitful cooperation, which resulted in our joint work on "Point sets with distinct distances" [53]. My research about point sets in general position was initiated by a question of Raimund Seidel, who posed problem (2) mentioned above as an open question at a seminar on computational geometry in Dagstuhl (1993). I thank Richard K. Guy for inviting me to publish my results on this problem [68].

I am grateful to Kent D. Booklan for giving me the right pointers to the literature about the Hardy-Littlewood method. Finally, the countless publications of Paul Erdős, which are a treasure trove for beautiful results and unsolved problems, stimulated much of my research.

Chapter 1

Point sets in general position

In this chapter we are concerned with points sets in general position. There is no unique definition of what general position is since it depends on the context. Roughly speaking, general position is a property that is generic for point sets in the sense that a randomly chosen set of say n points from the unit cube $[0,1]^d$ satisfies with probability 1, where d is the dimension under consideration. A point set is called *degenerate* if it is not in general position. The following two typical examples are the ones we are dealing with:

(i) There are no $d+1$ points on a common hyperplane (a plane with codimension 1).

(ii) There are no $d+2$ points on a common sphere.

Throughout this chapter we will consider lines and hyperplanes as special cases of circles and spheres respectively, namely those with infinite radius. In other words, property (ii) implies that there are no $d+2$ points on a common hyperplane. An instance of a point set that is "highly" degenerate is the set of points of a grid $[n]^d$ in d-space. This raises the interesting combinatorial question: what is the maximum size of a subset of the $[n]^d$-grid that is in general position?

In the context of example (i) with $d=2$ this is the well-known *no-three-in-line problem*, which originated in a problem of Dudeney [22] in 1917. It is immediate that $2n$ is an upper bound because otherwise there would be three points in the same column. It is still an open problem if it is always possible to select $2n$ points from the $[n]^2$-grid with no three points in a line. Several computer tests have been made supporting this conjecture (see [39, 49]). A lower bound of $(1-\epsilon)\cdot n$ for every $\epsilon > 0$ and $n \geq n_0(\epsilon)$ was proved by Erdős [58] based on the following construction: Let p be the largest prime not exceeding n and take the subset $\{(t, t^2 \bmod p) : 0 \leq t < p\}$. This bound was improved by Hall et al [47] to $(3/2-\epsilon)\cdot n$ for every $\epsilon > 0$ and $n \geq n_0(\epsilon)$ and this is the best that is known.

The upper bound in arbitrary dimension $d \geq 2$ is dn since we can partition the point set $[n]^d$ into n hyperplanes of the form $x_1 = k$ with $0 \leq k < n$ each containing at most d points. The construction of Erdős for the planar case extends to higher dimensions by taking the subset $\{(t, t^2 \bmod p, \ldots, t^d \bmod p) : 0 \leq t < p\}$, where p is the largest prime not exceeding n. This yields again a lower bound of $(1-\epsilon)\cdot n$ for every $\epsilon > 0$ and $n \geq n_0(\epsilon)$.

In [46] Erdős and Purdy ask: how many points can be chosen from the $[n]^2$-grid with no four of them on a common circle? This is the case $d = 2$ of example (ii) mentioned above. In Sections 1.1 and 1.2 we derive new bounds for this problem and its extension to higher dimensions.

Point sets in general position are of special interest in the context of geometric algorithms. Algorithms like for instance for constructing convex hulls or Voronoi diagrams are typically designed under the assumption that the input is in general position. It remains a nontrivial task for any implementor to cope with degenerate cases in a *consistent* way. In Sections 1.3 and 1.4 we will see that sets of lattice points in general position can be used to get rid of these problems.

1.1 The no–four–on–circle problem

Erdős and Purdy ask the following question [46]: consider the $[n]^2$-grid. How many points can you choose, s.t. there are no four of them on a common circle; in particular there are no four points on a line (a circle with infinite radius). They mentioned in [46] that it is easy to show that $n^{2/3-\epsilon}$ (for every $\epsilon > 0$ and $n \geq n_0(\epsilon)$) is a lower bound but they conjectured that more is possible. Let $C_2(n)$ denote the maximum number of such points. We will show the following.

Theorem 1.1 *Let n be a positive integer. Then, $C_2(n) > \frac{n}{4}$.*

This bound is optimal up to the constant. An obvious upper bound is given by $3n$ since each column contains at most 3 points. One may wonder what happens if we consider only circles of finite radius. Can we choose substantially more points from the $[n]^2$-grid with the weaker restriction that there are no 4 points on a circle with finite radius? A simple example for such a point set is $\{(x,0) \mid 0 \leq x < n\}$. But the answer is no.

Suppose there are k points with the desired property. Let a_i denote the number of points in column i, thus $\sum a_i = k$. Each pair of points in the same column defines a horizontal bisector. With B_i we denote the set of bisectors generated by pairs of points in column i. If for any $i \neq j$ the intersection $B_i \cap B_j$ is non-empty then there is a circle containing 2 points from column i and 2 points from column j which is impossible. So the sets B_i are pairwise disjoint. Moreover, $|B_i| \geq (a_i - 1) + (a_i - 2) = 2a_i - 3$. Note that a bisector maybe generated by several pairs of points in a fixed column. On the other hand there are only $(n-1) + (n-2) = 2n - 3$ possible horizontal bisectors. This implies

$$2k - 3n = \sum_{i=0}^{n-1} 2a_i - 3 \leq \sum_{i=0}^{n-1} |B_i| = \left| \bigcup_{i=0}^{n-1} B_i \right| \leq 2n - 3\,.$$

We infer the following upper bound.

Observation 1.2 *Let n be a positive integer. Then, $C_2(n) \leq \frac{5}{2}n - \frac{3}{2}$.*

This upper bound is valid even if we allow arbitrarily many points on a common line.

For the proof of the lower bound we will construct a subset of the $[n]^2$-grid with the desired property. In fact, our point set will have a further property: there are no *three* points on a line.

Let p be a prime number and $k \in \mathbb{Z}$. Define a set of lattice points $S(p,k)$ by

$$S(p,k) := \left\{(t, t^2 + k \bmod p) : 0 \le t \le (p+5)/4\right\} \subseteq [\lfloor \tfrac{p+5}{4} \rfloor + 1] \times [p]\,. \tag{1.1}$$

Note that the set $S(p,k)$ is a copy of $S(p,0)$, where each point is shifted upwards by k rows modulo p.

Lemma 1.3 *Let p be a prime number and $k \in \mathbb{Z}$. Then $S(p,k)$ contains no four points on a common circle and no three points on a common line.*

Proof. Assume first that there are three points $(t_i, t_i^2 + k \bmod p)$, $i = 1,2,3$, in $S(p,k)$ on a line. Then the determinant

$$\begin{vmatrix} 1 & t_1 & t_1^2 + k \bmod p \\ 1 & t_2 & t_2^2 + k \bmod p \\ 1 & t_3 & t_3^2 + k \bmod p \end{vmatrix}$$

would be zero. In particular, it would be zero modulo p. Thus,

$$0 \equiv \begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{vmatrix} = \prod_{i<j}(t_j - t_i) \pmod{p}\,.$$

But this is impossible since p is a prime and the t_i's are distinct modulo p.

Now assume there are four points $(t_i, t_i^2 + k \bmod p)$, $i = 1,2,3,4$, in $S(p,k)$ on a circle. Four points in the plane lie on a common circle if and only if their projections onto the unit paraboloid $\{(x, y, x^2 + y^2) : x, y \in \mathbb{R}\}$ lie on a common hyperplane (see [24]). Thus the determinant

$$\begin{vmatrix} 1 & t_1 & t_1^2 + k \bmod p & t_1^2 + (t_1^2 + k \bmod p)^2 \\ 1 & t_2 & t_2^2 + k \bmod p & t_2^2 + (t_2^2 + k \bmod p)^2 \\ 1 & t_3 & t_3^2 + k \bmod p & t_3^2 + (t_3^2 + k \bmod p)^2 \\ 1 & t_4 & t_4^2 + k \bmod p & t_4^2 + (t_4^2 + k \bmod p)^2 \end{vmatrix}$$

would be zero. In particular, we get

$$0 \equiv \begin{vmatrix} 1 & t_1 & t_1^2 & t_1^2 + t_1^4 \\ 1 & t_2 & t_2^2 & t_2^2 + t_2^4 \\ 1 & t_3 & t_3^2 & t_3^2 + t_3^4 \\ 1 & t_4 & t_4^2 & t_4^2 + t_4^4 \end{vmatrix} = \begin{vmatrix} 1 & t_1 & t_1^2 & t_1^4 \\ 1 & t_2 & t_2^2 & t_2^4 \\ 1 & t_3 & t_3^2 & t_3^4 \\ 1 & t_4 & t_4^2 & t_4^4 \end{vmatrix} = (t_1 + t_2 + t_3 + t_4)\prod_{i<j}(t_j - t_i) \pmod{p}\,.$$

The last equality is a special case of Lemma 1.6 ($d = k = 2$), that we will prove in Section 1.2. Since p is a prime and the t_i's are distinct (modulo p) the factor $t_1 + \ldots + t_4$ has to be zero modulo p. However, we have $0 < t_1 + t_2 + t_3 + t_4 \leq 4\,\frac{p+5}{4} - 6 = p - 1$, a contradiction. Thus, $S(p,k)$ contains no 3 points on a line and no 4 points on a circle. ❏

Proof of Theorem 1.1. Since the Theorem is obvious for $n \leq 4$ we can assume $n \geq 5$. Let p be a prime with $n \leq p \leq 4n - 9$. Such a prime exists, indeed Bertrand's Postulate [48] guarantees the existence of a prime with $n \leq p \leq 2n$. Note that the $[n]^2$-grid intersects all columns used by $S(p,k)$ since $n - 1 \geq \left\lfloor \frac{p+5}{4} \right\rfloor$.

Let C_k be the set of points of $S(p,k)$ contained in the $[n]^2$-grid, i.e.

$$C_k = [n]^2 \cap S(p,k) \quad \text{for } 0 \leq k \leq p-1\,.$$

Consider a fixed point of $S(p,0)$. As k varies from 0 to $p-1$ a copy of this point will be contained n times in the $[n]^2$-grid since $n \leq p$. We infer,

$$\sum_{k=0}^{p-1} |C_k| = n \cdot |S(p,0)| = n \cdot \left(\left\lfloor \tfrac{p+5}{4} \right\rfloor + 1 \right) > \frac{np}{4}\,.$$

Thus there exists a $k \in [p]$ with $|C_k| > \frac{n}{4}$. By construction, this C_k is a subset of the $[n]^2$-grid. Moreover, it contains no four points on a circle and no three points on a line by Lemma 1.3 since $C_k \subseteq S(p,k)$. ❏

1.2 Generalization to higher dimensions

In this section we will see how the construction given in the last section generalizes to dimension $d \geq 3$. Let $C_d(n)$ be the maximum number of points that can be chosen from the $[n]^d$-grid, s.t. there are no $d+2$ points on a common sphere. An upper bound for $C_d(n)$ is given by $(d+1) \cdot n$, since otherwise there would be a hyperplane (sphere with infinite radius) containing $d+2$ points.

Theorem 1.4 *For $d \geq 3$ and $n \in \mathbb{N}$ we have $C_d(n) > c \cdot n^{1/(d-1)}$, where $c > 0$ is a constant independent of n and d.*

A lower bound of $\Omega(n^{1/d})$ is implicitly given in [26] (see Section 1.3). The proof of this theorem uses the following construction. For a prime number p define the set

$$S_d(p) := \left\{ (t, t^2 \bmod p, \ldots, t^d \bmod p) : 0 \leq t \leq \frac{3}{16}\, p^{1/(d-1)} \right\}\,,$$

which is a subset of the $[p]^d$-grid.

Lemma 1.5 *For all $d \geq 3$ and p a prime number the set $S_d(p)$ does not contain a set of $d+2$ points on a common sphere or a set of $d+1$ points on a common hyperplane.*

It is known that $d+2$ points $p_1, p_2, \ldots, p_{d+2}$ in $\mathbb{R}^d$ lie on a common sphere if and only if their projections $f(p_i) = (p_{i1}, p_{i2}, \ldots, p_{id}, p_{i1}^2 + p_{i2}^2 + \ldots + p_{id}^2)$ onto the unit paraboloid in $\mathbb{R}^{d+1}$ lie on a common hyperplane (see [24]). So in order to test whether $p_1, \ldots, p_{d+2}$ lie on a sphere we have to check if the determinant

$$\begin{vmatrix} 1 & p_{1,1} & p_{1,2} & \ldots & p_{1,d} & p_{1,1}^2 + p_{1,2}^2 + \ldots + p_{1,d}^2 \\ 1 & p_{2,1} & p_{2,2} & \ldots & p_{2,d} & p_{2,1}^2 + p_{2,2}^2 + \ldots + p_{2,d}^2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & p_{d+2,1} & p_{d+2,2} & \ldots & p_{d+2,d} & p_{d+2,1}^2 + p_{d+2,2}^2 + \ldots + p_{d+2,d}^2 \end{vmatrix}$$

is zero. We will show that

$$M_d := \begin{vmatrix} 1 & t_1 & t_1^2 & \ldots & t_1^d & t_1^2 + t_1^4 + \ldots + t_1^{2d} \\ 1 & t_2 & t_2^2 & \ldots & t_2^d & t_2^2 + t_2^4 + \ldots + t_2^{2d} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & t_{d+2} & t_{d+2}^2 & \ldots & t_{d+2}^d & t_{d+2}^2 + t_{d+2}^4 + \ldots + t_{d+2}^{2d} \end{vmatrix}$$

is always nonzero modulo p for any choice of distinct $0 \le t_i \le \frac{3}{16} p^{1/(d-1)}$ which implies that $S_d(p)$ contains no $d+2$ points on a sphere. Define

$$D(d,k) := \begin{vmatrix} 1 & t_1 & t_1^2 & \ldots & t_1^d & t_1^{d+k} \\ 1 & t_2 & t_2^2 & \ldots & t_2^d & t_2^{d+k} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & t_{d+2} & t_{d+2}^2 & \ldots & t_{d+2}^d & t_{d+2}^{d+k} \end{vmatrix} \qquad \text{for } d \ge 0,\ k \ge 1.$$

For $1 \le j \le d$ we multiply the j-th column with t_{d+2} and subtract it from the $(j+1)$-st column. Also we multiply the $(d+1)$-st column with t_{d+2}^k and subtract it from the $(d+2)$-nd column. We cancel the last row and the first column leaving a factor of $(-1)^{d+1}$ and factor $t_i - t_{d+2}$ from the i-th row for $1 \le i \le d+1$. This gives

$$\begin{aligned} D(d,k) &= (-1)^{d+1} \prod_{1 \le i \le d+1} (t_i - t_{d+2}) \begin{vmatrix} 1 & t_1 & t_1^2 & \ldots & t_1^{d-1} & t_1^d \cdot \frac{t_1^k - t_{d+2}^k}{t_1 - t_{d+2}} \\ 1 & t_2 & t_2^2 & \ldots & t_2^{d-1} & t_2^d \cdot \frac{t_2^k - t_{d+2}^k}{t_2 - t_{d+2}} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & t_{d+1} & t_{d+1}^2 & \ldots & t_{d+1}^{d-1} & t_{d+1}^d \cdot \frac{t_{d+1}^k - t_{d+2}^k}{t_{d+1} - t_{d+2}} \end{vmatrix} \\ &= \prod_{1 \le i \le d+1} (t_{d+2} - t_i) \begin{vmatrix} 1 & t_1 & t_1^2 & \ldots & t_1^{d-1} & \sum_{l=1}^k t_{d+2}^{k-l} \cdot t_1^{(d-1)+l} \\ 1 & t_2 & t_2^2 & \ldots & t_2^{d-1} & \sum_{l=1}^k t_{d+2}^{k-l} \cdot t_2^{(d-1)+l} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & t_{d+1} & t_{d+1}^2 & \ldots & t_{d+1}^{d-1} & \sum_{l=1}^k t_{d+2}^{k-l} \cdot t_{d+1}^{(d-1)+l} \end{vmatrix} \\ &= \prod_{1 \le i \le d+1} (t_{d+2} - t_i) \sum_{l=1}^{k} t_{d+2}^{k-l} D(d-1,l)\,. \end{aligned} \tag{1.2}$$

Lemma 1.6

$$D(d,k) = \prod_{i<j}(t_j - t_i) \cdot \sum_{|e|=k-1} \prod_{i=1}^{d+2} t_i^{e_i},$$

where $|e|$ *denotes the sum* $|e| := e_1 + e_2 + \ldots + e_{d+2}$.

Proof. By induction on d. For $d = 0$ we have $D(0,k) = \begin{vmatrix} 1 & t_1^k \\ 1 & t_2^k \end{vmatrix} = t_2^k - t_1^k = (t_2 - t_1) \cdot \sum_{|e|=k-1} t_1^{e_1} t_2^{e_2}$.

Suppose the formula for $D(d-1,l)$ is correct for all l. Let k be arbitrary. According to equation (1.2) we know that

$$D(d,k) = \prod_{1 \le i \le d+1} (t_{d+2} - t_i) \sum_{l=1}^{k} t_{d+2}^{k-l} \, D(d-1,l).$$

The induction hypothesis implies

$$\prod_{1 \le i \le d+1} (t_{d+2} - t_i) \, t_{d+2}^{k-l} \, D(d-1,l) = \prod_{1 \le i < j \le d+2} (t_j - t_i) \cdot \sum_{|e|=l-1} t_{d+2}^{k-l} \cdot \prod_{i=1}^{d+1} t_i^{e_i}$$

for all l. Summation over all $1 \le l \le k$ gives

$$D(d,k) = \prod_{i<j}(t_j - t_i) \cdot \sum_{|e|=k-1} \prod_{i=1}^{d+2} t_i^{e_i}.$$

❏

Observe that

$$M_d = \begin{vmatrix} 1 & t_1 & \ldots & t_1^d & \sum_{l=\lfloor d/2 \rfloor+1}^{d} t_1^{2l} \\ 1 & t_2 & \ldots & t_2^d & \sum_{l=\lfloor d/2 \rfloor+1}^{d} t_2^{2l} \\ . & . & . & . & . \\ . & . & . & . & . \\ 1 & t_{d+2} & \ldots & t_{d+2}^d & \sum_{l=\lfloor d/2 \rfloor+1}^{d} t_{d+2}^{2l} \end{vmatrix} = \sum_{l=0}^{\lfloor \frac{d-1}{2} \rfloor} D(d, d-2l). \tag{1.3}$$

Proof of Lemma 1.5. Suppose we are given $d+2$ distinct points of $S_d(p)$ with parameters $0 \le t_1, \ldots, t_{d+2} \le T := \frac{3}{16} p^{1/(d-1)}$. In order to show that these points do not lie on a common sphere we will prove that the determinant M_d belonging to $t_1, \ldots, t_{d+2}$ is nonzero modulo p, which is sufficient. Let $Q(d,k) := \sum_{|e|=k-1} \prod_{i=1}^{d+2} t_i^{e_i}$. Then $D(d,k) = \prod_{i<j}(t_j - t_i) \cdot Q(d,k)$ by Lemma 1.6. We infer with (1.3) that

$$M_d = \prod_{i<j}(t_j - t_i) \cdot \sum_{l=0}^{\lfloor \frac{d-1}{2} \rfloor} Q(d, d-2l).$$

Since p is a prime $\prod(t_j - t_i)$ is nonzero modulo p, so we have to show that $\sum Q(d, d-2l)$ is nonzero modulo p. Observe that $Q(d,k) < T^{k-1} \cdot \binom{d+k}{k-1}$ since there are $\binom{d+k}{k-1}$ monomials of the form $\prod_{i=1}^{d+2} t_i^{e_i}$ with degree $|e| = \sum_{i=1}^{d+2} e_i = k-1$. Thus

$$0 < \sum_{l=0}^{\lfloor\frac{d-1}{2}\rfloor} Q(d, d-2l) < T^{d-1} \cdot \sum_{l=0}^{\lfloor\frac{d-1}{2}\rfloor} \binom{2d-2l}{d-1-2l}.$$

It is easy to see that $\binom{2d-2l}{d-1-2l} \leq (1/2)^{2l} \cdot \binom{2d}{d-1}$ which implies

$$\sum_{l=0}^{\lfloor\frac{d-1}{2}\rfloor} Q(d, d-2l) < T^{d-1} \cdot \binom{2d}{d-1} \sum_{l=0}^{\infty} \left(\frac{1}{4}\right)^l = \frac{4}{3} T^{d-1} \cdot \binom{2d}{d-1}.$$

Since $\binom{2d}{d-1} \leq 4^{d-1}$ we infer

$$0 < \sum_{l=0}^{\lfloor\frac{d-1}{2}\rfloor} Q(d, d-2l) < \frac{4}{3} \cdot 4^{d-1} T^{d-1} \leq \left(\frac{16}{3}\right)^{d-1} \left(\frac{3}{16} p\right)^{d-1} = p.$$

This proves that $M_d \not\equiv 0 \pmod{p}$.

Now suppose we are given $d+1$ points of $S_d(p)$ with parameters $0 \leq t_1, t_2, \ldots, t_{d+1} \leq T < p$. If these points lie on a common hyperplane then the Vandermonde determinant

$$\begin{vmatrix} 1 & t_1 & t_1^2 & \ldots & t_1^d \\ 1 & t_2 & t_2^2 & \ldots & t_2^d \\ . & . & . & . & . \\ . & . & . & . & . \\ 1 & t_{d+1} & t_{d+1}^2 & \ldots & t_{d+1}^d \end{vmatrix} = \prod_{i<j} (t_j - t_i)$$

would be zero modulo p which is impossible, since p is a prime and the t_i's are distinct. Hence the set $S_d(p)$ has the desired properties. ❑

Proof of Theorem 1.4. We will show that the lower bound is true for $c = 3/32$. For $n = 1$ there is nothing to prove. So suppose that $n \geq 2$. According to Bertrand's Postulate ([48]) there is a prime number p satisfying $n/2 < p \leq n$. Since $S_d(p)$ is clearly a subset of the $[n]^d$-grid we infer that

$$C_d(n) \geq |S_d(p)| > \frac{3}{16} p^{1/(d-1)} > \frac{3}{16} \left(\frac{n}{2}\right)^{1/(d-1)} \geq \frac{3}{32} n^{1/(d-1)}.$$

❑

1.3 Application to a perturbation technique

In this section we will briefly discuss an application of the sets S_d that we constructed in 1.1 and 1.2. Many geometric algorithms such as constructing the convex hull or the Voronoi diagram of a given set of n points in $\mathbb{R}^d$ (see for example [24]) are based on tests of the following predicates:

(i) Given a sequence of $d+1$ points $v_1, v_2, \ldots, v_{d+1} \in \mathbb{R}^d$, on which side of the hyperplane spanned by $v_1, \ldots, v_d$ does the point v_{d+1} lie? If the points $v_1, \ldots, v_{d+1}$ lie on a common hyperplane then we have a degeneracy.

(ii) Given a sequence of $d+2$ points $v_1, v_2, \ldots, v_{d+2} \in \mathbb{R}^d$. Does the point v_{d+2} lie inside or outside of the sphere determined by $v_1, \ldots, v_{d+1}$? If the points $v_1, \ldots, v_{d+2}$ lie on a common sphere (or hyperplane, i.e. a sphere with infinite radius) we have a degeneracy.

Typically, those geometric algorithms are designed under the assumption that the input data generates no degeneracy, avoiding annoying case studies. This leaves the task to cope with degenerate cases in a *consistent* way to the implementor, which turns out to be a nontrivial matter. Moreover, when working with finite precision, degeneracies are likely to occur. The first idea one may have is to "break ties arbitrarily" just the way it can be done in a sorting algorithm when two equal number are encountered. More precisely, this means in the context of predicate (i) that if $v_1, \ldots, v_{d+1}$ lie on a common hyperplane, we can flip a coin and the decide the answer of the predicate. But it is easy to see that this can lead to inconsistencies in the sense that there exists no point set realizing these orientations.

Edelsbrunner and Mücke [25] introduced a technique called *Simulation of Simplicity* (SoS) based on the following idea (see also [26]). We perturb the input set P in such a way that the resulting set P' is in general position and the amount of perturbation is so small that the predicates involved are unchanged for nondegenerate subsets of P. This perturbed point set P' is used only inside of the predicates to get consistent answers. The consistency is guaranteed by the physical existence of P'. An algorithm designed for point sets in general position can now be applied to P; the only things that have to be changed are the predicates. In a postprocessing step artificial geometric objects such as faces of measure zero (Voronoi diagrams) or nonextreme points on the boundary of the convex hull can be eliminated together with incident faces.

The perturbation is accomplished in the following way. Suppose we are given a set $P = \{p_1, p_2, \ldots, p_n\}$ of n points in $\mathbb{R}^d$ and a set $\Delta = \{\Delta_1, \Delta_2, \ldots, \Delta_n\} \subset \mathbb{R}^d$ that is in general position with respect to predicate (i) or (ii) respectively. The set P is then perturbed into the set $P(\epsilon) = \{p_1(\epsilon), \ldots, p_n(\epsilon)\}$ with $p_i(\epsilon) = p_i + \epsilon\Delta_i$. In [26] it is shown that $P(\epsilon)$ is in general position and the predicate is unchanged for nondegenerate subsets of P, given that $\epsilon > 0$ is sufficiently small. For an efficient implementation of this perturbation scheme as presented in [26] it is necessary that the bitcomplexity of the set Δ, i.e. the number of bits needed to represent the points of Δ, is small. For predicate

(ii) Emiris and Canny made use of the set $\Delta := \{(t, t^2, \ldots, t^d) : 0 \leq t < n\}$, which is a set of points from the $[n^d]^d$-grid. This set can now be replaced by $\Delta := S_d(p) = \{(t, t^2 \bmod p, \ldots, t^d \bmod p) : 0 \leq t \leq 3/16 \cdot p^{1/(d-1)}\} \subset [p]^d$, where p is the smallest prime with $3/16 \cdot p^{1/(d-1)} \geq n$. In Section 1.2 we proved that Δ is in general position with respect to predicate (ii). We infer that $p \leq c_d n^{d-1}$ by Bertrand's postulate and thus $\Delta \subset [c_d n^{d-1}]^d$ improving the bitcomplexity.

1.4 Perturbation of lattice points

In this section we investigate a different kind of perturbation. Consider the grid $[n]^2$. We are looking for a map f from this grid to some larger grid $[N]^2$ with the following two properties:

(i) $f([n]^2)$ contains no 3 points on a line.

(ii) f preserves the orientation of lattice triangles in the following sense: Suppose we are given 3 lattice points (x_i, y_i), $i = 1, 2, 3$ not on a common line, then the orientation of the image points $f(x_i, y_i)$ is the same as for (x_i, y_i), $i = 1, 2, 3$.

What is the minimal value of N – denoted by $N_2(n)$ – admitting a map f with these properties?

A lower bound is given by $N_2(n) \geq n^2/2$ since there are at most $2N$ points in the $[N]^2$ grid with no 3 points on a line.

Theorem 1.7 *Let $\epsilon > 0$ be given. Then, for n sufficiently large we have*

$$N_2(n) \leq (3 + \epsilon) \cdot n^3 .$$

The proof of this upper bound is constructive. Let $p \geq n$ be a prime number with $p \equiv 3 \pmod 4$. Consider the map $f : [n]^2 \to [\lambda n + p]^2$ given by

$$f(x, y) := \lambda \cdot (x, y) + (y, x^2 + y^2 \bmod p) \tag{1.4}$$

with $\lambda := 3np$.

Lemma 1.8 *The map f satisfies the conditions (i) and (ii).*

Proof. First we will prove condition (ii). Let (x_i, y_i), $i = 1, 2, 3$ be given points from $[n]^2$ not on a common line, i.e.

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \neq 0 .$$

W.l.o.g. we can assume that the determinant is positive (otherwise interchange (x_1, y_1) and (x_2, y_2)). Since all entries of the determinant are integers it follows that

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \geq 1 . \tag{1.5}$$

We have to show that the corresponding determinant D of the image points is positive too. Let $z_i := (x_i^2 + y_i^2) \bmod p$ for $i = 1,2,3$. Note that $|z_i - z_j| < p$ and similarly $|x_i - x_j|, |y_i - y_j| < n$. By (1.5) we infer for D

$$\begin{aligned}
\begin{vmatrix} 1 & \lambda x_1 + y_1 & \lambda y_1 + z_1 \\ 1 & \lambda x_2 + y_2 & \lambda y_2 + z_2 \\ 1 & \lambda x_3 + y_3 & \lambda y_3 + z_3 \end{vmatrix} &= \begin{vmatrix} 1 & \lambda x_1 & \lambda y_1 \\ 1 & \lambda x_2 & \lambda y_2 \\ 1 & \lambda x_3 & \lambda y_3 \end{vmatrix} + \begin{vmatrix} 1 & \lambda x_1 & z_1 \\ 1 & \lambda x_2 & z_2 \\ 1 & \lambda x_3 & z_3 \end{vmatrix} + \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix} \\
&\geq \lambda^2 + \lambda \begin{vmatrix} x_2 - x_1 & z_2 - z_1 \\ x_3 - x_1 & z_3 - z_1 \end{vmatrix} + \begin{vmatrix} y_2 - y_1 & z_2 - z_1 \\ y_3 - y_1 & z_3 - z_1 \end{vmatrix} \\
&> \lambda^2 - \lambda \cdot 2np - 2np \\
&= (3np - 2) \cdot np \\
&> 0\,.
\end{aligned}$$

Hence, f preserves the orientation of lattice triangles.

Now we consider condition (i). Let $p_i = (x_i, y_i)$, $i = 1,2,3$ be 3 given points from $[n]^2$. If they do not lie on a common line then the same is true for their image points by condition (ii). So we can assume that p_1, p_2, p_3 do lie on a common line g. We represent the points on g in the form

$$g(t) = (a,b) + t \cdot (c,d)\,, \quad t \in \mathbb{R}\,, \tag{1.6}$$

where (a,b) is the smallest point in $g \cap [n]^2$ with respect to lexicographic order and c and d are relative prime integers with $c \geq 0$. Clearly, $c, |d| < n$ since $|g \cap [n]^2| \geq 2$ by assumption. Furthermore, every point in $g \cap [n]^2$ corresponds to a parameter $t \in [n]$. Let t_i be the parameter corresponding to p_i for $i = 1,2,3$. We have to show that the image points $f(p_1), f(p_2), f(p_3)$ are not collinear. Assume this is not the case, i.e. they are collinear. In other words the determinant

$$D := \begin{vmatrix} 1 & \lambda x_1 + y_1 & \lambda y_1 + (x_1^2 + y_1^2 \bmod p) \\ 1 & \lambda x_2 + y_2 & \lambda y_2 + (x_2^2 + y_2^2 \bmod p) \\ 1 & \lambda x_3 + y_3 & \lambda y_3 + (x_3^2 + y_3^2 \bmod p) \end{vmatrix}$$

is zero. We distinguish to cases.

$\underline{d \neq 0}$: Since D is zero we have $D \equiv 0 \pmod p$. Using the fact that λ is a multiple of p this equation simplifies to

$$\begin{vmatrix} 1 & y_1 & x_1^2 + y_1^2 \\ 1 & y_2 & x_2^2 + y_2^2 \\ 1 & y_3 & x_3^2 + y_3^2 \end{vmatrix} \equiv 0 \pmod p\,.$$

By substituting the parameter form (1.6), we get

$$0 \equiv \begin{vmatrix} 1 & b + t_1 d & (a + t_1 c)^2 + (b + t_1 d)^2 \\ 1 & b + t_2 d & (a + t_2 c)^2 + (b + t_2 d)^2 \\ 1 & b + t_3 d & (a + t_3 c)^2 + (b + t_3 d)^2 \end{vmatrix} = d(c^2 + d^2) \prod_{i<j} (t_j - t_i) \pmod p\,.$$

Since p is a prime number one of these factors must be zero modulo p. The parameters t_i are pairwise distinct elements from $[n]$. Remember that we chose p to be greater than or equal to n. So $\prod_{i<j}(t_j - t_i) \not\equiv 0 \pmod p$. Similarly $d \not\equiv 0 \pmod p$, since $0 \neq |d| < n \leq p$. We infer that $c^2 + d^2 \equiv 0 \pmod p$ or equivalently

$$c^2 \equiv -d^2 \pmod p\,. \tag{1.7}$$

Now we recall that $p \equiv 3 \pmod 4$, so that $(p-1)/2$ is an odd integer. Taking both sides of (1.7) to the power of $(p-1)/2$ implies $c^{p-1} \equiv -d^{p-1} \equiv -1 \pmod p$ by Fermat's little theorem and since $d \not\equiv 0 \pmod p$. But this is impossible again by Fermat's little theorem. Thus, $f(p_1), f(p_2), f(p_3)$ can not be collinear if $d \neq 0$.

$\underline{d=0}$: In this case the line g is horizontal, i.e. the points p_1, p_2, p_3 are of the form (x_1, b), (x_2, b), (x_3, b) where $b \in [n]$. Since we assume that these points are collinear, we get

$$0 = \begin{vmatrix} 1 & \lambda x_1 + b & \lambda b + (x_1^2 + b^2 \bmod p) \\ 1 & \lambda x_2 + b & \lambda b + (x_2^2 + b^2 \bmod p) \\ 1 & \lambda x_3 + b & \lambda b + (x_3^2 + b^2 \bmod p) \end{vmatrix} = \lambda \begin{vmatrix} 1 & x_1 & x_1^2 + b^2 \bmod p \\ 1 & x_2 & x_2^2 + b^2 \bmod p \\ 1 & x_3 & x_3^2 + b^2 \bmod p \end{vmatrix}.$$

This implies

$$0 \equiv \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = \prod_{i<j}(x_j - x_i) \pmod p\,.$$

Again, this is a contradiction since p is a prime with $p \geq n$ and the x_i's are pairwise distinct members of $[n]$.

We conclude that the map f satisfies condition (i), which completes the proof. ❑

Proof of Theorem 1.7. Let $\epsilon > 0$ be given and define $\epsilon' := \epsilon/4$. Consider the arithmetic progression $\{4k+3 \mid k \in \mathbb{N}_0\}$ and let $\pi_4(x)$ denote the number of primes in this progression not exceeding x. By the prime number theorem for arithmetic progressions (see [9],[56]) we know that

$$\lim_{x\to\infty} \frac{\pi_4(x)\ln x}{x} = \frac{1}{\phi(4)} = \frac{1}{2},$$

where ϕ is the eulerian totient function. In particular we have

$$\pi_4(x) = \frac{x}{2\ln x} + o\left(\frac{x}{\ln x}\right).$$

This implies

$$\pi_4((1+\epsilon')\cdot n) - \pi_4(n) = \frac{\epsilon' n}{2\ln n} + o\left(\frac{n}{\ln n}\right) > 0$$

for n sufficiently large, i.e. $n \geq n_0(\epsilon)$. Thus, there is a prime number p of the form $4k+3$ with $n < p \leq (1+\epsilon')\cdot n$. According to Lemma 1.8 the map $f : [n]^2 \to [\lambda n + p]^2$ given by (1.4) satisfies the conditions (i) and (ii). We infer that

$$N_2(n) \leq \lambda n + p = 3np \cdot n + p \leq (3 + 4\epsilon')\cdot n^3 = (3+\epsilon)\cdot n^3$$

for n large enough depending on ϵ. ❑

Higher dimensions.

Given a sequence of $d+1$ points $v_1, v_2, \ldots, v_{d+1} \in \mathbb{R}^d$. Define

$$D_H(v_1, \ldots, v_{d+1}) := \begin{vmatrix} 1 & v_{1,1} & v_{1,2} & \ldots & v_{1,d} \\ 1 & v_{2,1} & v_{2,2} & \ldots & v_{2,d} \\ . & . & . & \ldots & . \\ 1 & v_{d+1,1} & v_{d+1,2} & \ldots & v_{d+1,d} \end{vmatrix}.$$

The orientation of the sequence $v_1, \ldots, v_{d+1}$ is given by the sign of this determinant. Note that $D_H(v_1, \ldots, v_{d+1}) = 0$ if and only if the points lie on a common hyperplane. Similar to the two-dimensional case we are looking for a function f that maps the points of the $[n]^d$-grid to some larger grid $[N]^d$ with the following two properties.

(i) There are no $d+1$ points in $f([n]^d)$ on a common hyperplane.

(ii) The map f preserves the orientation of lattice simplices in the following sense. Suppose we are given $d+1$ points $v_1, \ldots, v_{d+1} \in [n]^d$ not on a common hyperplane, then the sign of $D_H(f(v_1), \ldots, f(v_{d+1}))$ is the same as for $D_H(v_1, \ldots, v_{d+1})$.

Let $N_d(n)$ denote the minimal value of N admitting a map f with these properties. Clearly, a lower bound is given by $N_d(n) \geq n^d/d$ since there are at most dN points in the $[N]^d$-grid with no $d+1$ points on a common hyperplane.

Theorem 1.9 *For $d \geq 3$ and $n \in \mathbb{N}$ we have*

$$N_d(n) \leq 2^{d+1} d!\, n^{2d}.$$

Note that this bound is weaker than Theorem 1.7 for $d = 2$, where we made use of a slightly more sophisticated construction that seems to be hard to extend to higher dimensions. Here we will use the following construction. Let $p \geq n^d$ be a prime number and define $\lambda := (2^d - 1) \cdot d!\, n^{d-1} p$. Consider the map $f := [n]^d \to [\lambda n + p]^d$ given by

$$f(x) := \lambda x + \Delta(x) \tag{1.8}$$

with

$$\Delta(x) := (z(x), z^2(x) \bmod p, \ldots, z^d(x) \bmod p)$$

and

$$z(x) := \sum_{k=1}^{d} x_k n^{k-1} \qquad \text{for } x = (x_1, \ldots, x_d).$$

Observe that the i-th row of $D_H(f(v_1), \ldots, f(v_{d+1}))$ is given by

$$1\,,\ \lambda v_{i,1} + z(v_i)\,,\ \lambda v_{i,2} + (z^2(v_i) \bmod p)\,,\ \ldots\,,\ \lambda v_{i,d} + (z^d(v_i) \bmod p)\,.$$

Lemma 1.10 *The map f satisfies conditions (i) and (ii).*

Proof. Consider condition (i) first. Let $v_1, \dots, v_{d+1}$ be a sequence of $d+1$ distinct points from $[n]^d$. We have to show that $D_H(f(v_1), \dots, f(v_{d+1})) \neq 0$. We will prove $D_H(f(v_1), \dots, f(v_{d+1})) \not\equiv 0 \pmod{p}$ which is sufficient. Since λ is an integer multiple of p we have

$$D_H(f(v_1), \dots, f(v_{d+1})) \equiv D_H(\Delta(v_1), \dots, \Delta(v_{d+1})) \equiv \prod_{i<j}(z(v_j) - z(v_i)) \pmod{p}.$$

Observe that for $x \neq y \in [n]^d$ we have $z(x) \neq z(y)$ and $0 \leq z(x), z(y) \leq n^d - 1 < p$ by the definition of the function z. But this implies

$$D_H(f(v_1), \dots, f(v_{d+1})) \equiv \prod_{i<j}(z(v_j) - z(v_i)) \not\equiv 0 \pmod{p}$$

since p is a prime.

Now we consider condition (ii). Let $v_1, \dots, v_{d+1}$ be a sequence of $d+1$ points from $[n]^d$ with $D_H(v_1, \dots, v_{d+1}) \neq 0$. We can assume w.l.o.g. that $D_H(v_1, \dots, v_{d+1}) > 0$ (otherwise interchange v_1 and v_2). Since all entries of this determinant are integers we have $D_H(v_1, \dots, v_{d+1}) \geq 1$. By using the fact that a determinant is linear in each column we expand $D_H(f(v_1), \dots, f(v_{d+1}))$ to

$$D_H(f(v_1), \dots, f(v_{d+1})) = \lambda^d \cdot D_H(v_1, \dots, v_{d+1}) + \sigma \geq \lambda^d + \sigma\,,$$

where σ is a sum of $2^d - 1$ terms of the form λ^k times a determinant with $0 \leq k \leq d-1$. Furthermore, the absolute value of σ is bounded by

$$|\sigma| < \sum_{k=0}^{d-1} \binom{d}{k} d!\,(\lambda n)^k\, p^{d-k} \leq (2^d - 1) \cdot d!\,(\lambda n)^{d-1} p\,.$$

We conclude that

$$D_H(f(v_1), \dots, f(v_{d+1})) \geq \lambda^d + \sigma > \lambda^d - (2^d - 1) \cdot d!\,\lambda^{d-1} n^{d-1} p = 0$$

by the choice $\lambda = (2^d - 1) \cdot d!\, n^{d-1} p$. ❑

Proof of Theorem 1.9. Let $d \geq 3$ and $n \in \mathbb{N}$ be given. Furthermore, let p be the smallest prime number greater than n^d. Bertrand's postulate (see [48]) guarantees that $p \leq 2\,n^d$. By Lemma 1.10 the map $f : [n]^d \to [\lambda n + p]^d$ given by (1.8) with $\lambda = (2^d - 1) \cdot d!\, n^{d-1} p$ satisfies the desired properties. Thus,

$$N_d(n) \leq \lambda n + p \leq 2^{d+1} d!\, n^{2d}\,.$$

❑

Preserving the insphere predicate.

In this section we will discuss – for the sake of completeness – how to adapt the construction of the last section in order to perturb lattice points while preserving the insphere predicate *and* the orientation predicate. The bounds we will get still leave a large gap and it is an interesting open problem to derive better bounds.

Given a sequence of $d+2$ points $v_1, v_2, \dots, v_{d+1} \in \mathbb{R}^d$ not on a common hyperplane and a point $v_{d+2} \in \mathbb{R}^d$. Then there is a unique sphere through $v_1, \dots, v_{d+1}$ and the insphere predicate determines whether the point v_{d+2} lies inside or outside this sphere. If v_{d+2} lies on this sphere then we have a degeneracy. This predicate can be calculated by computing the sign of the determinant

$$D_S(v_1, \dots, v_{d+2}) := \begin{vmatrix} 1 & v_{1,1} & v_{1,2} & \dots & v_{1,d} & v_{1,1}^2 + v_{1,2}^2 + \dots + v_{1,d}^2 \\ 1 & v_{2,1} & v_{2,2} & \dots & v_{2,d} & v_{2,1}^2 + v_{2,2}^2 + \dots + v_{2,d}^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & v_{d+2,1} & v_{d+2,2} & \dots & v_{d+2,d} & v_{d+2,1}^2 + v_{d+2,2}^2 + \dots + v_{d+2,d}^2 \end{vmatrix}.$$

Note that $D_S(v_1, \dots, v_{d+2}) = 0$ if and only if $v_1, \dots, v_{d+2}$ lie on a common sphere (compare Section 1.2). As in the last section we will use the determinant

$$D_H(v_1, \dots, v_{d+1}) := \begin{vmatrix} 1 & v_{1,1} & v_{1,2} & \dots & v_{1,d} \\ 1 & v_{2,1} & v_{2,2} & \dots & v_{2,d} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 1 & v_{d+1,1} & v_{d+1,2} & \dots & v_{d+1,d} \end{vmatrix}$$

to test the orientation of the point sequence $v_1, \dots, v_{d+1}$.

We are now prepared to formulate the problem. We are looking for a map f from $[n]^d$ to some larger grid $[M]^d$ with the following two properties:

(i) The set $f([n]^d)$ contains no $d+1$ distinct points on a common hyperplane and no $d+2$ distinct points on a common sphere.

(ii) The map f preserves the orientation of lattice simplices (see Section 1.2). Furthermore, f preserves the insphere predicate in the following sense. Suppose we are given $d+2$ points $v_1, \dots, v_{d+2} \in [n]^d$ not on a common sphere, then the sign of $D_S(f(v_1), \dots, f(v_{d+2}))$ is the same as for $D_S(v_1, \dots, v_{d+2})$.

Denote with $M_d(n)$ the smallest value of M admitting such a map. Clearly, $M_d(n) \geq n^d/d$ since there are at most $d \cdot M$ points in the $[M]^d$-grid with no $d+1$ points on a hyperplane.

Let $p \geq (16\, n^d/3)^{d-1}$ be a prime and define $\lambda := (d\, 2^d - 1)(d+1)!\, n^{d+1}\, p$. Consider the map $f := [n]^d \to [\lambda n + p]^d$ given by

$$\begin{aligned} f(x) &:= \lambda x + \Delta(x) \\ \Delta(x) &:= (z(x), z^2(x) \bmod p, \dots, z^d(x) \bmod p) \\ z(x) &:= \sum_{k=1}^{d} x_k\, n^{k-1} \qquad \text{for } x = (x_1, \dots, x_d) \end{aligned}$$

This construction is the same as in the last section with the difference that p is chosen much larger with respect to n and λ is chosen much larger with respect to n and p. The i-th row of the determinant $D_S(f(v_1), \ldots, f(v_{d+2}))$ is given by

$$1, \lambda v_{i,1} + z(v_i), \ldots, \lambda v_{i,d} + (z^d(v_i) \bmod p), (\lambda v_{i,1} + z(v_i))^2 + \ldots + (\lambda v_{i,d} + (z^d(v_i) \bmod p))^2 .$$

Lemma 1.11 *The map f satisfies the properties (i) and (ii).*

Proof. Consider condition (i) first. Let $v_1, v_2, \ldots, v_{d+2}$ be given distinct points from $[n]^d$. We will show $D_H(f(v_1), \ldots, f(v_{d+1})) \not\equiv 0 \pmod p$ and $D_S(f(v_1), \ldots, f(v_{d+2})) \not\equiv 0 \pmod p$ which is sufficient. We have

$$D_S(f(v_1), \ldots, f(v_{d+2})) \equiv D_S(\Delta(v_1), \ldots, \Delta(v_{d+2})) \pmod p$$

and

$$D_H(f(v_1), \ldots, f(v_{d+1})) \equiv D_H(\Delta(v_1), \ldots, \Delta(v_{d+1})) \pmod p$$

since λ is an integer multiple of p. For $x \neq y \in [n]^d$ the choice of p and the definition of the function z implies $0 \leq z(x), z(y) < n^d \leq 3/16\, p^{1/(d-1)}$ and $z(x) \not\equiv z(y) \pmod p$. But this is precisely the situation as in the proof of Lemma 1.5, where we have shown that determinants of the form $D_H(\Delta(v_1), \ldots, \Delta(v_{d+1}))$ and $D_S(\Delta(v_1), \ldots, \Delta(v_{d+2}))$ are always nonzero modulo p. Hence,

$$D_H(f(v_1), \ldots, f(v_{d+1})) \equiv D_H(\Delta(v_1), \ldots, \Delta(v_{d+1})) \not\equiv 0 \pmod p$$

and

$$D_S(f(v_1), \ldots, f(v_{d+2})) \equiv D_S(\Delta(v_1), \ldots, \Delta(v_{d+2})) \not\equiv 0 \pmod p .$$

Now we consider condition (ii). It follows from the proof of Lemma 1.10 that f preserves the orientation of lattice simplices. So let $v_1, v_2, \ldots, v_{d+2} \in [n]^d$ be given with $D_S(v_1, \ldots, v_{d+2}) \neq 0$. We assume w.l.o.g. that $D_S(v_1, \ldots, v_{d+2}) > 0$, which implies $D_S(v_1, \ldots, v_{d+2}) \geq 1$. By using the fact that a determinant is linear in each column we expand $D_S(f(v_1), \ldots, f(v_{d+2}))$ to

$$D_S(f(v_1), \ldots, f(v_{d+2}) = \lambda^{d+2}\, D_S(v_1, \ldots, v_{d+2}) + \sigma \geq \lambda^{d+2} + \sigma ,$$

where σ is a sum of $d\,2^d - 1$ terms of the form λ^k times a determinant with $0 \leq k \leq d+1$. Furthermore, the absolute value of σ is bounded by

$$|\sigma| < (d\,2^d - 1)\,(d+1)!\,(\lambda n)^{d+1} p .$$

Thus, we infer that

$$D_S(f(v_1), \ldots, f(v_{d+2})) > \lambda^{d+2} - (d\,2^d - 1)\,(d+1)!\,(\lambda n)^{d+1} p = 0$$

by the choice $\lambda := (d\,2^d - 1)(d+1)!\, n^{d+1}\, p$, which concludes the proof. ❑

By taking p to be the smallest prime number exceeding $(16\, n^d/3)^{d-1}$ we get the following

Corollary 1.12 *For $d \geq 2$ and $n \in \mathbb{N}$ we have $M_d(n) \leq c_d\, n^{d^2+2}$.*

Chapter 2

Independent sets

In 1940 P. Turán [69] proved his celebrated theorem that gives a complete answer to the following question: How many edges can a graph on n vertices have without containing a complete graph on r vertices? This was the starting point of extremal graph theory, a discipline that attracted many mathematicians and that is still an active area of research. The complementary version of Turán's question is: How many edges must a graph on n vertices have without containing an independent set of size r? Turán's Theorem then provides us with a lower bound on the independence number of a graph on n vertices depending on the number of edges. This bound can be refined by considering the vertex degrees instead of the number of edges. Furthermore, these bounds can be extended to uniform hypergraphs. For the most general case, namely, arbitrary hypergraphs we will derive a new result that is a generalization of the results for graphs and uniform hypergraphs. All these bounds are essentially sharp. On the other hand they can be improved for graphs and hypergraphs that are "uncrowded".

In Chapters 3 and 4 we will see that many selection problems can be formulated in terms of the independence number of certain hypergraphs. The lower bounds on the independence number given in this chapter then provide us with lower bounds for these selection problems. Thus, Turán's Theorem not only was the starting point of extremal graph theory but also is a key to the solution of many extremal problems in areas like number theory, geometry and combinatorics.

2.1 Turán's Theorem and independence number of graphs

We will formulate Turán's famous theorem in a form that is complementary (but equivalent) to its original version (see also [14]). Let $S_{n,r}$ be the graph consisting of the union of r disjoint cliques of size $n_i = \lfloor \frac{n+i}{r} \rfloor$ for $i = 0, 1, \ldots, r-1$, i.e. the cliques are as equal as possible. Let $s_{n,r}$ denote the number of edges of $S_{n,r}$, thus $s_{n,r} = \sum \binom{n_i}{2}$. Observe that the independence number of $S_{n,r}$ is $\alpha(S_{n,r}) = r$ since an independent set contains at most 1 vertex from each clique. However, choosing 1 point per clique yields an independent set.

Theorem 2.1 (Turán [69]) *Let $n \geq r \geq 1$ be given integers and let G be a graph on n vertices with m edges and independence number $\alpha(G) = r$. Then, $m \geq s_{n,r}$ with equality if, and only if, G is isomorphic to $S_{n,r}$.*

Turán's Theorem implies a lower bound on the independence number in terms of the number of edges, and this bound is best possible.

Corollary 2.2 *Let G be a graph on n vertices and m edges. Then,*

$$\alpha(G) \geq \frac{n^2}{2m+n}. \text{ [1]}$$

Furthermore, equality holds if, and only if, G is the union of disjoint cliques of equal cardinality.

Proof. Let G be a graph on n vertices, m edges and independence number $r := \alpha(G)$. By Jensen's inequality[2],

$$s_{n,r} = \sum_{i=0}^{r-1} \binom{\lfloor \frac{n+i}{r} \rfloor}{2} \geq r \binom{\frac{n}{r}}{2} = \frac{n^2 - rn}{2r}.$$

Turán's Theorem implies $m \geq s_{n,r} \geq \frac{n^2-rn}{2r}$ and thus $\alpha(G) \geq \frac{n^2}{2m+n}$. Furthermore, equality holds if, and only if, G is isomorphic to $S_{n,r}$ and r is a divisor of n. ❏

The following theorem of Erdős is an extension of Turán's Theorem (for those not familiar with the hungarian language, see [15]).

Theorem 2.3 (Erdős [30]) *Let $n \geq r \geq 1$ be given integers. Suppose, $G = (V, E)$ is a graph on n vertices with $\alpha(G) \leq r$. Then there exists a graph H with vertex set V that is the union of r disjoint cliques such that $d_H(v) \leq d_G(v)$ for all $v \in V$. Moreover, if equality holds for all vertices $v \in V$ then G is isomorphic to H.*

Erdős' Theorem leads to the following lower bound for the independence number of a graph in terms of the degrees that is an improvement of Corollary 2.2. This bound was proved independently by Wei [71] and Caro [17] without using Theorem 2.3. Also, an elegant probabilistic proof is given in [8].

Theorem 2.4 (Wei/Caro [71, 17]) *Let $G = (V, E)$ be a graph. Then,*

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v)+1}.$$

[1] For an efficient parallel algorithm for computing independent sets of this size see [42]

[2] $\phi(\sum \lambda_i x_i) \leq \sum \lambda_i \phi(x_i)$ for ϕ convex, $\sum \lambda_i = 1$, $\lambda_i \geq 0$.

By Jensen's inequality this implies $\alpha(G) \geq \frac{n}{\overline{d}+1} = \frac{n^2}{2m+n}$, where $\overline{d} = \frac{2m}{n}$ is the average degree of G.

Proof. Let $r := \alpha(G)$. By Theorem 2.3, there is a graph H with vertex set V that is the union of r disjoint cliques such that $d_H(v) \leq d_G(v)$ for all $v \in V$. This implies $\sum_{v\in V} \frac{1}{d_G(v)+1} \leq \sum_{v\in V} \frac{1}{d_H(v)+1}$. Each clique of H contributes 1 to the second sum, thus

$$\sum_{v\in V} \frac{1}{d_G(v)+1} \leq \sum_{v\in V} \frac{1}{d_H(v)+1} = r = \alpha(G)\,.$$

❏

Theorem 2.3 tells us even more.

Observation 2.5 *Let $G = (V, E)$ be a graph. Then $\alpha(G) = \sum_{v\in V} \frac{1}{d(v)+1}$ if, and only if, G is the union of disjoint cliques.*

Proof. If $G = (V, E)$ is the union of r disjoint cliques, then $\alpha(G) = r = \sum_{v\in V} \frac{1}{d(v)+1}$ since each clique contributes 1 to the sum.

Now suppose $\alpha(G) = \sum_{v\in V} \frac{1}{d(v)+1}$ and define $r := \alpha(G)$. According to Theorem 2.3 there is a graph H with vertex set V that is the union of r disjoint cliques with $d_H(v) \leq d_G(v)$ for all $v \in V$. Since

$$r = \sum_{v\in V} \frac{1}{d_H(v)+1} \geq \sum_{v\in V} \frac{1}{d_G(v)+1} = \alpha(G) = r$$

we infer $d_H(v) = d_G(v)$ for all $v \in V$. By (the second part of) Theorem 2.3, this happens only if G is isomorphic to H. ❏

This observation tells us that the Wei/Caro bound is tight in terms of the degrees. On the other hand it can be arbitrarily bad: Take a complete bipartite graph $K_{l,l}$ on $n = 2l$ vertices, which has independence number $l = n/2$. The Wei/Caro bound yields only $\frac{2l}{l+1} < 2$. This is not surprising, because a graph is not uniquely determined by its degree sequence. Consider the following graph G: Let A and B be two cliques of size $l \geq 2$ and add l independent edges (matching) between A and B. Then, G has the same degree sequence as $K_{l,l}$ but independence number 2.

We include here a second proof of Theorem 2.4 (see also [44]) that supplies us with an efficient algorithm for computing independent sets of size as guaranteed by the lower bound. Furthermore, this proof illustrates the basic idea of the proof of the extension of the Wei/Caro bound to arbitrary hypergraphs given in Theorem 2.9.

Second proof of Theorem 2.4 and the MAX-algorithm. For a graph G and a vertex $x \in V(G)$ let $G \setminus x$ denote the resulting graph after removing x together with incident edges from G. Consider the following *sequential algorithm* for computing an independent set in a graph:

```
MAX(G):
WHILE E(G) ≠ ∅ DO
    Choose x ∈ V(G) with maximal degree;
    G := G \ x;
END;
Output independent set I = V(G).
```

Clearly, this algorithm computes an independent set for the input graph G. Furthermore, it can be implemented with linear running time $O(|V| + |E|)$.

For convenience, define $F(G) := \sum_{v \in V(G)} \frac{1}{d_G(v)+1}$ for any graph G. Suppose, x is a vertex of maximum degree $d(x) = \Delta(G)$. Let $N(x)$ (neighborhood of x) denote the set of vertices adjacent to x. Then,

$$\begin{aligned} F(G \setminus x) - F(G) &= \sum_{w \in N(x)} \left[\frac{1}{d(w)} - \frac{1}{d(w)+1}\right] - \frac{1}{\Delta + 1} \\ &= \sum_{w \in N(x)} \frac{1}{d(w)(d(w)+1)} - \frac{1}{\Delta+1} \\ &\geq \Delta \frac{1}{\Delta(\Delta+1)} - \frac{1}{\Delta+1} \\ &= 0\,. \end{aligned}$$

So we see that the value of F never decreases during the algorithm. On the other hand, its final value is $F(G[I]) = |I| \leq \alpha(G)$, where $G[I]$ denotes the induced subgraph on I. Thus, $F(G) \leq \alpha(G)$. More precisely, $F(G)$ is a lower bound for the size of the independent set generated by the MAX-algorithm. ❑

The original proof of Wei [71] uses a similar algorithm called MIN: Sucessively delete a vertex v with minimal degree together with its neighborhood and add v to the independent set.

2.2 Independence number of uniform hypergraphs

The Wei/Caro Theorem 2.4 for graphs raises the question if a similar lower bound can be found for the independence number of hypergraphs. Before stating the results we have to make some definitions.

A *hypergraph* is a pair $H = (V, \mathcal{E})$ where V is a finite set and $\mathcal{E}$ is a collection of non-empty subsets of V, i.e. $\mathcal{E} \subseteq 2^V \setminus \{\emptyset\}$. The *rank* r of a hypergraph $H = (V, \mathcal{E})$ is the maximal size of an edge in $\mathcal{E}$. The hypergraph H is k-*uniform* if all edges in $\mathcal{E}$ have size k. Thus, graphs are 2-uniform hypergraphs. The *degree* $d(v)$ of a vertex of a uniform hypergraph is the number of edges containing v. A set $I \subseteq V$ is called *independent* if $2^I \cap \mathcal{E} = \emptyset$, i.e. the set I contains no edge of $\mathcal{E}$. The maximal size of an independent set of H is defined as the *independence number* $\alpha(H)$.

We start with a bound due to Spencer including the nice probabilistic proof.

Theorem 2.6 (Spencer [64]) *Let $H = (V, \mathcal{E})$ be a k-uniform hypergraph on n vertices with average degree $d \geq 1$. Then,*

$$\alpha(H) \geq \left(1 - \frac{1}{k}\right) \cdot n \cdot d^{-\frac{1}{k-1}}\,.$$ [3]

Proof. Set $p := d^{-1/(k-1)}$. Since $d \geq 1$ we have $p \in [0, 1]$. Choose a subset $X \subset V$ by picking each vertex independently with probability p. Let Y be the set of edges from $\mathcal{E}$ contained in X. Deleting one vertex from each edge from Y yields an independent set of size at least $|X| - |Y|$. Since

$$\mathbf{E}(|X| - |Y|) = np - \frac{nd}{k} \cdot p^k = \left(1 - \frac{1}{k}\right) \cdot n \cdot d^{-\frac{1}{k-1}}$$

we conclude that there exists an independent set of size at least as large as claimed. ❑

Caro and Tuza proved the following result, which is an extension of Theorem 2.4.

Theorem 2.7 (Caro and Tuza [18]) *Let $H = (V, \mathcal{E})$ be a k-uniform hypergraph with $k \geq 2$. Then*

$$\alpha(H) \geq \sum_{v \in V} f(d(v))\,,$$

where $d(v)$ is the degree of v, i.e. the number of edges containing v and the function f is given by

$$f(d) := \prod_{i=1}^{d} \left(1 - \frac{1}{i\,(k-1) + 1}\right).$$

In fact, Caro and Tuza proved a lower bound for the l-independence number $\alpha_l(H)$, defined as the maximum size of a subset $I \subset V(H)$ with the property that the induced subhypergraph $H[I]$ has maximum degree less than l. In particular, $\alpha_1(H) = \alpha(H)$. The proof of Theorem 2.7 is similar to the second proof of the Wei/Caro Theorem given in the last section. Moreover, it is shown in [18] that the MAX-*algorithm* can be used to compute an independent set in a k-uniform hypergraph of size at least as large as guaranteed by the lower bound in time $O(k \cdot |\mathcal{E}| + |V|)$.

Remarks. (i) The function f in Theorem 2.7 can be simplified to $f(d) = \binom{d+1/(k-1)}{d}^{-1}$. Thus, we may rewrite the result as

$$\alpha(H) \geq \sum_{v \in V} \binom{d(v) + \frac{1}{k-1}}{d(v)}^{-1}.$$

For $k = 2$ (ordinary graphs) this is the Wei/Caro bound.

(ii) An asymptotic formula is given by $f(d) = (\frac{1}{k-1}! + o(1))\, d^{-1/(k-1)}$, using Euler's formula $\frac{1}{z!} = \lim_{n \to \infty} \binom{n+z}{n} n^{-z}$.

[3]For a parallel algorithm for computing an independent set of size $c \cdot nd^{-1/(k-1)}$ see [5]

(iii) We claim that $f(d)$ is a strictly convex function in d (k fixed). For $x, y > 0$ let

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

be Euler's Betafunction. Then, $f(d) = \frac{1}{k-1} \cdot B(d+1, \frac{1}{k-1})$. It is well known (see for example [12]) that $B(x,y)$ is log convex in each of its arguments. In particular, $B(x,y)$ is strictly convex in x and the claim follows.

The convexity of $f(d) = \binom{d+1/(k-1)}{d}^{-1}$ together with Jensen's inequality yields the following result.

Corollary 2.8 *Let H be a k-uniform hypergraph, $k \geq 2$, with n vertices, m edges and average degree $\bar{d} = \frac{km}{n}$. Then,*

$$\alpha(H) \geq n \cdot \binom{\bar{d} + \frac{1}{k-1}}{\bar{d}}^{-1}.$$

Corollary 2.2 is included as the special case $k = 2$. To compare this bound with Theorem 2.6 for average degree $d \geq 1$ define $f(d) = \binom{d+\frac{1}{k-1}}{d}^{-1}$ and $g(d) = (1 - 1/k) \cdot d^{-1/(k-1)}$. Observe that g satisfies the differential equation

$$g'(d) = -\frac{1}{(k-1) \cdot d} \cdot g(d) \tag{2.1}$$

while f satisfies the (corresponding) difference equation $(k-1)\, d\, [f(d) - f(d-1)] = -f(d)$. Since $f(d)$ is strictly convex we infer $(k-1)\, d\, f'(d) > -f(d)$ and so

$$f'(d) > -\frac{1}{(k-1)\, d} f(d)\,. \tag{2.2}$$

Now equations 2.1 and 2.2 together with the fact that $f(1) = g(1)$ imply $f(d) \geq g(d)$ for $d \geq 1$. In other words, the bound given in Corollary 2.8 is at least as good as the bound given in Theorem 2.6.

2.3 Arbitrary hypergraphs

In this section we will derive a new result that is a generalization of the theorem of Wei/Caro (Theorem 2.4) and the theorem of Caro and Tuza (Theorem 2.7) to arbitrary (non-uniform) hypergraphs. In order to do this we have to generalize the concept of the degree of a vertex. Let $H = (V, \mathcal{E})$ be a hypergraph of rank r. For every vertex $v \in V$ define the *degree vector* $\mathbf{d}(v) = (d_1(v), d_2(v), \ldots, d_r(v)) \in \mathbb{N}_0^r$ where $d_m(v)$ is the number of edges of size m containing v for $1 \leq m \leq r$.

Definition. Let $r \geq 1$ be an integer. Define the function $f_r : \mathbb{N}_0^r \to \mathbb{R}$ by

$$f_r(\mathbf{d}) = \sum_{\mathbf{i} \in \mathbb{N}_0^r} \left[\prod \binom{d_m}{i_m} \right] \frac{(-1)^{\sum i_m}}{\sum (m-1) \cdot i_m + 1} .$$

The product and the inner sums are taken over all $1 \leq m \leq r$. Note that the outer sum is finite since all summands are zero unless $\mathbf{i} \in [\mathbf{0}, \mathbf{d}] := \{\mathbf{j} \in \mathbb{N}_0^r : 0 \leq j_m \leq d_m \text{ for all } 1 \leq m \leq r\}$. Now we are in the position to state our main theorem.

Theorem 2.9 *Let $H = (V, \mathcal{E})$ be a hypergraph of rank r. Then*

$$\alpha(H) \geq \sum_{v \in V} f_r(\mathbf{d}(v)) .$$

Suppose $H = (V, \mathcal{E})$ is k-uniform, $k \geq 2$. Let $v \in V$ be arbitrary and let $\mathbf{e}_k$ denote the k-th unit vector. Since H is k-uniform $f_k(\mathbf{d}(v))$ reduces to

$$f_k(\mathbf{d}(v)) = f_k(d_k(v) \cdot \mathbf{e}_k) = \sum_{i=0}^{d_k(v)} \binom{d_k(v)}{i} \frac{(-1)^i}{(k-1) \cdot i + 1} = \binom{d_k(v) + \frac{1}{k-1}}{d_k(v)}^{-1}$$

(see Concrete Mathematics [43] p. 188). Thus the theorem is a generalization of the results of Wei/Caro and Caro/Tuza. Let us also consider the case $k = 1$, i.e. H is 1-uniform. Then $f_1(\mathbf{d}(v))$ reduces to

$$f_1(\mathbf{d}(v)) = f_1(d_1(v)) = \sum_{i=0}^{d_1(v)} \binom{d_1(v)}{i} (-1)^i = \begin{cases} 1 \text{ if } d_1(v) = 0 \\ 0 \text{ if } d_1(v) \geq 1 \end{cases} .$$

This is what we expect: The unique maximum independent set is given by the set of isolated vertices.

Observation 2.10 *Let $H = (V, \mathcal{E})$ be a matching of rank r, i.e. H is a hypergraph with the property $e \neq e' \in \mathcal{E} \Rightarrow e \cap e' = \emptyset$. Then,*

$$\alpha(H) = \sum_{v \in V} f_r(\mathbf{d}(v)) .$$

Proof. Since H is a matching the independence number of H is given by

$$\alpha(H) = \#\text{vertices of degree vector zero} + \sum_{e \in \mathcal{E}} (|e| - 1) .$$

On the other hand $f_r(\mathbf{0}) = 1$ and for every edge $e \in \mathcal{E}$ we have $\sum_{v \in e} f_r(\mathbf{d}(v)) = |e|(1 - 1/|e|) = |e| - 1$. Thus, $\alpha(H) = \sum_{v \in V} f_r(\mathbf{d}(v))$. ❏

In Section 2.1 we saw that the Wei/Caro bound is tight and the extremal graphs are given by unions of cliques. It is natural to ask how the bound $F(H) := \sum_v f_r(\mathbf{d}(v))$ given in Theorem 2.9 behaves for the union of uniform cliques, i.e. a hypergraph that is the union of disjoint hypergraphs H_j, each of which is a complete k_j-uniform hypergraph for some k_j. To this end, let H be a complete k-uniform hypergraph on $n \geq k$ vertices. Clearly, $\alpha(H) = k-1$ and each vertex has degree $d := \binom{n-1}{k-1} \geq 1$. By the inequality that we proved in the remark after Corollary 2.8,

$$F(H) = n \cdot \binom{d + \frac{1}{k-1}}{d}^{-1} \geq n \cdot \left(1 - \frac{1}{k}\right) d^{-\frac{1}{k-1}}.$$

On the other hand,

$$d^{-\frac{1}{k-1}} = \binom{n-1}{k-1}^{-\frac{1}{k-1}} \geq \frac{k-1}{en}.$$

We infer,

$$F(H) \geq \left(1 - \frac{1}{k}\right) e^{-1}(k-1) = \left(1 - \frac{1}{k}\right) e^{-1} \alpha(H).$$

Note that $F(H) = \alpha(H)$ for $k = 2$ by Observation 2.5. Hence, $F(H) \geq \frac{2}{3e}\alpha(H)$. This bound obviously holds also when $n < k$.

Observation 2.11 *Let H be the union of disjoint uniform cliques. Then $F(H) \geq \frac{2}{3e}\alpha(H)$.*

Proof. Let $H = H_1 + H_2 + \ldots + H_t$ be the union of disjoint uniform cliques. Then, $\alpha(H) = \alpha(H_1) + \ldots + \alpha(H_t)$ and $F(H) = F(H_1) + \ldots + F(H_t)$. For each $j = 1 \ldots t$, $F(H_j) \geq \frac{2}{3e}\alpha(H_j)$. Thus, $F(H) \geq \frac{2}{3e}\alpha(H)$. ❑

This shows that the bound $F(H)$ is tight up to a constant factor.

Lemma 2.12 *Let $r \in \mathbb{N}$, $C_1, C_2, \ldots, C_r \geq 0$ and $C_0 > 0$ be given. The function $g : \mathbb{N}_0^r \to \mathbb{R}$ given by*

$$g(\mathbf{d}) = \sum_{\mathbf{i}} \left[\prod \binom{d_m}{i_m}\right] \frac{(-1)^{\sum i_m}}{\sum C_m i_m + C_0}$$

is the solution of the recurrence

$$g(\mathbf{d}) = \frac{\sum_k C_k d_k g(\mathbf{d} - \mathbf{e}_k)}{\sum_k C_k d_k + C_0}$$

with $g(\mathbf{0}) = C_0^{-1}$. In particular, $g(\mathbf{d})$ is non-negative for all $\mathbf{d} \in \mathbb{N}_0^r$.

By this lemma we infer that our function f satisfies the recurrence

$$f(\mathbf{d}) = \frac{\sum (k-1) \cdot d_k f(\mathbf{d} - \mathbf{e}_k)}{\sum (k-1) \cdot d_k + 1}$$

with $f(\mathbf{0}) = 1$. In particular, $0 \leq f(\mathbf{d}) \leq 1$ for all $\mathbf{d} \in \mathbb{N}_0^r$. For later purposes we need the following equivalent partial difference equation for f

$$f(\mathbf{d}) = \sum_m (m-1) \cdot d_m \left[f(\mathbf{d} - \mathbf{e}_m) - f(\mathbf{d})\right] \tag{2.3}$$

for $\mathbf{d} \neq \mathbf{0}$.

For convenience let us define the function $F(H) := \sum_{v \in V} f(\mathbf{d}(v))$ for every hypergraph $H = (V, \mathcal{E})$, where $f = f_r$ and $r = \text{rank}(H)$. Suppose x is a vertex of H. Let $H \setminus x$ denote the resulting hypergraph after removing x together with all incident edges from H. The key to the proof of our main theorem is

Lemma 2.13 *Let $H = (V, \mathcal{E})$ be a hypergraph with $\mathcal{E} \neq \emptyset$. Then there exists a vertex $x \in V$ with $F(H \setminus x) \geq F(H)$.*

The main work will be the proof of this lemma.

Proof of Theorem 2.9. Lemma 2.13 enables us to use the following algorithm to find an independent set I in H.

```
WHILE E(H) ≠ ∅ DO
      Choose x ∈ V(H) with F(H\x) ≥ F(H);
      H := H \ x;
END;
Output independent set I = V(H).
```

Since $f(\mathbf{0}) = 1$ we know that $F(H[I]) = |I|$. The value of F never decreases by the choice of the deleted vertices. Thus, $F(H) \leq F(H[I]) = |I| \leq \alpha(H)$. ❑

We remark that the proof implies a polynomial *algorithm* that computes an independent set of size at least $F(H)$ in an arbitrary hypergraph H of constant rank. In particular, for uniform hypergraphs, this is the MAX-algorithm (see also [18, 44]): Successively remove vertices of maximum degree with all incident edges until no edges are left. It is easy to see that a vertex x with maximum degree in a uniform hypergraph has always the property $F(H \setminus x) \geq F(H)$.

2.4 Proof of lemmata

We start with the proof of Lemma 2.12.

Proof of Lemma 2.12. Let $r \in \mathbb{N}$, $C_1, C_2, \ldots, C_r \geq 0$ and $C_0 > 0$ be given. We have to show that the function $g : \mathbb{N}_0^r \to \mathbb{R}$ given by

$$g(\mathbf{d}) = \sum_{\mathbf{i}} \left[\prod \binom{d_m}{i_m} \right] \frac{(-1)^{\sum i_m}}{\sum C_m i_m + C_0}$$

satisfies the recurrence

$$g(\mathbf{d}) = \frac{\sum C_k d_k g(\mathbf{d} - \mathbf{e}_k)}{\sum C_k d_k + C_0}$$

with $g(\mathbf{0}) = C_0^{-1}$. It is easy to check that $g(\mathbf{0}) = C_0^{-1}$ holds.

Let us rewrite the recurrence as a partial difference equation

$$C_0 g(\mathbf{d}) = \sum_k C_k d_k [g(\mathbf{d} - \mathbf{e}_k) - g(\mathbf{d})]$$

for $\mathbf{d} \neq \mathbf{0}$. Suppose $d_k > 0$ then we have

$$\begin{aligned} g(\mathbf{d} - \mathbf{e}_k) - g(\mathbf{d}) &= -\sum_{\mathbf{i}} \binom{d_k - 1}{i_k - 1} \prod_{m \neq k} \binom{d_m}{i_m} \frac{(-1)^{\sum i_m}}{\sum C_m i_m + C_0} \\ &= -\frac{1}{d_k} \sum_{\mathbf{i}} \prod \binom{d_m}{i_m} (-1)^{\sum i_m} \frac{i_k}{\sum C_m i_m + C_0} . \end{aligned}$$

Hence,

$$C_k d_k [g(\mathbf{d} - \mathbf{e}_k) - g(\mathbf{d})] = -\sum_{\mathbf{i}} \prod \binom{d_m}{i_m} (-1)^{\sum i_m} \frac{C_k i_k}{\sum C_m i_m + C_0}$$

and therefore

$$\begin{aligned} \sum_k C_k d_k [g(\mathbf{d} - \mathbf{e}_k) - g(\mathbf{d})] &= -\sum_{\mathbf{i}} \prod \binom{d_m}{i_m} (-1)^{\sum i_m} \frac{\sum_k C_k i_k}{\sum C_m i_m + C_0} \\ &= -\sum_{\mathbf{i}} \prod \binom{d_m}{i_m} (-1)^{\sum i_m} \left(1 - \frac{C_0}{\sum C_m i_m + C_0}\right) \\ &= \underbrace{-\sum_{\mathbf{i}} \prod \binom{d_m}{i_m} (-1)^{\sum i_m}}_{=0 \text{ for } \mathbf{d} \neq \mathbf{0}} \\ &\quad + C_0 \underbrace{\sum_{\mathbf{i}} \prod \binom{d_m}{i_m} \frac{(-1)^{\sum i_m}}{\sum C_m i_m + C_0}}_{=g(\mathbf{d})} \\ &= C_0 g(\mathbf{d}) \end{aligned}$$

as desired. ❑

For the proof of Lemma 2.13 we need two additional lemmata.

Lemma 2.14 *For $r \in \mathbb{N}$, $1 \leq k, l \leq r$ and $\mathbf{d} \in \mathbb{N}_0^r$ with $d_k \geq 1$ we have*

$$f(\mathbf{d} - \mathbf{e}_k) - f(\mathbf{d}) \geq f((\mathbf{d} + \mathbf{e}_l) - \mathbf{e}_k) - f(\mathbf{d} + \mathbf{e}_l) .$$

Proof. We will show that

$$[f(\mathbf{d} - \mathbf{e}_k) - f(\mathbf{d})] - [f((\mathbf{d} + \mathbf{e}_l) - \mathbf{e}_k) - f(\mathbf{d} + \mathbf{e}_l)] \geq 0 .$$

Consider the case $k \neq l$ first.

$$\begin{aligned} f(\mathbf{d}-\mathbf{e}_k) - f(\mathbf{d}) &= \sum_{\mathbf{i}} \binom{d_k - 1}{i_k} \prod_{m \neq k} \binom{d_m}{i_m} \frac{(-1)^{\sum i_m}}{\sum (m-1)\, i_m + 1} \\ &\quad - \sum_{\mathbf{i}} \binom{d_k}{i_k} \prod_{m \neq k} \binom{d_m}{i_m} \frac{(-1)^{\sum i_m}}{\sum (m-1)\, i_m + 1} \\ &= -\sum_{\mathbf{i}} \binom{d_k - 1}{i_k - 1} \prod_{m \neq k} \binom{d_m}{i_m} \frac{(-1)^{\sum i_m}}{\sum (m-1)\, i_m + 1} \,. \end{aligned}$$

Similarly

$$f((\mathbf{d}+\mathbf{e}_l)-\mathbf{e}_k) - f(\mathbf{d}+\mathbf{e}_l) = -\sum_{\mathbf{i}} \binom{d_k - 1}{i_k - 1} \binom{d_l + 1}{i_l} \prod_{m \neq k,l} \binom{d_m}{i_m} \frac{(-1)^{\sum i_m}}{\sum (m-1)\, i_m + 1} \,.$$

Putting this together yields

$$\begin{aligned} [f(\mathbf{d}-\mathbf{e}_k) - f(\mathbf{d})] &- [f((\mathbf{d}+\mathbf{e}_l)-\mathbf{e}_k) - f(\mathbf{d}+\mathbf{e}_l)] \\ &= \sum_{\mathbf{i}} \binom{d_k - 1}{i_k - 1} \binom{d_l}{i_l - 1} \prod_{m \neq k,l} \binom{d_m}{i_m} \frac{(-1)^{\sum i_m}}{\sum (m-1)\, i_m + 1} \\ &= \sum_{\mathbf{i}} \binom{d_k - 1}{i_k} \prod_{m \neq k} \binom{d_m}{i_m} \frac{(-1)^{\sum i_m}}{\sum (m-1)\, i_m + \underbrace{[(k-1)+(l-1)+1]}_{=:C_0>0}} \\ &= g(\mathbf{d}-\mathbf{e}_k) \,, \end{aligned}$$

where g is given by the recurrence

$$g(\mathbf{d}) = \frac{\sum (m-1) \cdot d_m g(\mathbf{d}-\mathbf{e}_m)}{\sum (m-1) \cdot d_m + C_0}$$

with $g(\mathbf{0}) = C_0^{-1} > 0$ according to Lemma 2.12. In particular, $g(\mathbf{d}-\mathbf{e}_k)$ is non-negative which proofs the claim for $k \neq l$.

Now let $k = l$. We have to prove that $[f(\mathbf{d}-\mathbf{e}_k) - f(\mathbf{d})] - [f(\mathbf{d}) - f(\mathbf{d}+\mathbf{e}_k)] \geq 0$. Consider again

$$f(\mathbf{d}-\mathbf{e}_k) - f(\mathbf{d}) = -\sum_{\mathbf{i}} \binom{d_k - 1}{i_k - 1} \prod_{m \neq k} \binom{d_m}{i_m} \frac{(-1)^{\sum i_m}}{\sum (m-1)\, i_m + 1}$$

and similarly

$$f(\mathbf{d}) - f(\mathbf{d}+\mathbf{e}_k) = -\sum_{\mathbf{i}} \binom{d_k}{i_k - 1} \prod_{m \neq k} \binom{d_m}{i_m} \frac{(-1)^{\sum i_m}}{\sum (m-1)\, i_m + 1} \,.$$

We infer that

$$\begin{aligned}[f(\mathbf{d}-\mathbf{e}_k)-f(\mathbf{d})] &- [f(\mathbf{d})-f(\mathbf{d}+\mathbf{e}_k)] \\ &= \sum_{\mathbf{i}} \binom{d_k-1}{i_k-2} \prod_{m\neq k} \binom{d_m}{i_m} \frac{(-1)^{\sum i_m}}{\sum(m-1)\,i_m+1} \\ &= \sum_{\mathbf{i}} \binom{d_k-1}{i_k} \prod_{m\neq k} \binom{d_m}{i_m} \frac{(-1)^{\sum i_m}}{\sum(m-1)\,i_m+\underbrace{[2\,(k-1)+1]}_{=:C_0>0}} \\ &= g(\mathbf{d}-\mathbf{e}_k)\ ,\end{aligned}$$

where g is again given by the recurrence

$$g(\mathbf{d}) = \frac{\sum(m-1)\cdot d_m g(\mathbf{d}-\mathbf{e}_m)}{\sum(m-1)\cdot d_m + C_0}$$

with $g(\mathbf{0}) = C_0^{-1} > 0$ according to Lemma 2.12. In particular, $g(\mathbf{d}-\mathbf{e}_k)$ is non-negative and the claim follows also for $k = l$. ❏

Remark. Lemma 2.14 tells us that for any $\mathbf{d}$ and k the difference $f(\mathbf{d}-\mathbf{e}_k)-f(\mathbf{d})$ decreases whenever we increase any component of $\mathbf{d}$. This is essential for the proof of Lemma 2.15.

Lemma 2.15 *Let $r \in \mathbb{N}$, $\mathbf{d} \in \mathbb{N}_0^r$ and $\Delta \in [\mathbf{0}, \mathbf{d}]$ be given. Then*

$$f(\mathbf{d}-\Delta) - f(\mathbf{d}) \geq \sum_{m=1}^{r} \Delta_m \cdot [f(\mathbf{d}-\mathbf{e}_m) - f(\mathbf{d})]\ .$$

Proof. Let $r \in \mathbb{N}$, $\mathbf{d} \in \mathbb{N}_0^r$ and $\Delta \in [\mathbf{0}, \mathbf{d}]$ be given. Consider the points $(\mathbf{d}-\Delta)$ and $\mathbf{d}$ on the $\mathbb{N}_0^r$ grid. A *monotonical path* between these points is a sequence of grid points starting with $(\mathbf{d}-\Delta)$ and terminating with $\mathbf{d}$ where two neighboring points are of the form $(\mathbf{d}'-\mathbf{e}_m)$, $\mathbf{d}'$ for some $1 \leq m \leq r$. Each monotonical path between $(\mathbf{d}-\Delta)$ and $\mathbf{d}$ has length $\sigma := \sum \Delta_m$ and the number of such paths is given by the multinomial coefficient $\binom{\sum \Delta_m}{\Delta_1, \ldots, \Delta_r}$. Now let $P = \mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_\sigma$ be such a monotonical path, $\mathbf{p}_0 = \mathbf{d}-\Delta$ and $\mathbf{p}_\sigma = \mathbf{d}$. According to this path we rewrite $f(\mathbf{d}-\Delta)-f(\mathbf{d})$ as the telescoping sum

$$f(\mathbf{d}-\Delta) - f(\mathbf{d}) = \sum_{j=1}^{\sigma} [f(\mathbf{p}_{j-1}) - f(\mathbf{p}_j)]\ .$$

Note that all differences have the form $f(\mathbf{d}'-\mathbf{e}_m) - f(\mathbf{d}')$ for some $1 \leq m \leq r$ and $\mathbf{d}' \in [\mathbf{d}-\Delta+\mathbf{e}_m, \mathbf{d}]$.

For each $1 \leq m \leq r$ there are exactly Δ_m differences of the form $f(\mathbf{d}'-\mathbf{e}_m) - f(\mathbf{d}')$ in the telescoping sum since P is monotonic. By Lemma 2.14 we see that each such difference satisfies

$$f(\mathbf{d}'-\mathbf{e}_m) - f(\mathbf{d}') \geq f(\mathbf{d}-\mathbf{e}_m) - f(\mathbf{d})\ .$$

Thus we can estimate

$$f(\mathbf{d}-\Delta)-f(\mathbf{d})\geq\sum_m \Delta_m\left[f(\mathbf{d}-\mathbf{e}_m)-f(\mathbf{d})\right].$$

❏

We are now prepared for the proof of Lemma 2.13.

Proof of Lemma 2.13. Let $H=(V,\mathcal{E})$ be a hypergraph of rank r with $\mathcal{E}\neq\emptyset$. Define V^* to be the set of all non-isolated vertices, i.e. vertices x with $\mathbf{d}(x)\neq\mathbf{0}$. By assumption, $V^*\neq\emptyset$. Furthermore, for two distinct vertices $x,w\in V$ the *co-degree vector* is given by $\mathbf{d}(x,w)=(d_1(x,w),d_2(x,w),\dots,d_r(x,w))\in\mathbb{N}_0^r$, where $d_m(x,w)$ is the number of edges of size m containing both x and w. Set $\mathbf{d}(w,w):=0$. Now let $x\in V^*$ be arbitrary, then

$$F(H\setminus x)-F(H)=\sum_{w\in V^*}\left[f(\mathbf{d}(w)-\mathbf{d}(x,w))-f(\mathbf{d}(w))\right]-f(\mathbf{d}(x)).$$

Consider one summand. Lemma 2.15 implies

$$\left[f(\mathbf{d}(w)-\mathbf{d}(x,w))-f(\mathbf{d}(w))\right]\geq\sum_m d_m(x,w)\cdot\left[f(\mathbf{d}(w)-\mathbf{e}_m)-f(\mathbf{d}(w))\right].$$

Thus

$$F(H\setminus x)-F(H)\geq\sum_{w\in V^*}\sum_m d_m(x,w)\cdot\left[f(\mathbf{d}(w)-\mathbf{e}_m)-f(\mathbf{d}(w))\right]-f(\mathbf{d}(x)).$$

We sum these differences up over all $x\in V^*$:

$$\begin{aligned}
\sum_{x\in V^*}[F(H\setminus x) &- F(H)]\\
&\geq \sum_{x\in V^*}\sum_{w\in V^*}\sum_m d_m(x,w)\cdot\left[f(\mathbf{d}(w)-\mathbf{e}_m)-f(\mathbf{d}(w))\right]-\sum_{x\in V^*}f(\mathbf{d}(x))\\
&= \sum_m\sum_{x\in V^*}\sum_{w\in V^*} d_m(x,w)\cdot\left[f(\mathbf{d}(w)-\mathbf{e}_m)-f(\mathbf{d}(w))\right]-\sum_{x\in V^*}f(\mathbf{d}(x))\\
&= \sum_m\sum_{w\in V^*}\left(\sum_{x\in V^*} d_m(x,w)\right)\left[f(\mathbf{d}(w)-\mathbf{e}_m)-f(\mathbf{d}(w))\right]-\sum_{x\in V^*}f(\mathbf{d}(x))\\
&= \sum_m\sum_{w\in V^*}(m-1)\cdot d_m(w)\left[f(\mathbf{d}(w)-\mathbf{e}_m)-f(\mathbf{d}(w))\right]-\sum_{x\in V^*}f(\mathbf{d}(x))\\
&= \sum_{x\in V^*}\left(\sum_m(m-1)\cdot d_m(x)\left[f(\mathbf{d}(x)-\mathbf{e}_m)-f(\mathbf{d}(x))\right]-f(\mathbf{d}(x))\right)\\
&= 0\,.
\end{aligned}$$

There we made use of the following observation

$$\sum_{x\in V^*}d_m(x,w)=(m-1)\cdot d_m(w)$$

and the fact that $f(\mathbf{d})$ satisfies the partial difference equation (2.3) for $\mathbf{d}\neq\mathbf{0}$. By definition, $\mathbf{d}(x)\neq\mathbf{0}$ for all $x\in V^*$.

We infer that for a random $x\in V^*$ the expectation of $F(H\setminus x)-F(H)$ is non-negative. Thus there exists an $x\in V^*\subset V$ with $F(H\setminus x)\geq F(H)$. ❏

2.5 A probabilistic bound

In this section we will give a lower bound on the independence number of hypergraphs that is very useful for deriving asymptotic lower bounds for several selection problems (as we will see in Chapter 3).

Lemma 2.16 *Let $t \geq 1$ be a given integer. Suppose $H = (V, \mathcal{E})$ is a hypergraph on n vertices with edge set $\mathcal{E} = \mathcal{E}_1 \cup \ldots \cup \mathcal{E}_t$, where each edge in $\mathcal{E}_j$ has size $k_j \geq 2$ for $1 \leq j \leq t$. Then,*

$$\alpha(H) \geq \frac{n}{2} \cdot \min\left\{1, \min_j \left(\frac{n}{2t|\mathcal{E}_j|}\right)^{\frac{1}{k_j-1}}\right\}.$$

The proof is similar to the proof of Theorem 2.6.

Proof. Set with foresight

$$p := \min\left\{1, \min_j \left(\frac{n}{2t|\mathcal{E}_j|}\right)^{\frac{1}{k_j-1}}\right\}.$$

Choose a random subset $X \subseteq V$ by picking each vertex independently with probability p. Let Y denote the set of edges contained in X and y_j denote the number of edges from $\mathcal{E}_j$ contained in X. Thus, $|Y| = y_1 + \ldots + y_t$. Then,

$$\begin{aligned} \mathbf{E}(|X|) &= pn \\ \mathbf{E}(|Y|) &= \mathbf{E}(y_1) + \ldots + \mathbf{E}(y_t) \\ &= p^{k_1}|\mathcal{E}_1| + \ldots + p^{k_t}|\mathcal{E}_t|. \end{aligned}$$

By the choice of p we infer $p^{k_j-1}|\mathcal{E}_j| \leq \frac{n}{2t}$. Thus, $\mathbf{E}(y_k) = p^{k_j}|\mathcal{E}_j| \leq \frac{pn}{2t}$. Summation over all $1 \leq j \leq t$ gives $\mathbf{E}(|Y|) \leq t \cdot \frac{pn}{2t} = \frac{pn}{2}$. If we delete a vertex from each edge in Y we get an independent set of expected size at least

$$\mathbf{E}(|X| - |Y|) \geq \frac{pn}{2} = \frac{n}{2} \cdot \min\left\{1, \min_j \left(\frac{n}{2t|\mathcal{E}_j|}\right)^{\frac{1}{k_j-1}}\right\}.$$

This implies the existence of an independent set of the desired size. ❏

2.6 Uncrowded hypergraphs

The lower bounds on the independence number for graphs and hypergraphs given in the last sections are tight and the examples reaching these bounds are very crowded in the sense that they contain large cliques. So if we assume that a graph (or hypergraph) contains no large cliques there is some hope that better bounds can be found. For graphs, an appropriate assumption is that the graph is triangle-free. Indeed, Shearer proved the following result, which is an improvement of a result given in [3].

Theorem 2.17 (Shearer [59]) *Let G be a triangle-free graph on n vertices with average degree $d > 1$. Then,*

$$\alpha(G) \geq \frac{d \ln d - d + 1}{(d-1)^2} \cdot n \,.$$

A slight improvement of this bound is given in [60]. As mentioned in [59] this bound is tight up to a small constant factor. Using random graphs, it is possible to construct triangle-free graphs with average degree d whose independence number is bounded by approximately $\frac{2 \ln d}{d} \cdot n$. The proof of this theorem implies a polynomial *sequential algorithm* for computing an independent set in a triangle-free graph of size as guaranteed by the lower bound.

Recently, Shearer proved the following lower bound for the independence number of graphs which contain no complete graph K_r for $r \geq 4$.

Theorem 2.18 (Shearer [61]) *Let G be a graph on n vertices with average degree d which contains no K_r $(r \geq 4)$. Then for large d,*

$$\alpha(G) \geq c_r \, n \, \frac{\ln d}{d \ln \ln d} \,.$$

This improves the lower bound $\alpha(G) \geq c_r' \, n \, \frac{\ln \ln d}{d}$ given in [1]. Still it is an open question whether $\alpha(G) \geq c_r \, n \, \frac{\ln d}{d}$.

Now we consider "uncrowded" hypergraphs. Let $H = (V, \mathcal{E})$ be a k-uniform hypergraph with $k \geq 3$. A 2-*cycle* is a pair of distinct edges $e_1, e_2 \in \mathcal{E}$ that intersect in at least 2 vertices, i.e. $|e_1 \cap e_2| \geq 2$. A hypergraph is called *uncrowded* if it does not contain any 2-cycle. Note that this definition differs from the definition given in [2]. Duke, Lefmann and Rödl proved the following result, which is an extension of a result given in [2].

Theorem 2.19 (Duke, Lefmann and Rödl [23]) *Let $k \geq 3$ be a given integer. Then there exists a constant T_k such that the following holds. Suppose H is an uncrowded k-uniform hypergraph on n vertices and maximum degree at most T^{k-1}, where $T \geq T_k$. Then,*

$$\alpha(H) \geq \frac{0.98}{e} \cdot 10^{\frac{5}{k-1}} \cdot \frac{n}{T} (\ln T)^{\frac{1}{k-1}} \,.$$

A construction using random hypergraphs shows, that this bound is tight up to a constant factor depending only on k (see [2]). Unfortunately, no polynomial sequential algorithm (or even an NC algorithm) is known to compute independent sets of size given by this theorem.

We replace maximum degree by average degree and get the following result.

Corollary 2.20 *Let $k \geq 3$ be a given integer. Then there exist constants C_k, t_k such that the following holds. Suppose H is an uncrowded k-uniform hypergraph on n vertices and average degree at most t^{k-1}, where $t \geq t_k$. Then,*

$$\alpha(H) \geq C_k \cdot \frac{n}{t} (\ln t)^{\frac{1}{k-1}} \,.$$

Proof. Choose $t_k := 2^{-1/(k-1)}\,T_k$, where T_k is the constant given in Theorem 2.19. Let H be an uncrowded k-uniform hypergraph on n vertices and average degree at most t^{k-1}, where $t \geq t_k$. Clearly, H contains less than $n/2$ vertices of degree larger than twice the average degree. Deleting these vertices from H yields a new hypergraph H' with at least $n/2$ vertices and maximum degree at most $2\,t^{k-1} = T^{k-1}$, where $T := 2^{1/(k-1)}\,t \geq T_k$. By Theorem 2.19,

$$\alpha(H) \geq \alpha(H') \geq \frac{0.98}{e} \cdot 10^{\frac{5}{k-1}} \cdot \frac{n/2}{T}\,(\ln T)^{\frac{1}{k-1}} \geq C_k \cdot \frac{n}{t}\,(\ln t)^{\frac{1}{k-1}}\,.$$

❑

We further simplify this corollary by removing the restriction $t \geq t_k$.

Lemma 2.21 *Let H be an uncrowded k-uniform hypergraph ($k \geq 3$) on n vertices and average degree at most t^{k-1}. Then*

$$\alpha(H) \geq c_k \cdot \frac{n}{t}\,(\ln t)^{\frac{1}{k-1}}\,.$$

Proof. The statement of the theorem is obvious for $t \leq 1$. So assume $t > 1$. Let H be an uncrowded k-uniform hypergraph ($k \geq 3$) on n vertices and average degree at most t^{k-1}. If $t \geq t_k$ then Corollary 2.20 implies

$$\alpha(H) \geq C_k \cdot \frac{n}{t}\,(\ln t)^{\frac{1}{k-1}}\,.$$

For values $1 < t < t_k$ we rely on Theorem 2.6 to yield

$$\alpha(H) \geq \left(1 - \frac{1}{k}\right)\frac{n}{t}\,.$$

Define

$$c_k := \min\left\{C_k, \left(1 - \frac{1}{k}\right) \cdot (\ln t_k)^{-\frac{1}{k-1}}\right\}\,.$$

Then,

$$\alpha(H) \geq c_k \cdot \frac{n}{t}\,(\ln t)^{\frac{1}{k-1}} \qquad \text{for } t \geq t_k$$

and

$$\alpha(H) \geq \left(1 - \frac{1}{k}\right)\frac{n}{t} \geq \left(1 - \frac{1}{k}\right)\frac{n}{t}\left(\frac{\ln t}{\ln t_k}\right)^{\frac{1}{k-1}} \geq c_k \cdot \frac{n}{t}\,(\ln t)^{\frac{1}{k-1}} \qquad \text{for } 1 < t < t_k.$$

Thus, $\alpha(H) \geq c_k \cdot \frac{n}{t}\,(\ln t)^{\frac{1}{k-1}}$ without any restriction on t. ❑

Chapter 3

Solving selection problems with hypergraphs

Several extremal problems in areas like combinatorics, graph theory, geometry and number theory can be formulated roughly in the following way: Given a structure and a collection of forbidden configurations. How many elements of the structure can be chosen, such that none of the forbidden configurations occur? Problems of this type will be called *selection problems*. In this chapter we will see how to attack selection problems using the results of Chapter 2 about the independence number of hypergraphs. Two introductory examples will illustrate this. Furthermore, we will derive bounds on the maximum size of a subset of the $[n]^d$-grid that determines distinct slopes.

Many selection problems have shown to be very difficult – even asymptotic exact results seem to be out of reach – and the best known lower bounds are based on a random construction called the "deletion method" that we already used for the proof of Lemma 2.16: Choose a random subset of the given structure (with an appropriate probability distribution) and delete one element from each occurring forbidden configuration. The average size of the resulting set yields the lower bound. But this means restating the proof of Lemma 2.16 for a particular hypergraph. Thus, the deleting method can be replaced by an application of Lemma 2.16, which simplifies the proofs. Furthermore, the bounds based on the deleting method can be substantially improved when the structure of the hypergraph corresponding to the given selection problem enables us to apply Lemma 2.21.

We will study so-called anti-Ramsey type results that form a suitable framework for the problems considered in Chapter 4. In addition to existence results we investigate asymptotic problems concerning threshold functions. Moreover, we will consider algorithmic aspects.

3.1 Two examples

In this section we demonstrate how to use hypergraphs in the context of selection problems by giving alternative proofs for two known results that have been proved originally by the deletion method.

We begin with an example from extremal set theory (see also [4, 50]).

Theorem 3.1 (Erdős and Füredi [34]) *For every integer $n \geq 1$ there is a family $\mathcal{F} \subseteq 2^{[n]}$ of size $|\mathcal{F}| \geq c \cdot \left(\frac{2}{\sqrt{3}}\right)^n$, $c > 0$ constant, such that there are no three distinct members $A, B, C \in \mathcal{F}$ with*

$$A \cap B \subseteq C \subseteq A \cup B \tag{3.1}$$

Proof. Form a 3-uniform hypergraph $H = (V, \mathcal{E})$, where $V = 2^{[n]}$ and $\{A, B, C\} \in \mathcal{E}$ if (3.1) holds. Clearly, an independent set in H is a family $\mathcal{F}$ with the desired properties. We identify each set $X \subseteq 2^{[n]}$ with its characteristic vector $(x_1, \ldots, x_n) \in \{0,1\}^n$ with $x_i = 1$ if $i \in X$. Then (3.1) holds if and only if the following two conditions are satisfied for all $i \in [n]$: (i) $c_i = 1 \Rightarrow a_i = 1$ or $b_i = 1$ and (ii) $c_i = 0 \Rightarrow a_i = 0$ or $b_i = 0$.

To count $|\mathcal{E}|$ we fix a set C. Then the number of pairs $\{A, B\}$ satisfying (3.1) is bounded by $3^n/2$. Thus, $|\mathcal{E}| \leq 2^n 3^n/2$. By Theorem 2.6 we infer

$$\alpha(H) \geq \frac{2}{3}\left(\frac{2^{3n}}{3 \cdot 2^n 3^n/2}\right)^{1/2} = \left(\frac{2}{3}\right)^{1.5}\left(\frac{2}{\sqrt{3}}\right)^n$$

and the claim follows with $c := (2/3)^{1.5}$. ❑

A striking corollary of Theorem 3.1 is the following geometric result (for a proof, see [4]).

Corollary 3.2 (Erdős and Füredi [34]) *For every integer $n \geq 1$ there is a set of at least $c \cdot \left(\frac{2}{\sqrt{3}}\right)^n$ points in n-dimensional Euclidean space, such that all angles determined by three points from the set are strictly less than $\pi/2$.* ❑

This disproves an old conjecture of Danzer and Grünbaum (see [20, 4]), that the maximum cardinality of such a set is at most $2n - 1$.

Our second example is from extremal graph theory. Suppose we are given a family of graphs $\mathcal{L} = \{G_1, G_2, \ldots, G_t\}$. By $\mathrm{EX}(n, \mathcal{L})$ we denote the set of graphs on n vertices that do not contain an isomorphic copy of any of the graphs in $\mathcal{L}$. Moreover, $\mathrm{ex}(n, \mathcal{L})$ is defined as the maximum number of edges of a graph in $\mathrm{EX}(n, \mathcal{L})$. The determination of $\mathrm{ex}(n, \mathcal{L})$ is one of the central problems in extremal graph theory. Turán's Theorem gives a complete answer for the case $\mathcal{L} = \{K_r\}$. Unfortunately, for many other graphs this problem is very difficult and even the asymptotics of $\mathrm{ex}(n, \mathcal{L})$ is unknown. The best known lower bounds are often (for example for complete bipartite graphs or for even cycles) given by the following result of Erdős (see also [62]).

Theorem 3.3 (Erdős [28]) *Let $\mathcal{L} = \{G_1, G_2, \ldots, G_t\}$ be a family of non-empty graphs, and let*

$$\gamma := \max_j \min_{H \subseteq G_j} \frac{|V(H)| - 2}{|E(H)| - 1}. \tag{3.2}$$

Then, for some $c > 0$, $\mathrm{ex}(n, \mathcal{L}) \geq c \cdot n^{2-\gamma}$.

Proof. For each G_j let H_j be a subgraph of G_j attaining the inner minimum in (3.2). Denote with v_j the number of vertices of H_j and with k_j the number of edges of H_j. Consider the complete graph on n vertices K_n. We form a hypergraph $H = (V, \mathcal{E})$, whose vertices are the $\binom{n}{2}$ edges of K_n. The edge set is given by $\mathcal{E} = \mathcal{E}_1 \cup \ldots \cup \mathcal{E}_t$, where $\mathcal{E}_j$ is the family of those $e \subseteq V$, which form (the set of edges of) a copy of H_j. The independent sets in this hypergraph are in one-to-one correspondence with (the set of edges of) graphs in $\mathrm{EX}(n, \mathcal{L})$. This implies $\mathrm{ex}(n, \mathcal{L}) = \alpha(H)$. Our aim is to apply Lemma 2.16.

The number of copies of H_j contained in K_n is given by

$$|\mathcal{E}_j| = \binom{n}{v_j} \cdot \frac{v_j!}{|\mathrm{Aut}\, H_j|} \leq n^{v_j} ,$$

where $\mathrm{Aut}\, H_j$ is the automorphism group of H_j. We infer,

$$\min_j \left(\frac{\binom{n}{2}}{|\mathcal{E}_j|} \right)^{\frac{1}{k_j - 1}} \geq c_1 \min_j \left(\frac{n^2}{n^{v_j}} \right)^{\frac{1}{k_j - 1}} = c_1\, n^{-\gamma} .$$

Furthermore, $c_1\, n^{-\gamma} < 1$ for large n. Now Lemma 2.16 implies

$$\mathrm{ex}(n, \mathcal{L}) = \alpha(H) \geq c_2 \binom{n}{2} c_1\, n^{-\gamma} \geq c\, n^{2-\gamma} .$$

❏

3.2 Point sets with distinct slopes

Let $s_d(n)$ be the maximum cardinality of a subset $\mathcal{P}$ of the $[n]^d$-grid with the property that all $\binom{|\mathcal{P}|}{2}$ difference vectors have distinct slope, i.e. the difference vectors span pairwise distinct 1-dimensional subspaces of $\mathbb{R}^d$. We say $\mathcal{P}$ is free of parallelities.

Erdős et al [35] considered this problem in the case $d = 2$ showing a lower bound of $\Omega(n^{1/2})$ and upper bound $5n^{4/5}$. Our generalization improves the lower bound. The lower bound for $d = 2$ has also been shown by Zhang [73] using a probabilistic construction.

Theorem 3.4 *For every integer $n \geq 2$,*

$$\begin{aligned} s_2(n) &\geq c_2 \frac{n^{2/3}}{\log^{1/3} n} \qquad \text{and} \\ s_d(n) &\geq c_d\, n^{d/3} \qquad \text{for } d \geq 3 , \end{aligned}$$

where $c_2, c_d > 0$ are constants.

Proof of Theorem 3.4. We show the existence of a set $\mathcal{P}$ of the given size which is free of parallelities. Let us formulate the problem in a different way by means of a hypergraph $\mathcal{H} = (V, \mathcal{E})$. Its node set $V = V(n,d)$ is the set of all grid points, thus $|V| = n^d$. The hypergraph is 4-uniform and its edge set $\mathcal{E} = \mathcal{E}(n,d)$ consists of the parallelities of the grid. A parallelity is defined as a 4-set of the $[n]^d$-grid that induces two difference vectors with equal slope.

An independent set of $\mathcal{H}$ is a set of grid points that is free of parallelities. Thus we have

$$s_d(n) = \alpha(\mathcal{H}).$$

We want to estimate $|\mathcal{E}|$. There are two types of parallelities. Either a parallelity contains three collinear points and an arbitrary fourth point or it does not contain a collinearity. The number of parallelities with a collinearity is $\Theta\left(n^d K(n,d)\right)$, where $K(n,d)$ is the number of collinear 3-sets of the grid.

A parallelity without collinearity defines two parallel lines each containing two of the four points. By an appropriate translation of one line onto the other we can map the parallelity to a collinear 3-set of the $[2n]^d$-grid. To see that, let $\{p_1, p_2, q_1, q_2\}$ be a parallelity, where the vectors $p_2 - p_1$ and $q_2 - q_1$ are parallel with the same orientation. Then we map this 4-set to $\{p_1, p_2, p_2 + (q_2 - q_1)\} \subseteq [2n]^d$. Such a collinearity occurs $\Theta(n^d)$ times as an image of a parallelity, hence

$$|\mathcal{E}| = \Theta\left(n^d K(n,d)\right).$$

Because of the high symmetry of the grid we have

$$K(n,d) = \Theta\left(n^d K_0(n,d)\right),$$

where $K_0(n,d)$ is the number of collinear 3-sets that contain the point $(0, \ldots, 0)$.

We can write $K_0(n,d)$ as

$$K_0(n,d) = \sum_{m=1}^{n-1} \Phi(m) \binom{\left\lfloor \frac{n-1}{m} \right\rfloor}{2},$$

where we define

$$\Phi(m) := \left| \left\{ (x_1, x_2, \ldots, x_d) \in \mathbb{N}^d : \max_i x_i = m,\ \gcd(x_1, x_2, \ldots, x_d) = 1 \right\} \right|.$$

We can think of $\Phi(m)$ as the number of lines that contain the origin and $\left\lfloor \frac{n-1}{m} \right\rfloor$ other grid points.

A lower bound for $\Phi(m)$ is $\phi(m)(m+1)^{d-2}$, because we can choose $x_1 = m$, x_2 relative prime to m and $x_3, \ldots, x_d$ arbitrary between 0 and m, where $\phi(m)$ is the Eulerian totient function. On the other hand we have an obvious upper bound of $\Phi(m) \leq d(m+1)^{d-1}$. Hence we get

$$K_0(n,d) \leq \sum_{m=1}^{n-1} d(m+1)^{d-1} \frac{n^2}{m^2} = dn^2 \sum_{m=1}^{n-1} \frac{(m+1)^{d-1}}{m^2},$$

which is of order n^d for $d \geq 3$ and of order $n^2 \log n$ for $d = 2$, and a lower bound

$$K_0(n,d) \geq \sum_{m=1}^{n-1} \phi(m)(m+1)^{d-2} \binom{\lfloor \frac{n-1}{m} \rfloor}{2},$$

which is also of order n^d for $d \geq 3$ and of order $n^2 \log n$ for $d = 2$ (see [9] for summation properties of the function ϕ).

We conclude that

$$|\mathcal{E}| = \begin{cases} \Theta(n^6 \log n) & \text{for } d = 2 \\ \Theta(n^{3d}) & \text{for } d \geq 3 \end{cases}$$

By the Lemma 2.6 there is an independent set of size at least

$$s_d(n) = \alpha(\mathcal{H}) \geq c \left(\frac{|V|^4}{|\mathcal{E}|} \right)^{1/3} \geq \begin{cases} c_2 \frac{n^{2/3}}{\log^{1/3} n} & \text{for } d = 2 \\ c_d n^{d/3} & \text{for } d \geq 3 \end{cases}$$

❑

Theorem 3.5 *For every integer $d \geq 2$ there exists a constant $c_d > 0$ such that*

$$s_d(n) \leq c_d \, n^{d^2/(2d+1)}.$$

The following proof uses similar ideas as the proof of Erdős et al [35] for the case $d = 2$.

Proof. Let $\mathcal{P}$ be a set of P points of the $[n]^d$-grid that is free of parallelities. For integers m and Δ we define

$$A_{m,\Delta} := \{\Delta \cdot (x_1, x_2, \ldots, x_d) : 0 \leq x_i \leq m-1, \; i = 1, 2, \ldots, d\}$$

a set of m^d points. With Δ^d translated copies of $A_{m,\Delta}$ we can cover a grid of size $(\Delta m)^d$, so

$$t := \Delta^d \left(1 + \frac{n}{\Delta m}\right)^d = \left(\Delta + \frac{n}{m}\right)^d \leq 2^d \frac{n^d}{m^d}$$

disjoint copies are sufficient to cover the $[n]^d$-grid for

$$\Delta \leq \frac{n}{m}. \tag{3.3}$$

Let $\omega_1, \omega_2, \ldots, \omega_t$ be the number of points of $\mathcal{P}$ in the copies. Then we have $P = \sum_{i=1}^{t} \omega_i$ and

$$D := \sum_{i=1}^{t} \binom{\omega_i}{2} \geq t \, \frac{P/t(P/t-1)}{2} = \frac{P(P-t)}{2t} \geq \frac{P^2}{4t} \geq \frac{P^2 m^d}{2^{d+2} n^d}$$

for $t \leq \frac{P}{2}$, which holds for

$$m^d \geq 2^{d+1} n^d / P. \tag{3.4}$$

There are D difference vectors induced by the set $\mathcal{P}$ whose coordinates are divisible by Δ. By dividing these vectors by Δ we get vectors with coordinates between 0 and $m-1$. No two such vectors coincide because $\mathcal{P}$ is free of parallelities. This holds even for distinct values of Δ.

We have found $\frac{P^2 m^d}{2^{d+2} n^d}$ vectors for every Δ between 1 and $\left\lfloor \frac{n}{m} \right\rfloor$ (see equation 3.3), so we can estimate

$$\frac{P^2 m^d}{2^{d+2} n^d} \left\lfloor \frac{n}{m} \right\rfloor \le m^d$$

and therefore

$$P^2 \le \frac{2^{d+2} n^d}{\left\lfloor \frac{n}{m} \right\rfloor}.$$

To optimize this estimate we choose m as small as possible according to equation (3.4), that is

$$m = 1 + \left\lfloor \left(\frac{2^{d+1}}{P} \right)^{1/d} \cdot n \right\rfloor \ , \ \left\lfloor \frac{n}{m} \right\rfloor \ge \frac{P^{1/d}}{3}$$

for sufficiently large n. Altogether we get $P^{(2d+1)/d} \le 3 \cdot 2^{d+2} n^d$ which implies

$$P \le c'_d n^{d^2/(2d+1)},$$

with a constant

$$c'_d := \left(3 \cdot 2^{d+2}\right)^{d/(2d+1)}.$$

❑

3.3 Anti-Ramsey type results

The slope problem that we considered in the last section is a special case of the following more general problem. Suppose we are given a coloring f of the edges of the complete graph on n vertices with colors $t \in T$, i.e. $f : E(K_n) \longrightarrow T$ (adjacent edges with equal color are allowed). A complete subgraph K_k of K_n is called a *rainbow* (or multicolored clique) if the restriction $f|E(K_k)$ to the edge set of K_k is an injection. What is the maximum order of a rainbow with respect to f? Denote this maximum with $r(f)$, the *rainbow number*. This problem is in some sense complementary to Ramsey type problems, which deal with the existence of monochromatic subgraphs of given edge colorings.

The slope problem of the last section fits into this framework by considering the complete graph on the vertex set $[n]^d$. The color $f(\{v,w\})$ of an edge $\{v,w\}$ will be defined as the slope of the difference vector, i.e. the 1-dimensional subspace of $\mathbb{R}^d$ spanned by $v-w$. A rainbow in this graph corresponds to a subset of vertices of the $[n]^d$-grid that determines pairwise distinct slopes, thus, $s_d(n) = r(f)$.

Lemma 3.6 *Let $f : E(K_n) \longrightarrow T$ be a coloring of the edges of the complete graph on n vertices. Let $\mathcal{E}_3$ be the family of 3-sets of $V(K_n)$ that determine two equal colors and let $\mathcal{E}_4$ be the family of 4-sets of $V(K_n)$ that determine two equal colors but do not contain a 3-set from $\mathcal{E}_3$. Then,*

$$r(f) \geq c\,n \cdot \min\left\{1, \left(\frac{n}{|\mathcal{E}_3|}\right)^{1/2}, \left(\frac{n}{|\mathcal{E}_4|}\right)^{1/3}\right\},$$

where $c > 0$ is a constant.

This lemma is an easy consequence of Lemma 2.16.

Proof. Form a hypergraph $H = (V, \mathcal{E})$, with $V = V(K_n)$ and $\mathcal{E} = \mathcal{E}_3 \cup \mathcal{E}_4$. An independent set in this hypergraph corresponds to a rainbow with respect to the coloring f. Thus, $\alpha(H)$ is the maximum size of a multicolored clique. By Lemma 2.16 we infer

$$r(f) = \alpha(H) \geq c\,n \cdot \min\left\{1, \left(\frac{n}{|\mathcal{E}_3|}\right)^{1/2}, \left(\frac{n}{|\mathcal{E}_4|}\right)^{1/3}\right\}$$

and the claim follows. ❏

Remark. The lower bound given in Lemma 3.6 is tight up to a constant. To see this consider the following edge coloring f. Partition the complete graph K_n into l cliques $C_1, \ldots, C_l$ of size n/l and color all edges inside clique C_i with the color $t_i := i$. All edges between different monochromatic cliques get pairwise distinct colors from $T \setminus \{1, \ldots, l\}$. The maximum size of a rainbow is $r(f) = 2l$, since we can choose at most 2 vertices from each C_i. To compare this with Lemma 3.6 we observe that

$$\begin{aligned} |\mathcal{E}_3| &= l \cdot \binom{n/l}{3} \sim \frac{n^3}{l^2} \\ |\mathcal{E}_4| &= l \cdot \binom{n/l}{4} \sim \frac{n^4}{l^3}. \end{aligned}$$

Thus, the lower bound in Lemma 3.6 gives $r(f) \geq c\,l$, which is tight.

Applications of Lemma 3.6 will be given in Chapter 4. An easy upper bound is given by the following observation.

Observation 3.7 (Upper bound) *Let $f : E(K_n) \longrightarrow T$ be a coloring of the edges of the complete graph on n vertices. Then, $r(f) = O\left(|T|^{1/2}\right)$.*

Proof. A rainbow of size k induces $\binom{k}{2}$ distinct colors, which implies $\binom{k}{2} \leq |T|$ and the claim follows. ❏

Clearly, this upper bound is useless for $|T| > \binom{n}{2}$. On the other hand, it is easy to find edge colorings f with $r(f) = \Theta(|T|^{1/2})$ in the case $|T| = O(n^2)$.

In some cases, when the given edge coloring has a special structure, better bounds can be found. An important example is the problem of Sidon-sets in abelian groups. Suppose we are given a finite subset V of an abelian group $\mathcal{G}$. A subset $S \subseteq V$ is called a Sidon-set if all pairwise sums $a + b$ ($a \neq b \in S$) are distinct. What is the maximum size of a Sidon-set in V? (For an overview about Sidon-sets see [63]). This problem can be formulated in terms of an edge coloring in a natural way. Consider the complete graph with vertex set V and color each edge $\{v, w\}$ with the color $f(\{v, w\}) = v + w$. A Sidon-set in V forms a rainbow with respect to the coloring f. Due to the underlying algebraic structure f is a *proper coloring*, i.e. f has the property that adjacent edges have different colors. Therefore, the following result of Alon, Lefmann and Rödl can be applied.

Theorem 3.8 (Alon, Lefmann and Rödl [7]) *There exist constants $c_1, c_2 > 0$ such that the following holds. Let f be a proper edge coloring of K_n. Then,*

$$r(f) \geq c_1 n^{1/3} (\ln n)^{1/3} .$$

Furthermore, there exists a proper edge coloring g of K_n with

$$r(g) \leq c_2 n^{1/3} (\ln n)^{1/3} .$$

The theorem given in [7] is in fact more general, since it considers "proper" edge colorings of uniform hypergraphs. The following corollary is a direct consequence of Theorem 3.8 and Observation 3.7.

Corollary 3.9 *Let $\mathcal{G}$ be an abelian group of order n. Then there exists a Sidon-set in $\mathcal{G}$ with at least $c_1 n^{1/3} (\ln n)^{1/3}$ elements. Moreover, the size of any Sidon-set S in $\mathcal{G}$ is bounded by $|S| = O(n^{1/2})$.*

The gap of 1/3 and 1/2 in the exponent is quite typical for problems dealing with Sidon-sets. Only for some particular classes of groups tight bounds are known. In Chapter 4 we will study Sidon-sets of square numbers.

In some cases the assumption that the given edge coloring of K_n is a proper coloring is too restrictive but still a better lower bound than given by Lemma 3.6 can be found. Before stating the theorem that we will use in Chapter 4 for the proofs of Theorems 4.2 and 4.4 we introduce some further notations.

Given an edge coloring $f : E(K_n) \longrightarrow T$. For $t \in T$, $f^{-1}(t)$ is the set of all edges colored by color t. By $\bar{d}_t = \frac{2 \cdot |f^{-1}(t)|}{n}$ we denote the average degree of color $t \in T$. Let Δ_t be the maximum degree of color t, i.e. the maximum number of edges in color t, incident at some vertex, and let $\Delta = \max \{\Delta_t \mid t \in T\}$.

Theorem 3.10 *For every $\gamma > 0$ there exists a constant $C = C(\gamma) > 0$, such that for all integers $n \geq 2$ the following holds.*

Let $f\colon E(K_n) \longrightarrow T$ be a coloring and suppose τ satisfies the following conditions *(i)* $\tau \geq \sum_{t \in T} \bar{d_t}^2$, *and*

(ii) $\tau \geq n^{1/2+\gamma} \cdot \Delta^{3/2}$.

Then,

$$r(f) \geq C \cdot \left(\frac{n^2}{\tau}\right)^{1/3} \cdot (\ln n)^{1/3} . \tag{3.5}$$

Proof: It is sufficient to prove the theorem for sufficiently large n, say $n \geq n_0$. To see this assume the theorem holds for $n \geq n_0$ for some n_0 and some constant $C > 0$. For values of n less than n_0 the lower bound of the theorem is less than $C\, n_0^{1/2}\, (\ln n_0)^{1/3}$. By adapting the constant C the inequality (3.5) holds for all n. Thus we can assume that n is sufficiently large throughout the proof.

Let $V = \{1, 2, \ldots, n\}$ be the vertex set of a complete graph K_n and let $f\colon E(K_n) \longrightarrow T$ be an edge coloring. Let τ satisfy requirements (i) and (ii) in Theorem 3.10. We can also assume that τ satisfies

$$\tau < n^2 \ln n , \tag{3.6}$$

since otherwise the assertion (3.5) is trivial and we are done.

We will construct i-uniform hypergraphs $\mathcal{G}_i = (V, \mathcal{E}_i)$, $i = 3, 4$, with the same vertex set as follows:

$$\begin{aligned} \{v_1, v_2, v_3\} \in \mathcal{E}_3 &\;\Leftrightarrow\; f(\{v_1, v_2\}) = f(\{v_1, v_3\}) \\ \{v_1, v_2, v_3, v_4\} \in \mathcal{E}_4 &\;\Leftrightarrow\; f(\{v_1, v_2\}) = f(\{v_3, v_4\}). \end{aligned}$$

Observe that a subset $X \subseteq V$ yields a rainbow if and only if X is an independent set in both $\mathcal{G}_3$ and $\mathcal{G}_4$. Our aim will be to give a lower bound for the maximum size of such an independent set based on Lemma 2.21. We cannot apply Lemma 2.21 directly, as the $\mathcal{G}_i$, $i = 3, 4$, are in general not uncrowded. To come to such an uncrowded situation we will pick a random subset of the vertex set V, and show that an induced subhypergraph can be made uncrowded.

First we will give upper bounds for the cardinalities of $\mathcal{E}_3$ and $\mathcal{E}_4$. For $|\mathcal{E}_3|$ note that every pair $\{v, w\}$ of vertices can be extended in at most $2\Delta - 2$ ways to an edge $E \in \mathcal{E}_3$. Thus,

$$|\mathcal{E}_3| < \binom{n}{2} \cdot 2 \cdot \Delta < n^2 \cdot \Delta . \tag{3.7}$$

Concerning the size of $\mathcal{E}_4$ we obviously have

$$|\mathcal{E}_4| \leq \sum_{t \in T} \binom{|f^{-1}(t)|}{2} .$$

As $2 \cdot |f^{-1}(t)| = \bar{d}_t \cdot n$, it follows with (i) that

$$|\mathcal{E}_4| \leq \sum_{t \in T} \binom{\frac{\bar{d}_t \cdot n}{2}}{2} < \frac{n^2}{8} \cdot \sum_{t \in T} \bar{d}_t^{\,2} \leq \frac{1}{8} \cdot n^2 \cdot \tau \; . \tag{3.8}$$

Next we will count the number of 2-cycles in $\mathcal{G}_4$. Let $c_2(\mathcal{G})$ denote the number of 2-cycles in a hypergraph $\mathcal{G} = (V, \mathcal{E})$. We will count the 2-cycles more carefully: for $j = 2, 3$ let $c_{2,j}(\mathcal{G})$ be the number of $(2, j)$-cycles, i.e. the number of pairs $\{E, E'\} \in [\mathcal{E}]^2$ with $|E \cap E'| = j$. Clearly, $c_2(\mathcal{G}) = c_{2,2}(\mathcal{G}) + c_{2,3}(\mathcal{G})$.

Concerning $c_{2,2}(\mathcal{G}_4)$, choose an edge $E \in \mathcal{E}_4$ and then pick a pair $\{v, w\} \subset E$ of vertices. The number of edges $E' \in \mathcal{E}_4$ with $E \cap E' = \{v, w\}$ is less than the number of pairs $\{x, y\}$ with $f(\{v, w\}) = f(\{x, y\})$ or $f(\{v, x\}) = f(\{w, y\})$. There are at most $2n\Delta$ such pairs, hence with (3.8) we have

$$c_{2,2}(\mathcal{G}_4) \leq |\mathcal{E}_4| \cdot \binom{4}{2} \cdot 2 \cdot n \cdot \Delta \leq \frac{3}{2} \cdot n^3 \cdot \tau \cdot \Delta \; . \tag{3.9}$$

To count the number of $(2, 3)$-cycles, we fix an edge $E \in \mathcal{E}_4$ and a three-element subset $S \subset E$. Then S can be extended in at most $\binom{3}{2} \cdot \Delta$ ways to an edge $E' \in \mathcal{E}_4$, hence

$$c_{2,3}(\mathcal{G}_4) \leq |\mathcal{E}_4| \cdot \binom{4}{3} \cdot \binom{3}{2} \cdot \Delta \leq \frac{3}{2} \cdot n^2 \cdot \tau \cdot \Delta \; . \tag{3.10}$$

Now we choose a random subset of V by picking each vertex independently with probability

$$p = n^{-1/3+\epsilon} \cdot \tau^{-1/3} \; ,$$

where $0 < \epsilon < \gamma/12$. For the random subset $V' \subseteq V$ consider the induced random subhypergraphs $\mathcal{G}'_i = (V', \mathcal{E}'_i)$ for $i = 3, 4$, where $\mathcal{E}'_i = \mathcal{E}_i \cap [V']^i$. Moreover, let $c_{2,i}(V')$, $i = 2, 3$ be random variables counting the number of $(2, i)$-cycles in $\mathcal{G}'_4$.

Assumption (3.6) makes sure that $pn \to \infty$ as $n \to \infty$, so we have

$$\mathbf{Prob}(|V'| \geq 0.99\, pn) = 1 - o(1) \tag{3.11}$$

by Chernoff's inequality.

From (3.7) and (ii) we obtain for $\epsilon < \frac{\gamma}{3}$ that

$$\mathbf{E}(|\mathcal{E}'_3|) = p^3 \cdot |\mathcal{E}_3| < p^3 \cdot n^2 \cdot \Delta = pn \cdot \frac{n^{1/3+2\epsilon} \cdot \Delta}{\tau^{2/3}} \leq pn \cdot \frac{1}{n^{2/3 \cdot \gamma - 2\epsilon}} = o(pn) \; . \tag{3.12}$$

By (3.9) and (3.10) we infer for $\epsilon < \frac{\gamma}{12}$

$$\begin{aligned} \mathbf{E}(c_2(\mathcal{G}_4')) &= p^6 \cdot c_{2,2}(\mathcal{G}_4) + p^5 \cdot c_{2,3}(\mathcal{G}_4) \leq \frac{3}{2} \cdot p^6 \cdot n^3 \cdot \tau \cdot \Delta + \frac{3}{2} \cdot p^5 \cdot n^2 \cdot \tau \cdot \Delta \\ &= \frac{3pn}{2} \cdot \left(\frac{n^{1/3+5\epsilon} \cdot \Delta}{\tau^{2/3}} + \frac{\Delta}{n^{1/3-4\epsilon} \cdot \tau^{1/3}} \right) \\ &\leq \frac{3pn}{2} \cdot \left(n^{5\epsilon - 2/3 \cdot \gamma} + \frac{\Delta^{1/2}}{n^{1/2+\gamma/3-4\epsilon}} \right) = o(pn) \; . \end{aligned} \tag{3.13}$$

Moreover, we have

$$\mathbf{E}(|\mathcal{E}_4'|) = p^4 \cdot |\mathcal{E}_4| \, . \tag{3.14}$$

Using Markov's inequality, we infer with (3.11), (3.12), (3.13) and (3.14) that there exists a subset $V' \subseteq V$ with $|V'| \geq 0.99\,pn$, such that the induced hypergraphs $\mathcal{G}_i' = (V', \mathcal{E}_i')$, $i = 3, 4$, satisfy the following: $|\mathcal{E}_3'| = o(pn)$ and $c_2(\mathcal{G}_4') = o(pn)$ and also $|\mathcal{E}_4'| \leq 2 \cdot p^4 \cdot |\mathcal{E}_4|$. Now, delete one vertex from each triple $E \in \mathcal{E}_3'$ and from each 2-cycle in $\mathcal{G}_4'$. For n sufficiently large, we obtain a subset $V^* \subseteq V'$ of at least $\frac{pn}{2}$ vertices containing no edge from $\mathcal{E}_3$ and such that the induced subhypergraph $\mathcal{G}_4^* = (V^*, [V^*]^4 \cap \mathcal{E}_4)$ has no 2-cycle and has average degree at most

$$\frac{2\,p^4\,|\mathcal{E}_4|}{pn/2} = \frac{4 \cdot p^3 \cdot |\mathcal{E}_4|}{n} \leq \frac{1}{2}\,p^3 n \tau = \frac{1}{2}\,n^{3\epsilon} = t^3 \, ,$$

by (3.8), with t defined by the equation. We apply Lemma 2.21 to the hypergraph $\mathcal{G}_4^*$ and infer

$$\alpha(\mathcal{G}_4^*) \geq c_4 \cdot \frac{pn/2}{2^{-1/3} \cdot pn^{1/3}\tau^{1/3}} \cdot \left(\ln\left(2^{-1/3} \cdot n^{\epsilon}\right)\right)^{1/3} \geq C \cdot \left(\frac{n^2}{\tau}\right)^{1/3} \cdot (\ln n)^{1/3} \, .$$

❑

Corollary 3.11 *Let $f\colon E(K_n) \longrightarrow T$ be a coloring of the edges of the complete graph on n vertices, where $\Delta = O(n^{1-\beta})$ for a fixed $\beta > 0$. Then,*

$$r(f) \geq c \cdot \left(\frac{n}{\Delta}\right)^{1/3} \cdot (\ln n)^{1/3} \, ,$$

where $c = c(\beta) > 0$ is a constant.

Proof: By Theorem 3.10 with $\tau = n \cdot \Delta$ and taking $\gamma < \frac{\beta}{2}$. ❑

In particular, this implies the lower bound given in Theorem 3.8 by taking $\Delta = 1$ for proper colorings.

3.4 Threshold functions for rainbows

So far we considered only existence results for rainbows of given edge colorings. Another problem is the following. Suppose we are given a sequence $f_1, f_2, \ldots$ of edge colorings, where f_n is an edge coloring of the complete graph on n vertices K_n. For a given function $k = k(n)$, we say that *almost all* k-sets determine a rainbow, if a k-set $X \subseteq V(K_n)$ – chosen uniformly at random from the set of all k-sets of $V(K_n)$ – determines a rainbow with respect to f_n with probability tending to 1 as n goes to infinity.

A function $\tau(n)$ is called a *threshold function* for the rainbow property if the following two conditions are satisfied.

(i) Suppose $k = o(\tau(n))$, then almost all k-sets of $V(K_n)$ determine a rainbow.

(ii) Suppose $k = \omega(\tau(n))$, then almost all k-sets of $V(K_n)$ do not determine a rainbow.

The following theorem enables us to determine such threshold functions.

Theorem 3.12 *Given a sequence $f_1, f_2, \dots,$ where f_n is an edge coloring of K_n. For f_n let Δ_n be the maximum number of monochromatic edges incident to a common vertex, and let t_n denote the number of 4-sets in $V(K_n)$ that determine two disjoint monochromatic edges. If*

$$\Delta_n \leq c \cdot \frac{{t_n}^{3/4}}{n^2} \quad \textit{for all } n \in \mathbb{N} \tag{3.15}$$

for some constant $c > 0$, then $\tau(n) = n \cdot t_n^{-1/4}$ is a threshold function for the rainbow property.

Proof. For convenience we will write t instead of t_n and Δ instead of Δ_n throughout the proof, dropping the subscript n. Consider the hypergraph $H = (V, \mathcal{E}_3 \cup \mathcal{E}_4)$ corresponding to the edge coloring f_n, i.e. $V = V(K_n)$, $\mathcal{E}_3$ is the collection of 3-sets of V that determine two monochromatic edges and $\mathcal{E}_4$ is the collection of 4-sets of V that determine two disjoint monochromatic edges with respect to f_n. By definition, $|\mathcal{E}_4| = t$. A rainbow with respect to f_n corresponds to an independent set in this hypergraph.

(i) Suppose $k = k(n) = o(n \cdot t^{-1/4})$. Let $X \subseteq V$ be a k-set chosen uniformly at random from the set of all k-sets of V, and let Z be the number of edges from $\mathcal{E}_3 \cup \mathcal{E}_4$ contained in X. Then,

$$\mathbf{Prob}(X \text{ forms a rainbow}) = \mathbf{Prob}(Z = 0) \geq 1 - \mathbf{E}(Z)\,.$$

The expectation of Z is given by

$$\mathbf{E}(Z) = \frac{|\mathcal{E}_3| \cdot \binom{n-3}{k-3} + |\mathcal{E}_4| \cdot \binom{n-4}{k-4}}{\binom{n}{k}}\,.$$

To bound $|\mathcal{E}_3|$, choose two vertices $u \neq v \in V$. Then there are at most $\Delta - 1$ vertices $w \in V$ such that $f(\{u,v\}) = f(\{u,w\})$. This way, each edge from $\mathcal{E}_3$ will be counted at least twice. Thus, $|\mathcal{E}_3| \leq n\,(n-1)(\Delta - 1)/2 < n^2\,\Delta$. We infer using (3.15)

$$\begin{aligned}
\mathbf{E}(Z) &\leq n^2\,\Delta \cdot \frac{k^3}{n^3} + t \cdot \frac{k^4}{n^4} \\
&\leq c\,n^2\,\frac{t^{3/4}}{n^2} \cdot \frac{k^3}{n^3} + t \cdot \frac{k^4}{n^4} \\
&= o\left(n^2\,\frac{t^{3/4}}{n^2} \cdot \frac{n^3\,t^{-3/4}}{n^3}\right) + o\left(t \cdot \frac{n^4\,t^{-1}}{n^4}\right) \\
&= o(1)
\end{aligned}$$

This implies,

$$\mathbf{Prob}(X \text{ forms a rainbow}) = \mathbf{Prob}(Z=0) \geq 1 - \mathbf{E}(Z) = 1 - o(1)\,.$$

In other words, almost all k-sets are rainbows.

(ii) Now suppose $k = \omega(n \cdot t^{-1/4})$. Let Z denote the number of edges of $\mathcal{E}_4$ contained in a random k-set $X \subseteq V$ – this time ignoring the edges from $\mathcal{E}_3$. Then

$$\mathbf{Prob}(X \text{ forms a rainbow}) \leq \mathbf{Prob}(Z=0)\,.$$

By Chebychev's inequality we estimate

$$\mathbf{Prob}(Z=0) \leq \frac{\mathbf{Var}(Z)}{\mathbf{E}^2(Z)}\,.$$

Furthermore,

$$\mathbf{E}(Z) = t \cdot \frac{\binom{n-4}{k-4}}{\binom{n}{k}} = t \cdot \frac{k^{\underline{4}}}{n^{\underline{4}}}\,,$$

where $x^{\underline{r}} = x\,(x-1)\ldots(x-r+1)$ denotes the falling factorial. Suppose $\mathcal{E}_4 = \{S_1, \ldots, S_t\}$. Let z_i be the indicator variable for the event $S_i \subseteq X$. Thus $Z = \sum_i z_i$. Moreover, for $s = 5,6,7,8$ let a_s denote the number of pairs $\{S_i, S_j\}$ with $|S_i \cup S_j| = s$. By the definition of the variance we have

$$\begin{aligned}
\mathbf{Var}(Z) &= \mathbf{E}\left((Z - \mathbf{E}(z))^2\right) \\
&= \mathbf{E}(Z) + 2 \sum_{1 \leq i < j \leq t} \mathbf{E}(z_i z_j) - \mathbf{E}^2(Z) \\
&= \mathbf{E}(Z) + 2 \sum_{s=5}^{8} a_s \frac{k^{\underline{s}}}{n^{\underline{s}}} - \mathbf{E}^2(Z)\,.
\end{aligned}$$

To bound a_5, choose an edge $S \in \mathcal{E}_4$ and a 3-element subset $R \subseteq S$. Then R can be extended to an edge $\in \mathcal{E}_4 \setminus \{S\}$ in at most 3Δ ways. Thus,

$$a_5 \leq 12t\,\Delta\,.$$

To bound a_6, choose an edge $S \in \mathcal{E}_4$, a 2-set $R \subseteq S$ and an arbitrary vertex $v \in V \setminus S$. Then there are at most 3Δ ways to extend $R \cup \{v\}$ to an edge in $\mathcal{E}_4$. Thus,

$$a_6 \leq 18t\Delta\,.$$

Similar considerations for a_7 yield

$$a_7 \leq 12tn^2\Delta\,.$$

An obvious upper bound for a_8 is given by

$$a_8 \leq \binom{t}{2}\,.$$

Putting this together we get

$$\begin{aligned}
\mathbf{Var}(Z) &= \mathbf{E}(Z) + 2\sum_{s=5}^{8} a_s \frac{k^{\underline{s}}}{n^{\underline{s}}} - \mathbf{E}^2(Z) \\
&\le t\,\frac{k^{\underline{4}}}{n^{\underline{4}}} + 24t\Delta\,\frac{k^{\underline{5}}}{n^{\underline{5}}} + 36t\,n\Delta\,\frac{k^{\underline{6}}}{n^{\underline{6}}} + 24t\,n^2\Delta\,\frac{k^{\underline{7}}}{n^{\underline{7}}} + 2\binom{t}{2}\frac{k^{\underline{8}}}{n^{\underline{8}}} - \left(t\,\frac{k^{\underline{4}}}{n^{\underline{4}}}\right)^2 \\
&\le t\,\frac{k^{\underline{4}}}{n^{\underline{4}}} + c_1\,t\,n^2\Delta\,\frac{k^{\underline{7}}}{n^{\underline{7}}} \\
&\le t\,\frac{k^{\underline{4}}}{n^{\underline{4}}} + c_2\,t^{7/4}\,\frac{k^{\underline{7}}}{n^{\underline{7}}} \qquad \text{(by equation (3.15))}\,.
\end{aligned}$$

We apply Chebychev's inequality

$$\begin{aligned}
\mathbf{Prob}(X \text{ forms a rainbow}) &\le \mathbf{Prob}(Z = 0) \\
&\le \frac{\mathbf{Var}(Z)}{\mathbf{E}^2(Z)} \\
&\le \frac{t\,\frac{k^{\underline{4}}}{n^{\underline{4}}} + c_2\,t^{7/4}\,\frac{k^{\underline{7}}}{n^{\underline{7}}}}{\left(t\,\frac{k^{\underline{4}}}{n^{\underline{4}}}\right)^2} \\
&\le c_3\,\frac{n^4}{t\,k^4} + c_4\,\frac{n}{t^{1/4}\,k} \\
&= o(1)\,.
\end{aligned}$$

The last inequality follows from the assumption $k = \omega(n \cdot t^{-1/4})$. Thus, almost all k-sets do not determine a rainbow. ❑

To illustrate the use of this theorem we go back to the problem of Sidon sets in abelian groups. As mentioned earlier precise bounds for the largest Sidon set contained in a given abelian group are unknown, apart from a few special cases. Nevertheless, the following result is an easy consequence of Theorem 3.12.

Theorem 3.13 *Let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be a sequence of abelian groups, where $\mathcal{G}_n$ has order n. Then the following holds:*

(i) If $k = o(n^{1/4})$, then almost all k-set of $\mathcal{G}_n$ form a Sidon set.

(ii) If $k = \omega(n^{1/4})$, then almost all k-sets of $\mathcal{G}_n$ do not form a Sidon set.

Proof. Consider the edge coloring $f : E(K_n) \longrightarrow \mathcal{G}_n$ corresponding to the group $\mathcal{G}_n$ given by $f(\{a,b\}) = a + b$. We observe that a rainbow with respect to the coloring f is a Sidon set with respect to $\mathcal{G}_n$. As in Theorem 3.12 let Δ denote the maximum number of monochromatic edges incident to a common vertex and let t denote the number of 4-sets determining two disjoint monochromatic edges. Since $\mathcal{G}_n$ is an abelian group we have $\Delta = 1$ and $t = \Theta(n^3)$. Therefore, equation (3.15) is satisfied. Now Theorem 3.12 implies that $\tau(n) = n \cdot t^{-1/4} \sim n^{1/4}$ is a threshold function for the rainbow property and the claim follows. ❑

Further applications of Theorem 3.12 will be given in Chapter 4.

3.5 Computing large rainbows

In Section 3.3, Lemma 3.6, we derived a lower bound on the maximum size of a rainbow for a given edge coloring of a complete graph on n vertices. This raises the question how to compute rainbows of size as guaranteed by this lemma. In particular, such an algorithm enables us to find large solutions to selection problems for which lower bounds are based on Lemma 3.6 (see Chapter 4).

A polynomial algorithm for computing a maximum rainbow cannot exists, unless P = NP. To see this, assume that we have a polynomial algorithm A for this problem. We will show that under this assumption we are able to approximate the independent set problem for graphs up to an additive constant of 1 in polynomial time. Then it follows from the following result of Arora et al ([10], see also [11]) that P = NP.

Theorem 3.14 (Arora et al [10]) *There exists a constant $c > 0$ such that approximating the independence number of a graph on n vertices within a factor of n^c is NP-hard.*

To this end, let $G = (V, E)$ be a graph with $|V| = n$. Let K_n be the complete graph with the same vertex set V. We color all edges of K_n contained in G with the same color denoted by t_0. All other edges of K_n will be colored with distinct colors $t_1, t_2, \ldots$ different from t_0. Denote this edge coloring with f. We observe that an independent set in G corresponds to a rainbow with respect to f. Thus, $\alpha(G) \leq r(f)$. Moreover, a rainbow with respect to f contains at most one edge with color t_0, i.e. an edge contained in G. This implies $r(f) - 1 \leq \alpha(G)$. We conclude that we are able to compute an independent set in G of size at least $\alpha(G) - 1$ in polynomial time by using algorithm A.

However, we will see that there is a polynomial algorithm for computing a rainbow of guaranteed size. To this end, let f be an edge coloring of the complete graph K_n. With this edge coloring f we associate a hypergraph $H = (V(K_n), \mathcal{E}_3 \cup \mathcal{E}_4)$, where $\mathcal{E}_3$ is the family of 3-sets of vertices determining two equal edge colors and $\mathcal{E}_4$ is the family of 4-sets of vertices that determine two equal colors but do not contain a 3-set from $\mathcal{E}_3$. Hence we have $r(f) = \alpha(H)$, where $\alpha(H)$ is the independence number of H.

We define the *probabilistic bound*

$$\tilde{\alpha}(H) := \max_{p \in [0,1]} (pn - p^3|\mathcal{E}_3| - p^4|\mathcal{E}_4|),$$

which is a lower bound for $\alpha(H)$, since we can pick each vertex independently with probability p and then delete one vertex from each edge occurring in the resulting subhypergraph. This gives an independent set of size at least $pn - p^3|\mathcal{E}_3| - p^4|\mathcal{E}_4|$ in the average. This bound is at least as good as the bound given in Lemma 3.6 since the bound given there is based on a particular choice for the probability p (see proof of Lemma 3.6 and Lemma 2.16).

By using derandomization techniques (see [8]) we can turn this probabilistic argument into a deterministic algorithm that computes an independent set of the hypergraph and thus a rainbow for the given edge coloring of size at least $\tilde{\alpha}(H)$. In the following, we will describe the algorithm.

Let the vertex set of K_n be $V = \{1, 2, \ldots, n\}$. Let $f : E(K_n) \longrightarrow T$ be an edge coloring and assume that T is totally ordered. For $t \in T$ let $m_t = |f^{-1}(t)|$ be the number of edges in color t. In a preprocessing we form our hypergraph $H = (V, \mathcal{E}_3 \cup \mathcal{E}_4)$ by collecting pairs of edges of the same color. By first sorting the set of edges with respect to their colors this can be done in time $O(n^2 \ln n + \sum_{t \in T} m_t^2)$. Moreover, we use the following data structure. There is a list of the vertices $v \in V$ and a list of the edges $e \in \mathcal{E}_3 \cup \mathcal{E}_4$. For each vertex $v \in V$ there are pointers to all edges containing v. For each edge there are pointers to all vertices contained in it.

Knowing $|\mathcal{E}_3|$ and $|\mathcal{E}_4|$, we can compute that value $p \in [0, 1]$, which maximizes the expression $pn - p^3|\mathcal{E}_3| - p^4|\mathcal{E}_4|$. Fix this value of p.

In the following we will examine the vertices of V one by one and decide whether each vertex belongs to our independent set or not.

Set $\mathcal{E} = \mathcal{E}_3 \cup \mathcal{E}_4$. Suppose that we already made a partial selection of vertices and let $\epsilon_1, \epsilon_2, \ldots, \epsilon_j$ be the 0,1-sequence representing this selection, that is, for some vertices we determined whether they do ($\epsilon_i = 1$) or do not ($\epsilon_i = 0$) belong to our independent set. Define weight functions f_j and F_j depending on $\epsilon_1, \epsilon_2, \ldots, \epsilon_j$ as follows. For vertices $v \in V$ let

$$f_j(v) = \begin{cases} \epsilon_v & \text{if } v \leq j \\ p & \text{if } v > j. \end{cases}$$

For edges $e \in \mathcal{E}$ set

$$f_j(e) = \prod_{v \in e} f_j(v) \ .$$

Finally, set

$$F_j = F_j(\epsilon_1, \epsilon_2, \ldots, \epsilon_{j-1}) = \sum_{v \in V} f_j(v) - \sum_{e \in \mathcal{E}} f_j(e) \ .$$

Observe that F_j is the expected value of the number of vertices minus the number of edges in a random extension of the selection $\epsilon_1, \epsilon_2, \ldots, \epsilon_j$.

At the beginning, for $j = 0$, we have $f_0(v) = p$ and $f_0(e) = p^{|e|}$ for $v \in V$ and $e \in \mathcal{E}$, thus $F_0 = \tilde{\alpha}(H)$. We will construct a $0,1$-sequence $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ such that the values $F_j = F_j(\epsilon_1, \ldots, \epsilon_j)$ are nondecreasing for $j = 0, 1, \ldots, n$. In particular, we will have $F_n \geq F_0$.

Now assume that $\epsilon_1, \epsilon_2, \ldots, \epsilon_{j-1}$ and $F_{j-1} \geq F_{j-2} \geq \ldots \geq F_0$ are given. To determine ϵ_j and thus F_j, we compute the two values

$$\begin{aligned} W_j^0 &= F_j(\epsilon_1, \epsilon_2, \ldots, \epsilon_{j-1}, 0) \\ W_j^1 &= F_j(\epsilon_1, \epsilon_2, \ldots, \epsilon_{j-1}, 1). \end{aligned}$$

This can be done in time $O(1 + \deg_H(j))$. If $W_j^0 \geq W_j^1$, then we set $\epsilon_j = 0$ and $F_j = W_j^0$. Otherwise, if $W_j^1 > W_j^0$, then set $\epsilon_j = 1$ and $F_j = W_j^1$.

By straightforward calculations or by interpreting F_j as an expected value, we derive

$$F_{j-1} = (1-p) \cdot W_j^0 + p \cdot W_j^1 \ . \tag{3.16}$$

This implies $F_j \geq F_{j-1}$.

Continuing in this way, we obtain a 0, 1-sequence $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ with $F_n \geq F_0$. Set $I = \{v \in V \mid \epsilon_v = 1\}$. We claim that I is an independent set in H. Assume this is not the case. Thus there is an edge $e \in \mathcal{E}$ contained in I. Let j be the last vertex of e chosen by our algorithm. Since j was chosen, we know that $W_j^1 > W_j^0$, which with (3.16) implies that

$$F_j = W_j^1 > F_{j-1} \ .$$

On the other hand, since j is the *last* chosen vertex of e, we infer that

$$F_j - F_{j-1} \leq (f_j(j) - f_{j-1}(j)) - (f_j(e) - f_{j-1}(e)) = (1-p) - (1-p) = 0 \ ,$$

a contradiction. Thus I is an independent set with

$$|I| = F_n \geq F_0 = \tilde{\alpha}(H)$$

as desired.

Without the preprocessing this algorithm has a linear running time of $O(n + |\mathcal{E}_3| + |\mathcal{E}_4|)$. Since $|\mathcal{E}_3|, |\mathcal{E}_4| < \sum_{t \in T} m_t^2$, we have an overall running time of $O(n^2 \ln n + \sum_{t \in T} m_t^2)$.

Chapter 4

Point sets with distinct distances

In [36] Erdős and Guy considered the following problem: Determine the maximum size of a subset X of the $[n]^2$-grid, such that all mutual euclidean distances between different points of X are distinct, compare also [46]. Denoting the cardinality of such a set X by $f_2(n)$, they proved the following:

Theorem 4.1 (Erdős and Guy [36]) *For every integer $n \geq 3$,*

$$n^{\frac{2}{3}-\frac{c_1}{\ln\ln n}} \leq f_2(n) \leq c_2 \cdot \frac{n}{(\ln n)^{1/4}}\,, \tag{4.1}$$

where $c_1, c_2 > 0$ are constants.

To obtain the lower bound for $f_2(n)$, Erdős and Guy used Greedy-type arguments. The upper bound for $f_2(n)$ follows easily by a result from number theory. Namely, consider the edge coloring $f : E(K_{n^2}) \longrightarrow [2n^2]$, where an edge $\{(x_1, y_1), (x_2, y_2)\}$ receives color $(x_2 - x_1)^2 + (y_2 - y_1)^2$, the square of the euclidean distance. A rainbow with respect to f is the same as a subset of the $[n]^2$-grid with distinct distances. By a result of Landau [52], the number of integers not exceeding x, which are representable as a sum of two squares, is asymptotically $c \cdot \frac{x}{\ln^{1/2} x}$, where c is a positive constant. Thus our mapping f is actually a coloring $f : E(K_{n^2}) \longrightarrow T$, where $T \subseteq [2n^2]$ is of size $|T| = \Theta(\frac{n^2}{\ln^{1/2} n})$. It follows by Observation 3.7 that the maximum size of a rainbow is bounded by $r(f) = O(|T|^{1/2}) = O(\frac{n}{\log^{1/4} n})$.

Here we will improve the lower bound on $f_2(n)$, namely we will show the following.

Theorem 4.2 *For integers $n \geq 1$,*

$$f_2(n) \geq c \cdot n^{2/3}\,,$$

where $c > 0$ is a constant.

The main tools for the proof of Theorem 4.2 are an anti-Ramsey result from Chapter 3 and a result of Ramanujan in number theory. This anti-Ramsey result, together with a number theoretic result concerning the representation of integers by a fixed number of squares, which we deduce in Section 4.1, also yields lower bounds for the analog of the problem of Erdős and Guy in higher dimensions. Let $f_d(n)$ denote the maximum size of a subset X of the d-dimensional grid $[n]^d$ such that all mutual euclidean distances within X are distinct.

We remark that for $d = 1$ one has $f_1(n) = \Theta(\sqrt{n})$ by using perfect difference sets, cf. [36]. For $d \geq 3$, Erdős and Guy showed the following:

Theorem 4.3 (Erdős and Guy [36]) *Let $d \geq 3$ be an integer and $\epsilon > 0$ a fixed real. Then, for n sufficiently large,*

$$n^{2/3-\epsilon} \leq f_d(n) \leq c \cdot \sqrt{d} \cdot n , \tag{4.2}$$

where $c > 0$ is a constant.

As before, the upper bound follows easily by Observation 3.7. Indeed, in [36] it has been conjectured that

$$¿ \qquad f_d(n) \leq c \cdot d^{2/3} \cdot n^{2/3} \cdot (\ln n)^{1/3} \qquad ?$$

for $d \geq 3$.

In Section 4.2 we will derive an improved lower bound on $f_d(n)$ that matches the conjectured upper bound (apart from the constant factor).

Theorem 4.4 *Let $d \geq 3$ be a positive integer. Then for every integer $n \geq 1$,*

$$f_d(n) \geq c_d \cdot n^{2/3} \cdot (\ln n)^{1/3} ,$$

where $c_d > 0$ is a constant only dependent on d.

In Section 4.3 we will consider the corresponding selection problems for points in the plane in arbitrary position. We will show that every n-point set in the euclidean plane $\mathbb{R}^2$ contains a subset X with mutual distinct distances such that $|X| \geq c \cdot n^{1/4}$ for some constant $c > 0$. This gives a partial answer to a question raised by Erdős and Guy [36]. Furthermore, this result improves a lower bound of $c \cdot n^{1/5}$ which follows from [13, proof of Theorem 4.2], as communicated to the author by J. Pach. Moreover, we will show that under the assumption that the n points in $\mathbb{R}^2$ are in general position (no three on a line) the lower bound on $|X|$ can be improved to $c \cdot n^{1/3}$. Also, generalizations to higher dimensions will be given.

To do so we will answer a question of Erdős and Fishburn [38],[33] concerning the distribution of distances in planar point sets. Namely, we will show the following: if n points in general position in $\mathbb{R}^2$ are given with distinct distances $d_1, d_2, \ldots, d_t$, where d_i occurs with multiplicity m_i, $i = 1, 2, \ldots, t$, then $\sum_{i=1}^{t} m_i^2 \leq c \cdot n^3$ for some positive constant c. The regular n-gon shows that this bound is tight up to a small constant factor. The upper bound $\sum m_i^2 = O(n^3)$ also holds for n points in general position in $\mathbb{R}^d$ (no $d+1$ points on a common hyperplane). Moreover, we will show that for the corresponding problem for n arbitrary points in $\mathbb{R}^2$ one has $\sum_{i=1}^t m_i^2 \leq c \cdot n^{3.25}$, where c is a positive constant.

Threshold results for point sets with distinct distances will be derived in Section 4.4. Improving a result of Avis, Erdős and Pach [13], we will show that almost all k-sets of a set of n points in the plane determines $\binom{k}{2}$ distinct distances, if $k = o(n^{3/16})$. A similar result will be given for point sets in general position and this result is sharp. Furthermore, we will determine threshold functions for point sets with distinct distances contained in the $[n]^d$-grid.

In Section 4.5 we will give the new lower bound $c \cdot n^{2/3}$ ($c > 0$ constant) on the size of a $\mathbb{B}_2$-subset of the set $\{1^2, 2^2, \ldots, n^2\}$. This improves earlier results of Alon and Erdős [6].

4.1 Representations of integers by sums of squares

The distribution of distances between lattice points is closely related to problems in additive number theory, namely, the representation of integers by a fixed number of squares. To illustrate this, consider an arbitrary point p of the integer grid $\mathbb{Z}^d$ in dimension d. For any point $q \in \mathbb{Z}^d$, the euclidean distance between p and q is of the form $d(p,q) = \sqrt{x_1^2 + \ldots + x_d^2}$, where the x_i's are integers. Therefore, $m := d^2(p,q)$ is an integer that is representable as a sum of d squares. Furthermore, the number of grid points on the sphere with radius $\sqrt{m}$ around p is given by the number of representations of m as a sum of d squares.

In this section we will be concerned with two number theoretic results (Theorems 4.8 and 4.15) that we will need for the proofs of Theorem 4.2 and 4.4. For convenience we will use Vinogradov's notation $f(n) \ll g(n)$ for $f(n) = O(g(n))$.

Definition 4.5 For integers $m \geq 0$ and $d \geq 1$ let $r_d(m)$ be the number of representations of m as a sum of d squares, i.e.

$$r_d(m) = \left|\left\{(x_1, \ldots, x_d) \in \mathbb{Z}^d : m = x_1^2 + \ldots + x_d^2\right\}\right| .$$

Consider the case $d = 2$ first. The function $r_2(m)$ oscillates rather wildly. For infinitely many $m \in \mathbb{N}$ we have $r_2(m) = 0$. An upper bound is given by the following result.

Theorem 4.6 (Wigert [48])

$$r_2(m) \leq m^{\frac{c}{\ln \ln m}} ,$$

where $c > 0$ is a constant.

This bound cannot be improved as there is an infinite sequence of values of m with $r_2(m) \geq m^{c'/\ln\ln m}$, $c' > 0$ constant (see [48]). The average behaviour of $r_2(m)$ is easy to determine.

Theorem 4.7 (Gauss [48])

$$\sum_{m=1}^{n} r_2(m) = \pi n + O(\sqrt{n}) .$$

The determination of the order of the error term has received the attention of many mathematicians and there is a long history of improvements on Theorem 4.7 (see [51]).

The following higher moment result, which plays a central role in the proof of Theorem 4.2, is due to Ramanujan (see also [72]).

Theorem 4.8 (Ramanujan [57])

$$\sum_{m=1}^{n}(r_2(m))^2 = \Theta(n\ln n)\,.$$

In fact Ramanujan determines also the leading constant.

Here we will give an alternative proof for the upper bound. Our approach uses simple geometric considerations and might be of interest by itself. For doing so and for later purposes we will use the following definition and lemma.

Definition 4.9 For two points $p, q \in \mathbb{R}^d$ the hyperplane formed by the set of points in $\mathbb{R}^d$ that are equidistant to p and q is called the *perpendicular bisector* of p and q.

Let P be a finite set of points in R^d. Consider the bipartite graph $\mathcal{B} = ([P]^2 \cup P, I)$ with

$$(\{p,q\},z) \in I \iff z \text{ lies on the perpendicular bisector of } p \text{ and } q.$$

Then define $\Delta(P) := |I|$.

Roughly speaking, $\Delta(P)$ is the number of incidences between perpendicular bisectors determined by P and points of P (each bisector can be generated by several pairs of points!). Note that $\Delta(P)$ is nearly the same as the number of isosceles triangles determined by P apart from the fact that equilateral triangles are counted 3 times.

Lemma 4.10 *Let P be a set of n points in $\mathbb{R}^d$. Let the points of P determine distinct distances $d_1, d_2, \ldots, d_t$, where d_i occurs with multiplicity m_i for $i = 1, 2, \ldots, t$. Then,*

$$\sum_{i=1}^{t} m_i^2 \leq \frac{n}{2}\cdot\left(\Delta(P) + \binom{n}{2}\right)\,.$$

The idea of the proof is related to an argument of Szemerédi (see [32]).

Proof. Let P be the set of n given points in $\mathbb{R}^d$. Note that a point z lies on the bisector of p and q if and only if p and q have the same distance from z. For $z \in P$ and $i = 1, 2, \ldots, t$ let $m_i(z)$ denote the number of points in P, which have distance d_i from z. Using Jensen's inequality we infer that

$$\begin{aligned}\Delta(P) &= \sum_{z\in P}\sum_{i=1}^{t}\binom{m_i(z)}{2} = \sum_{i=1}^{t}\sum_{z\in P}\binom{m_i(z)}{2} \geq \sum_{i=1}^{t} n\cdot\binom{\frac{2m_i}{n}}{2} = \frac{2}{n}\cdot\sum_{i=1}^{t} m_i^2 - \sum_{i=1}^{t} m_i \\ &= \frac{2}{n}\cdot\sum_{i=1}^{t} m_i^2 - \binom{n}{2}\,.\end{aligned}$$

Thus,

$$\sum_{i=1}^{t} m_i^2 \le \frac{n}{2} \cdot \left(\Delta(P) + \binom{n}{2} \right) .$$

❏

Lemma 4.11

$$\Delta([n]^2) \le c \cdot n^4 \cdot \ln n ,$$

for some positive constant c.

Note that $\Delta([n]^2)$ is equal to the number of isosceles triangles since $[n]^2$ contains no equilateral triangle.

Proof. For distinct points p_1, p_2 in the $[n]^2$-grid, where $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$, let l be the line through p_1 and p_2 and define

$$s(l) = \max \left\{ \frac{1}{g} \cdot |x_2 - x_1|, \frac{1}{g} \cdot |y_2 - y_1| \right\} ,$$

where $g = \gcd(x_2 - x_1, y_2 - y_1)$. Note that $s(l)$ only depends on l and is independent of the choice of p_1 and p_2. We observe that $|l \cap [n]^2| \le \frac{n}{s(l)}$. Let l' be the perpendicular bisector of p_1 and p_2, then we also have $|l' \cap [n]^2| \le \frac{n}{s(l)}$.

To bound $\Delta([n]^2)$ we fix an arbitrary point $p_1 \in P$ and an integer $1 \le s \le n$. Choose a line l through p_1 with $s(l) = s$. The number of such lines is at most $4s$. Then we select a point $p_2 \in l \cap [n]^2$ and a point $z \in l' \cap [n]^2$, where l' is the perpendicular bisector of p_1 and p_2. For both p_2 and z there are at most $\frac{n}{s}$ possibilities. Thus

$$\Delta([n]^2) \le n^2 \cdot \sum_{s=1}^{n} 4s \cdot \frac{n}{s} \cdot \frac{n}{s} = 4n^4 \cdot \sum_{s=1}^{n} \frac{1}{s} \le 4n^4 \cdot \left(1 + \int_1^n \frac{1}{x} dx \right) \le c \cdot n^4 \cdot \ln n.$$

❏

Corollary 4.12 *For $i = 1, 2, \ldots, 2(n-1)^2$ let m_i denote the occurrence of distance $\sqrt{i}$ between different points of the $[n]^2$-grid. Then,*

$$\sum_{i=1}^{2(n-1)^2} m_i^2 \le c_2 \cdot n^6 \cdot \ln n$$

for some positive constant c_2.

Proof. By Lemmata 4.10 and 4.11 we obtain

$$\sum_{i=1}^{2(n-1)^2} m_i^2 \le \frac{n^2}{2} \cdot \left(c \cdot n^4 \cdot \ln n + \binom{n^2}{2} \right) \le c_2 \cdot n^6 \cdot \ln n .$$

❏

Corollary 4.13 *There exists a positive constant* c_3 *such that for every positive integer* n

$$\sum_{i=1}^{n}(r_2(i))^2 \leq c_3 \cdot n \cdot \ln n \; .$$

Proof. Consider a circle of radius at most $\frac{n-1}{2}$ around an arbitrary point of the $[n]^2$-grid. Then at least a quarter of this circle lies inside the $[n]^2$-grid. Thus

$$\sum_{i=1}^{(\frac{n-1}{2})^2} \left(n^2 \cdot \frac{r_2(i)}{4}\right)^2 \leq \sum_{i=1}^{2(n-1)^2} m_i^2 \; ,$$

where m_i denotes the occurrence of distance $\sqrt{i}$ between different points in the $[n]^2$ grid. By Corollary 4.12 we deduce $\sum_{i=1}^{n}(r_2(i))^2 \leq c_3 \cdot n \cdot \ln n$ for some positive constant c_3. ❏

Remark. By using the same ideas it is also possible to prove the lower bound in Theorem 4.8.

Next we study the case $d \geq 3$. Like $r_2(m)$ the functions $r_3(m)$ and $r_4(m)$ oscillate rather wildly. The situation changes for $d \geq 5$ as the following result holds (see [45, p. 122]).

Theorem 4.14 [45] *Let* $d \geq 5$ *be an fixed integer. Then*

$$r_d(m) = \Theta(m^{d/2-1}) \, .$$

For our purposes we will show the following theorem, which we will need for the proof of Theorem 4.4.

Theorem 4.15 *Let* $d \geq 3$ *be a fixed integer. Then*

$$\sum_{m=1}^{n}(r_d(m))^2 = \Theta(n^{d-1}) \, .$$

For $d \geq 5$, this theorem follows immediately from Theorem 4.14. Still some work has to be done for the cases $d = 3, 4$. It is known (see for example [40, p. 7]) that

$$\sum_{m=1}^{n} r_d(m) \sim \frac{\pi^{d/2}}{\Gamma(d/2+1)} \, n^{d/2} \, ,$$

i.e. the number of lattice points inside or on the sphere S of radius $\sqrt{n}$ around the origin in $\mathbb{R}^d$ is asymptotically the same as the volume of S. Thus, Cauchy's inequality[1] implies

$$\sum_{m=1}^{n}(r_d(m))^2 \geq \frac{1}{n} \cdot \left(\sum_{m=1}^{n} r_d(m)\right)^2 \sim \frac{\pi^d}{\Gamma^2(d/2+1)} \, n^{d-1} \, ,$$

[1] $(\sum a_i b_i)^2 \leq \sum a_i^2 \sum b_i^2$

which proves the lower bound of Theorem 4.15.

For the proof of the upper bound of Theorem 4.15 we will use the *Hardy-Littlewood method* as presented in [21] and [70]. Throughout the remaining part of this section let $N := \lfloor\sqrt{n}\rfloor$ and suppose $d \geq 3$. Define $R_d(m, N)$ as the number of representations of m as the sum of d squares of *positive* integers not exceeding N, i.e.

$$R_d(m) = \left|\left\{(x_1, \ldots, x_d) \in \mathbb{Z}^d : m = x_1^2 + x_2^2 + \ldots + x_d^2\ ,\ 1 \leq x_i \leq N\right\}\right|.$$

Clearly, $r_d(m) \geq R_d(m, N)$. On the other we have the following estimate.

Lemma 4.16 *Let $\epsilon > 0$ and $N = \lfloor\sqrt{n}\rfloor$, then*

$$\sum_{m=1}^{n} (r_d(m))^2 \leq c_d \sum_{m=1}^{n} (R_d(m, N))^2 + O(n^{d-3/2+\epsilon})$$

with a constant c_d depending only on d.

Proof. It is easy to see that we can estimate

$$R_d(m, N) \leq r_d(m) \leq 2^d \cdot R_d(m, N) + d \cdot r_{d-1}(m)$$

for $m \leq n$. By Theorem 4.6, $r_2(m) = O(m^{\epsilon'})$ for every $\epsilon' > 0$. Therefore, we have

$$r_s(m) \leq (\sqrt{m})^{s-2} \cdot O(m^{\epsilon'}) = O(m^{s/2-1+\epsilon'}) \tag{4.3}$$

for $s \geq 2$. We can then estimate

$$(r_d(m))^2 \leq 2^{2d} \cdot (R_d(m, N))^2 + O(m^{d-5/2+\epsilon})$$

and the assertion follows. ❏

By Lemma 4.16, it is sufficient to show $\sum_{m=1}^{n}(R_d(m, N))^2 \ll n^{d-1}$ in order to prove Theorem 4.15.

The first step of the Hardy-Littlewood method is to transform the sum into an integral. To this end we define $e(y) := e^{2\pi i y}$ and $f(\alpha) := \sum_{x=1}^{N} e(\alpha x^2)$ for real numbers α.

Lemma 4.17 *Let $N = \lfloor\sqrt{n}\rfloor$. Then,*

$$\sum_{m=1}^{dn} (R_d(m, N))^2 = \int_0^1 |f(\alpha)|^{2d}\, d\alpha.$$

Proof. First we observe that

$$(f(\alpha))^d = \sum_{x_1=1}^{N} \cdots \sum_{x_d=1}^{N} e\left(\alpha \cdot (x_1^2 + \ldots + x_d^2)\right) = \sum_{m=1}^{dn} R_d(m, N) \cdot e(\alpha m)\,.$$

Since $|z|^2 = z\bar{z}$ for $z \in \mathbb{C}$,

$$\left|(f(\alpha))^d\right|^2 = \sum_{m_1=1}^{dn} R_d(m_1, N) \cdot e(\alpha m_1) \sum_{m_2=1}^{dn} R_d(m_2, N) \cdot e(-\alpha m_2).$$

If we integrate this we obtain

$$\int_0^1 |f(\alpha)|^{2d}\, d\alpha = \sum_{1 \le m_1, m_2 \le dn} R_d(m_1, N) \cdot R_d(m_2, N) \cdot \int_0^1 e(\alpha \cdot (m_1 - m_2))\, d\alpha.$$

The integral on the right hand side is 0 for $m_1 \neq m_2$ and 1 for $m_1 = m_2$, thus

$$\int_0^1 |f(\alpha)|^{2d}\, d\alpha = \sum_{m=1}^{dn} (R_d(m, N))^2 .$$

❑

In order to estimate $\int_0^1 |f(\alpha)|^{2d}\, d\alpha$ we break the integration interval into two parts called major-arcs and minor-arcs. This is the second step of the Hardy-Littlewood method. The major-arcs will determine the order of magnitude while the contribution of the minor-arcs will be negligible.

In the following calculations let $0 < \delta < 1/5$ be a fixed real number. For integers a, q with $1 \le a \le q \le N^\delta$ and $(a, q) = 1$ define the *major-arcs* as

$$\mathcal{M}_{a,q} := \{\alpha \in \mathbb{R} : |\alpha - a/q| < N^{-2+\delta}\}.$$

We observe that these major-arcs are disjoint since their lengths are much smaller than the distances between their centers. Let $\mathcal{M}$ denote the union of the $\mathcal{M}_{a,q}$. It is convenient to shift the integration interval $(0, 1]$ to the right to $U := (N^{-2+\delta}, 1 + N^{-2+\delta}]$. As $f(\alpha) = f(\alpha + 1)$ we have

$$\int_0^1 |f(\alpha)|^{2d}\, d\alpha = \int_U |f(\alpha)|^{2d}\, d\alpha.$$

Now $\mathcal{M} \subset U$ by definition. The set $\mathrm{M} := U \setminus \mathcal{M}$ forms the *minor-arcs*.

To see that the minor-arcs can be neglected in order to prove Theorem 4.15 we use the following

Lemma 4.18 **[21]** *Suppose $s \ge 5$, then*

$$\int_{\mathrm{M}} |f(\alpha)|^s\, d\alpha = O(n^{s/2-1-\delta'}) ,$$

where $\delta' > 0$ is a constant depending on δ.

Using this, we immediately get for $d \geq 3$, that

$$\int_{\mathrm{M}} |f(\alpha)|^{2d} \, d\alpha = O(n^{d-1-\delta'}) \ , \tag{4.4}$$

where δ' is a constant depending on δ.

Thus it remains to prove that

$$\int_{\mathcal{M}} |f(\alpha)|^{2d} \, d\alpha = O(n^{d-1}) \ .$$

Define

$$S_{a,q} := \sum_{z=1}^{q} e(az^2/q) \qquad \text{and} \qquad I(\beta) := \int_0^N e(\beta\xi^2) \, d\xi.$$

For the treatment of the major-arcs we use the following lemmata.

Lemma 4.19 **[21]** *If $\alpha \in \mathcal{M}_{a,q}$, then*

$$f(\alpha) = V(\alpha) + O(N^{2\delta})$$

with $V(\alpha) := q^{-1} \cdot S_{a,q} \cdot I(\alpha - a/q)$.

The notation $V(\alpha)$ is a bit sloppy because V also depends on a and q. Nevertheless there should be no confusion since α determines the major-arc $\mathcal{M}_{a,q}$ in which it lies.

Lemma 4.20 **[21]** *Let a, q be relative prime integers with $q > 0$. Then for every $\epsilon > 0$,*

$$|S_{a,q}| = O(q^{1/2+\epsilon}).$$

Now let $\alpha \in \mathcal{M}_{a,q}$. Lemma 4.19 and the obvious fact that

$$|V(\alpha)| = |q^{-1} \cdot S_{a,q} \cdot I(\alpha - a/q)| \leq N$$

yields

$$(f(\alpha))^{2d} = (V(\alpha))^{2d} + O(N^{2d-1+2\delta}).$$

Since the measure of $\mathcal{M}$ is bounded by $O(N^{-2+3\delta})$ and since $\delta < 1/5$ we can again neglect the O-term and consider only $\int_{\mathcal{M}} |V(\alpha)|^{2d} \, d\alpha$.

By Lemma 4.20 we see that

$$|V(\alpha)|^{2d} \leq |q^{-1} S_{a,q}|^{2d} \cdot |I(\alpha - a/q)|^{2d} \ll q^{-d+\epsilon'} \cdot |I(\alpha - a/q)|^{2d}$$

for $\alpha \in \mathcal{M}_{a,q}$. Substituting $\alpha = a/q + \beta$ implies

$$\int_{\mathcal{M}_{a,q}} |V(\alpha)|^{2d} \, d\alpha \ll q^{-d+\epsilon'} \cdot \int_{|\beta|<N^{-2+\delta}} |I(\beta)|^{2d} \, d\beta.$$

Summing up over all major-arcs gives

$$\sum_{q=1}^{N^\delta} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathcal{M}_{a,q}} |V(\alpha)|^{2d} \, d\alpha \ll \sum_{q=1}^{N^\delta} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-d+\epsilon'} \cdot \int_{|\beta|<N^{-2+\delta}} |I(\beta)|^{2d} \, d\beta$$

$$\leq \left(\sum_{q=1}^{\infty} q^{1-d+\epsilon'} \right) \cdot \int_{|\beta|<N^{-2+\delta}} |I(\beta)|^{2d} \, d\beta = \text{const} \cdot \int_{|\beta|<N^{-2+\delta}} |I(\beta)|^{2d} \, d\beta .$$

The infinite sum converges since $d \geq 3$. By substituting $\xi = N \cdot t$ and $\beta = N^{-2} \cdot \gamma$ in $I(\beta) = \int_0^N e(\beta \xi^2) \, d\xi$ we obtain

$$\begin{aligned} \int_{|\beta|<N^{-2+\delta}} |I(\beta)|^{2d} \, d\beta &= N^{2d-2} \cdot \int_{|\gamma|<N^\delta} \left| \int_0^1 e(\gamma t^2) \, dt \right|^{2d} d\gamma \\ &\leq n^{d-1} \cdot \int_{|\gamma|<N^\delta} \left| \int_0^1 e(\gamma t^2) \, dt \right|^{2d} d\gamma \ , \end{aligned} \tag{4.5}$$

since $N = \lfloor \sqrt{n} \rfloor$.

We want to show that the outer integral is bounded from above independent of n. By substituting $t = \gamma^{-1/2} z$ and by the fact that $\int_0^\infty \cos(x^2) \, dx$ and $\int_0^\infty \sin(x^2) \, dx$ are bounded [16] it follows that

$$\left| \int_0^1 e(\gamma t^2) \, dt \right| = \left| \gamma^{-1/2} \cdot \int_0^{\gamma^{1/2}} e(z^2) \, dz \right| \ll \gamma^{-1/2} \ , \tag{4.6}$$

for $\gamma \geq 0$. On the other hand, obviously

$$\left| \int_0^1 e(\gamma t^2) \, dt \right| \leq 1 \ . \tag{4.7}$$

This enables us to extend the integration in (4.5) to infinity:

$$\begin{aligned} \int_{-\infty}^{+\infty} \left| \int_0^1 e(\gamma t^2) \, dt \right|^{2d} d\gamma &= \int_{-1}^{+1} \left| \int_0^1 e(\gamma t^2) \, dt \right|^{2d} d\gamma + 2 \cdot \int_1^{+\infty} \left| \int_0^1 e(\gamma t^2) \, dt \right|^{2d} d\gamma \\ &\ll 2 \cdot 1 + 2 \cdot \int_1^{+\infty} \gamma^{-d} \, d\gamma = O(1) \end{aligned}$$

since $d \geq 3$. There we made use of (4.6) and (4.7). We infer that $\int_{\mathcal{M}} |V(\alpha)|^{2d} \, d\alpha \ll n^{d-1}$. Thus we have proved

Lemma 4.21

$$\int_{\mathcal{M}} |f(\alpha)|^{2d}\, d\alpha = O(n^{d-1}).$$

Because of the fact that $\int_0^1 = \int_{\mathcal{M}} + \int_{\mathrm{M}}$ and by (4.4) we have

Corollary 4.22

$$\int_0^1 |f(\alpha)|^{2d}\, d\alpha = O(n^{d-1}).$$

Now Lemma 4.16 and 4.17 and Corollary 4.22 imply Theorem 4.15. ❑

4.2 Grid points

In this section we will give the proofs for Theorems 4.2 and 4.4. First we will show Theorem 4.2.

Proof of Theorem 4.2. Consider the $[n]^2$-grid. We form a complete graph K_{n^2} with vertex set $[n]^2$. We color the edges $\{x, y\}$ by the square of the euclidean distance of their endpoints x and y. For fixed positive integer t and every grid point v, notice that the number of grid points w with euclidean distance $\sqrt{t}$ from v is bounded from above by the number of representations of t as a sum of two squares. Hence, the average degrees $\bar{d}_t$ of color t satisfy $\bar{d}_t \leq r_2(t)$, and for the maximum degree Δ we have that

$$\Delta \leq n^{c_1/\ln\ln n}\ , \tag{4.8}$$

where c_1 is a positive constant, by Theorem 4.6. By Theorem 4.8 we have

$$\sum_{t=1}^{2(n-1)^2} \bar{d}_t^{\,2} \leq \sum_{t=1}^{2(n-1)^2} (r_2(t))^2 = c_2 \cdot n^2 \cdot \ln n\ ,$$

where c_2 is a positive constant. Setting $\tau = C \cdot n^2 \cdot \ln n$ for a positive constant C, which is large enough, we see with (4.8) that (i) and (ii) in Theorem 3.10 are satisfied (notice that the number of vertices is n^2). Hence, there exists a totally multicolored complete subgraph on k vertices with

$$k \geq c' \cdot \left(\frac{n^4}{n^2 \cdot \ln n}\right)^{1/3} \cdot (\ln n^2)^{1/3} \geq c \cdot n^{2/3}\ .$$

The vertices of this subgraph determine k points in the $[n]^2$-grid with mutual distinct distances. We remark that we could have also used Corollary 4.12 to obtain the same conclusion. ❑

Next we will prove Theorem 4.4.

Proof of Theorem 4.4. Fix a positive integer $d \geq 3$. We proceed as in the proof given above by coloring the edges of the complete graph on n^d vertices by the square of the euclidean distances of the corresponding endpoints. Clearly, $\bar{d}_t \leq r_d(t)$ and

$$\Delta \leq n^{d/2-1+\epsilon'}$$

for any fixed $\epsilon' > 0$ by (4.3).

Now,

$$\sum_{t=1}^{d(n-1)^2} \bar{d}_t^{\,2} \leq \sum_{t=1}^{d(n-1)^2} (r_d(t))^2 = O(n^{2d-2}) \tag{4.9}$$

by Theorem 4.15. With $\tau = C \cdot n^{2d-2}$, where $C > 0$ is a large enough constant, requirements (i) and (ii) of Theorem 3.10 are satisfied. Hence there exists a totally multicolored complete subgraph on k vertices with

$$k \geq c(d) \cdot \left(\frac{n^{2d}}{n^{2d-2}}\right)^{1/3} \cdot \left(\ln n^d\right)^{1/3} \geq c_d \cdot n^{2/3} \cdot (\ln n)^{1/3} ,$$

which yields the desired result. ❏

4.3 Points in arbitrary position

In this section we will study the maximum cardinality of a subset of n arbitrary points in the euclidean space $\mathbb{R}^d$, such that the mutual distances among the points of X are distinct. More formally, we want to determine

$$\min_{\substack{P \subset \mathbb{R}^d \\ |P|=n}} \max\{|X| : X \subseteq P \text{ determines distinct distances}\} .$$

This question was raised by Erdős and Guy in [36]. For $d = 1$ this problem is asymptotically solved as the following is known [31]: Given a set P of n arbitrary points on a line. Then there exists a subset $X \subseteq P$ with distinct distances of size $|X| \geq c\,n^{1/2}$. By considering n equidistant points on a line, an upper bound of $c'\,n^{1/2}$ follows.

We will also consider the same question for n points in *general position* (no $d+1$ points on a common hyperplane) in $\mathbb{R}^d$.

Our focus is the planar case. For this we need to study the *distribution of distances* of point sets in $\mathbb{R}^2$. One of the famous problems in combinatorial geometry is the following old question of Erdős [27]: How often can the same distance (w.l.o.g. the unit distance) occur in a set of n points in the plane? This innocent looking question has attracted many researchers and still there is no satisfying answer. Suppose we are given a set P of n points in the plane that determine distinct distances $d_1, \ldots, d_t$, where distance d_i occurs with multiplicity m_i for $i = 1, 2, \ldots, t$. Clearly, we have $\sum m_i = \binom{n}{2}$. Erdős' question is thus to determine

$$u(n) = \max_{\substack{P \subset \mathbb{R}^2 \\ |P|=n}} \max_i m_i(P)\,.$$

As already observed by Erdős [27], the $[\sqrt{n}]^2$-grid yields a lower bound of $u(n) \geq n^{1+c/\ln\ln n}$, where $c > 0$ is a constant. No improvement has been made on this bound. The best upper bound is the following.

Theorem 4.23 (Spencer et al [65]) *Let $u(n)$ denote the maximum number of unit distances occurring in a set of n points in the plane. Then,*

$$u(n) \leq c\, n^{4/3}\,,$$

where $c > 0$ is a constant.

Remark. Let $u_d(n)$ denote the maximum number of unit distances that can occur in a set of n points in $\mathbb{R}^d$. The current bounds for $d = 3$ are

$$c_1 n^{4/3} \log\log n \leq u_3(n) \leq c_1' n^{3/2} \beta(n) \tag{4.10}$$

where $\beta(n)$ is an extremely slowly growing function depending on the inverse of Ackermann's function. The lower bound, which is due to Erdős [27], follows from the $[n^{1/3}]^3$-grid. The upper bound has been proved by Clarkson et al [19] by using bounds on incidences between points and unit spheres in $\mathbb{R}^3$. Surprisingly, the problem seems to be easier for $d \geq 4$ as the following precise result of Erdős [29] is known.

$$u_d(n) = \frac{n^2}{2}\left(1 - \frac{1}{\lfloor d/2 \rfloor} + o(1)\right) \qquad \text{for } d \geq 4\,. \tag{4.11}$$

The lower bound is based on a construction of H. Lenz, while the upper bound uses results from extremal graph theory. For a good overview on this and related problems see [54].

The following new result, which we will need for our purposes, provides us with additional information about the distribution of distances in planar point sets.

Theorem 4.24 *Let n arbitrary points in the plane be given. Let the n points determine distinct distances $d_1, d_2, \ldots, d_t$, where distance d_i occurs with multiplicity m_i for $i = 1, 2, \ldots, t$. Then,*

$$\sum_{i=1}^{t} m_i^2 \leq c \cdot n^{13/4}\,,$$

where $c > 0$ is a constant.

Remarks. (1) Theorem 4.23 shows that under the assumptions of Theorem 4.24 one has $m_i \leq c' \cdot n^{4/3}$ for $i = 1, 2, \ldots, t$, where c' is a positive constant. Their result applied in the straightforward way yields $\sum_{i=1}^{t} m_i^2 \leq \max\{m_1, m_2, \ldots, m_t\} \cdot \sum_{i=1}^{t} m_i \leq c' \cdot n^{4/3} \cdot \binom{n}{2} \leq cn^{10/3}$ for a positive constant c. Another way to get this upper bound is by using the result of Pach and Sharir [55] that the number of isosceles triangles is bounded by $O(n^{7/3})$. Then by Lemma 4.10 one obtains that $\sum_{i=1}^{t} m_i^2 \leq c_2 n^{10/3}$, where c_2 is a positive constant.

(2) For the points of the $[\sqrt{n}]^2$-grid we have $\sum m_i^2 = \Theta(n^3 \ln n)$ by Theorem 4.8. One might conjecture that this upper bound holds for any set of n points in the euclidean plane.

(3) For the 3-dimensional case, we have the upper bound $\sum m_i^2 \leq u_3(n) \cdot \binom{n}{2} \leq c \cdot n^{7/2}\beta(n)$ by (4.10). Furthermore, if $d \geq 4$ the sum $\sum m_i^2$ can be of order $\Omega(n^4)$, since $u_d(n) = \Theta(n^2)$ by (4.11).

For the proof of Theorem 4.24 we will use

Lemma 4.25 *Let $0 = S_0 \leq S_1 \leq \ldots \leq S_t$ and $m_1 \geq m_2 \geq \ldots \geq m_t \geq 0$ be sequences of real numbers such that*

$$\sum_{i=1}^{j} m_i \leq S_j \qquad \text{for } j = 1, 2, \ldots, t, \tag{4.12}$$

and let $\phi : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ be a strictly convex[2] *monotonically increasing function. Then*

$$\sum_{i=1}^{t} \phi(m_i) \leq \sum_{i=1}^{t} \phi(S_i - S_{i-1}) . \tag{4.13}$$

Proof. We will apply induction on t. For $t = 1$ the assertion is trivial since ϕ is monotonically increasing, hence assume $t > 1$ and that the conclusion of the lemma holds for all values $1, 2, \ldots, t-1$. For given $S_0, S_1, \ldots, S_t$ it suffices to show (4.13) for any sequence $m_1 \geq m_2 \geq \ldots \geq m_t$ satisfying (4.12) and maximizing the expression $\sum_{i=1}^{t} \phi(m_i)$. Let $m_1, m_2, \ldots, m_t$ be such a sequence. If $m_t = 0$, then (4.13) is clearly satisfied by the induction hypothesis, hence we can assume that $m_t > 0$.

Suppose first that $\sum_{i=1}^{j} m_i < S_j$ for all $j = 1, 2, \ldots, t-1$. Then define a new sequence $m_1^*, m_2^*, \ldots, m_t^*$ as follows

$$\begin{aligned} m_1^* &= m_1 + \epsilon \\ m_i^* &= m_i \qquad \text{for } i = 2, 3, \ldots, t-1 \\ m_t^* &= m_t - \epsilon \end{aligned}$$

where $\epsilon = \min\{m_t, S_1 - m_1, S_2 - (m_1 + m_2), \ldots, S_{t-1} - \sum_{i=1}^{t-1} m_i\}$. Since $\epsilon > 0$ by assumption and since ϕ is strictly convex we have $\sum_{i=1}^{t} \phi(m_i^*) > \sum_{i=1}^{t} \phi(m_i)$, which contradicts the maximality of the sequence $m_1, m_2, \ldots, m_t$.

Thus there is a k, $1 \leq k \leq t-1$, with

$$\sum_{i=1}^{k} m_i = S_k . \tag{4.14}$$

[2] i.e. $\phi(tx + (1-t)\,y) < t\,\phi(x) + (1-t)\,\phi(y)$ for $t \in (0,1)$ and $x, y \in \mathbb{R}_0^+$

By the induction assumption it follows that

$$\sum_{i=1}^{k} \phi(m_i) \le \sum_{i=1}^{k} \phi(S_i - S_{i-1}) \ . \tag{4.15}$$

By (4.12) and (4.14) we have $\sum_{i=k+1}^{j} m_i \le S_j - S_k$ for $j = k+1, k+2, \ldots, t$. Using again the induction assumption we infer that

$$\sum_{i=k+1}^{t} \phi(m_i) \le \sum_{i=k+1}^{t} \phi((S_i - S_k) - (S_{i-1} - S_k)) = \sum_{i=k+1}^{t} \phi(S_i - S_{i-1}) \ , \tag{4.16}$$

thus combining (4.15) and (4.16) we obtain $\sum_{i=1}^{t} \phi(m_i) \le \sum_{i=1}^{t} \phi(S_i - S_{i-1})$, which finishes the induction step. ❏

Let P be a set of points in the plane and let C be a set of circles in $\mathbb{R}^2$. Define a bipartite graph G with vertex set $P \cup C$ and edge set E, where $(p, c) \in E$, $p \in P$ and $c \in C$, if and only if p lies on the circle c. Let $I(P, C)$ denote the number of *incidences* between points and circle, that is, the number of edges in this bipartite graph G. In our arguments we will use the following result of Clarkson, Edelsbrunner, Guibas, Sharir and Welzl:

Theorem 4.26 [19] *Let P be a set of points in $\mathbb{R}^2$ and let C be a set of circles in $\mathbb{R}^2$. Then*

$$I(P, C) = O\left(|P|^{3/5} \cdot |C|^{4/5} + |P| + |C|\right) \ . \tag{4.17}$$

Now we are ready to prove Theorem 4.24.

Proof of Theorem 4.24. Let $P \subset \mathbb{R}^2$ be a set of n points in the plane. Assume that $m_1 \ge m_2 \ge \ldots \ge m_t$. Around each point $p \in P$ draw circles with radius $d_1, d_2, \ldots, d_t$. For $j = 1, 2, \ldots, t$ let C_j be the set of all such circles with radius $d_1, d_2, \ldots, d_j$. Then we have $|C_j| = jn$ and

$$I(P, C_j) = 2 \cdot \sum_{i=1}^{j} m_i \ . \tag{4.18}$$

Thus, by (4.17) we have

$$I(P, C_j) \le c_1 \cdot n^{3/5} \cdot (jn)^{4/5} \ , \tag{4.19}$$

where $c_1 > 0$ is a constant.

Combining (4.18) and (4.19), we infer that

$$\sum_{i=1}^{j} m_i \le \frac{c_1}{2} \cdot n^{7/5} \cdot j^{4/5}$$

for $j = 1, 2, \ldots, t$.

On the other hand, we have

$$\sum_{i=1}^{l} m_i \leq \binom{n}{2} < \frac{n^2}{2} \qquad (4.20)$$

for $l = 1, 2, \ldots, t$.

Put $c_2 = \max\{1/2, c_1/2\}$ and

$$S_j = \begin{cases} c_2 \cdot n^{7/5} \cdot j^{4/5} & \text{if } 0 \leq j \leq n^{3/4} \\ c_2 \cdot n^2 & \text{if } n^{3/4} < j \leq t. \end{cases}$$

Clearly, the sequences $S_0, S_1, \ldots, S_t$ and $m_1, m_2, \ldots, m_t$ satisfy the assumptions of Lemma 4.25 with $\phi(x) := x^2$. Hence,

$$\sum_{i=1}^{t} m_i^2 \leq \sum_{i=1}^{t} (S_i - S_{i-1})^2 \leq \sum_{i=1}^{n^{3/4}} (S_i - S_{i-1})^2 \leq c_2^2 \cdot n^{14/5} \cdot \sum_{i=1}^{n^{3/4}} \left(i^{4/5} - (i-1)^{4/5}\right)^2 . \quad (4.21)$$

Since $i^{4/5} - (i-1)^{4/5} \leq i^{-1/5}$ for $i \geq 1$, (4.21) becomes

$$\sum_{i=1}^{t} m_i^2 \leq c_2^2 \cdot n^{14/5} \cdot \sum_{i=1}^{n^{3/4}} i^{-2/5} . \qquad (4.22)$$

The function $g(x) = x^{-2/5}$, $x > 0$, is decreasing, thus

$$\sum_{i=1}^{n^{3/4}} i^{-2/5} \leq 1^{-2/5} + \int_1^{n^{3/4}} x^{-2/5} dx = 1 + \frac{5}{3}(n^{9/20} - 1) < \frac{5}{3} n^{9/20} .$$

We infer with (4.22) that

$$\sum_{i=1}^{t} m_i^2 \leq c \cdot n^{13/4}$$

for some constant $c > 0$. ❏

Remark. Using Lemma 4.25 in a similar way as in the proof of Theorem 4.24 one can also derive the following upper bounds on higher moments for the distribution of distances. As before, let $m_1, \ldots, m_t$ denote the multiplicities of the distances occurring in a given set of n points in the plane. Then,

$$\begin{aligned} \sum_{i=1}^{t} m_i^k &= O\left(n^{\frac{5k+3}{4}}\right) && \text{for } k = 3, 4 \\ \sum_{i=1}^{t} m_i^5 &= O(n^7 \log n) \\ \sum_{i=1}^{t} m_i^k &= O\left(n^{\frac{7k}{5}}\right) && \text{for } k \geq 6 . \end{aligned}$$

Note that the bound $\max m_i = O(n^{4/3})$ given in Theorem 4.23 implies $\sum m_i^k = O(n^{(4k+2)/3})$, which is weaker for $2 \le k \le 9$.

Surprisingly, the situation changes radically if our point set is in *general position*. More generally, we have

Theorem 4.27 *Let P be a set of n points in $\mathbb{R}^d$ such that at most s points are on a common hyperplane. If these points determine distinct distances $d_1, d_2, \ldots, d_t$ with corresponding multiplicities $m_1, m_2, \ldots, m_t$, then*

$$\sum_{i=1}^{t} m_i^2 \le \frac{(s+1)n}{2} \cdot \binom{n}{2} . \tag{4.23}$$

Proof. We will give an upper bound for $\Delta(P)$ (see Definition 4.9). By assumption the perpendicular bisector of any two points p and q contains at most s points of P. Thus,

$$\Delta(P) \le s \cdot \binom{n}{2} . \tag{4.24}$$

By Lemma 4.10 we infer that

$$\sum_{i=1}^{t} m_i^2 \le \frac{n}{2} \cdot \left(\Delta(P) + \binom{n}{2}\right) \le \frac{(s+1)n}{2} \cdot \binom{n}{2} .$$

❏

Theorem 4.28 *Let P be a set of n points in the plane in general position. If these points determine the distinct distances $d_1, d_2, \ldots, d_t$ with multiplicities $m_1, m_2, \ldots, m_t$, then*

$$\sum_{i=1}^{t} m_i^2 \le \frac{3}{4} n^2 (n-1) . \tag{4.25}$$

If we further assume that the n points are in convex position, then

$$\sum_{i=1}^{t} m_i^2 \le \frac{3}{4} n^2 (n-1) - \frac{n^2}{2} .$$

This answers a question of Erdős and Fishburn [38],[33]. In their paper [33] an even stronger statement is conjectured namely that for convex n-gons one has

$$¿ \qquad \sum_{i=1}^{t} m_i^2 \le \frac{n^2 \cdot (n-1)}{2} \qquad ?$$

for $n \ge 3$ being an odd integer. For $n \ge 10$ an even integer it is stated in [33] that perhaps

$$¿ \qquad \sum_{i=1}^{t} m_i^2 \le \frac{n^2 \cdot (2n-3)}{4} \qquad ?$$

In both cases the regular convex n-gon would be an extremal configuration attaining the upper bounds. For $n = 4, 6, 8$ they proved that this bound does not hold. However, the regular n-gons show that the bounds given in Theorem 4.28 are tight up to a small constant factor.

Remarks. (1) Füredi proved in [41] that under the assumptions of Theorem 4.28, i.e., P is a convex n-gon, one has $m_i \leq 12n \log n$ for $i = 1, 2, \ldots, t$. Applying this in the straightforward way, one gets $\sum_{i=1}^t m_i^2 \leq \sum_{i=1}^t m_i \cdot \max\{m_1, m_2, \ldots, m_t\} \leq 6n^3 \cdot \log n$.

(2) According to the second remark after Theorem 4.24 we see, that it really makes a difference to assume general position.

Proof of Theorem 4.28. If the points of P are in general position, then (4.25) follows by Theorem 4.27 with $s = 2$. So assume that the points of P determine a convex n-gon. Observe that the bisector of x and y contains at most one point from P if x and y are adjacent along the boundary of the convex hull of P. Thus,

$$\Delta(P) \leq 2\binom{n}{2} - n\ .$$

By Lemma 4.10 we infer that

$$\sum_{i=1}^t m_i^2 \leq \frac{n}{2}\left(2\binom{n}{2} - n + \binom{n}{2}\right) = \frac{3n^2(n-1)}{4} - \frac{n^2}{2}\ .$$

❑

It might be worth noting that the following upper bounds for the sum $\sum_{i=1}^t m_i^2$ match the conjectured upper bounds of Erdős and Fishburn stated above.

Proposition 4.29 *Let P be a set of n points in $\mathbb{R}^2$, which has the following property:*
(*) *no circle with center $p \in P$ contains three or more other points of P.*

Let these n points determine distinct distances $d_1, d_2, \ldots, d_t$ with corresponding multiplicities $m_1, m_2, \ldots, m_t$. Then,

$$\sum_{i=1}^t m_i^2 \leq \begin{cases} \frac{n^2(n-1)}{2} & \text{if } n \text{ is odd} \\ \frac{n^2(2n-3)}{4} & \text{if } n \text{ is even.} \end{cases}$$

Proof. By (*) we have $m_i \leq n$ for $i = 1, 2, \ldots, t$. Thus

$$\sum_{i=1}^t m_i^2 \leq \frac{\binom{n}{2}}{n} n^2 = \frac{n^2(n-1)}{2}\ ,$$

which shows the assertion for n being an odd integer.

For n even, consider as above the bipartite graph $G = ([P]^2 \cup P, E)$ with $\{\{x,y\}, z\} \in E$ if and only if z lies on the perpendicular bisector of x and y. To determine $|E| = \Delta(P)$, fix a point $p \in P$. Let $d(p)$ denote the degree of p in G. Then $d(p)$ is equal to the number of unordered pairs $\{x, y\} \in [P \setminus \{p\}]^2$ such that p lies on the perpendicular bisector of x and y. Property (*) implies that these pairs form a matching. As n is even, we obtain $d(p) \leq (n-2)/2$ and therefore

$$\Delta(P) = \sum_{p \in P} d(p) \leq \frac{n(n-2)}{2} .$$

With Lemma 4.10 we infer that

$$\sum_{i=1}^{t} m_i^2 \leq \frac{n}{2} \left(\Delta(P) + \binom{n}{2} \right) \leq \frac{n^2(2n-3)}{4} .$$

❑

Next we will consider the corresponding selection problems.

Theorem 4.30 *Let P be a set of n points in general position in $\mathbb{R}^d$. Then there exists a subset $X \subseteq P$ with mutual distinct distances such that*

$$|X| \geq c_d \cdot n^{1/3}$$

for some positive constant $c_d > 0$.

Remark. For $d = 2$, an upper bound of $O(n^{1/2})$ is given by the regular n-gon.

Proof. Let $P = \{p_1, p_2, \dots, p_n\}$ be given as above. Let $d_1, d_2, \dots, d_t$ be the occurring distinct distances with corresponding multiplicities $m_1, m_2, \dots, m_t$. Consider the complete graph with vertex set P. Each edge $\{p, q\} \in [P]^2$ is colored by the euclidean distance of its endpoints. A rainbow with respect to this coloring, denoted by f, corresponds to a subset $X \subseteq P$ with distinct distances. Our aim is to apply Lemma 3.6. To this end, let $\mathcal{E}_3$ be the family of 3-sets of P that determine two equal distances and let $\mathcal{E}_4$ denote the family of 4-sets of P that determine two equal distances not incident to a common vertex.

To bound $|\mathcal{E}_3|$ consider two points $p, p' \in P$ and let $Q = \{q \in P : d(q,p) = d(q,p')\}$. We observe that all points in Q lie on the perpendicular bisector of p and p'. Since P is in general position we know that $|Q| \leq d$, thus

$$|\mathcal{E}_3| \leq \binom{n}{2} \cdot d < \frac{d}{2} \cdot n^2 . \tag{4.26}$$

Concerning $|\mathcal{E}_4|$, we have by Theorem 4.27 that

$$|\mathcal{E}_4| \leq \sum_{i=1}^{t} \binom{m_i}{2} \leq c'_d \cdot n^3 . \tag{4.27}$$

By Lemma 3.6 we infer

$$r(f) \geq c_d \, n^{1/3} .$$

Hence there exists a set $X \subseteq P$ of the desired size with distinct distances. ❑

Using Theorem 4.27 and Lemma 3.6 one can show in a similar fashion the following

Theorem 4.31 *Let P be set of n points in $\mathbb{R}^d$ such that at most s points of P lie on a common hyperplane. Then there exists a subset $X \subseteq P$ with mutual distinct distances such that*

$$|X| \geq c \cdot \left(\frac{n}{s}\right)^{1/3},$$

where $c > 0$ is an absolute constant (independent of s, d).

For n points in arbitrary position in the plane we have the following result.

Theorem 4.32 *Let P be a set of n points in the plane. Then there exists a subset $X \subseteq P$ with mutual distinct distances such that*

$$|X| \geq c \cdot n^{1/4},$$

where $c > 0$ is a constant.

This gives a partial answer to a question raised by Erdős and Guy [36]. Furthermore, this theorem improves a former result of [13, proof of Theorem 4.2], where the lower bound $|X| \geq c \cdot n^{1/5}$ has been implicitly given, as communicated to the author by J. Pach. An upper bound of $O(n^{1/2}/(\log n)^{1/4})$ follows from the $[\sqrt{n}]^2$-grid.

Proof. The arguments are similar to those used in the proof of Theorem 4.30. We consider the complete graph with vertex set P together with the edge coloring based on the euclidean distance of the endpoints. Pach and Sharir have shown in [55] that n points in the plane determine $O(n^{7/3})$ isosceles triangles, hence

$$|\mathcal{E}_3| \leq c_1 \cdot n^{7/3}$$

and by Theorem 4.24

$$|\mathcal{E}_4| \leq c_2 \cdot n^{13/4}.$$

By Lemma 3.6 we deduce,

$$r(f) \geq c n^{1/4}.$$

Thus, there exists a subset $X \subseteq P$ with distinct distances of size $|X| \geq c \cdot n^{1/4}$. ❑

For n arbitrary points in higher dimensions we have the following result.

Theorem 4.33 *Suppose $d \geq 2$. Let P be a set of n points in $\mathbb{R}^d$. Then there exists a subset $X \subseteq P$ with distinct distances of size*

$$|X| \geq c \cdot n^{\frac{1}{3d-2}},$$

where $c > 0$ is a constant (independent of d).

Remark. An upper bound of $O(n^{1/d})$ is given by the $[n^{1/d}]^d$-grid.

Proof. By induction on d. For $d = 2$ the assertion is given by Theorem 4.32. So suppose $d \geq 3$. Define $s := n^{(3d-5)/(3d-2)}$. We distinguish two cases:

(i) There is a hyperplane containing s points from P.

(ii) There is no hyperplane containing s points from P.

Consider case (i) first. Let $P' \subseteq P$ be a set of at least s points that is contained in a hyperplane. By the induction hypothesis, there is a subset $X \subseteq P'$ with distinct distances of size

$$|X| \geq c_1 \, s^{\frac{1}{3(d-1)-2}} = c_1 \, n^{\frac{1}{3d-2}}$$

proving the induction step for case (i).

Now consider case (ii). Since there are no s points on a common hyperplane we infer by Theorem 4.31 that there exists a subset $X \subseteq P$ with distinct distances of size

$$|X| \geq c_2 \left(\frac{n}{s}\right)^{1/3} = c_2 \, n^{\frac{1}{3d-2}} ,$$

which completes the induction step. ❑

4.4 Threshold functions

With Theorem 4.24 we see that for an arbitrary n-point set P in $\mathbb{R}^2$ the fraction of those k-element subsets of P which determine less than $\binom{k}{2}$ distinct distances is bounded from above by

$$\frac{\sum_{i=1}^{t} m_i^2 \cdot \binom{n-4}{k-4} + O(n^{7/3}) \cdot \binom{n-3}{k-3}}{\binom{n}{k}} = O\left(\frac{k^4}{n^{3/4}} + \frac{k^3}{n^{2/3}}\right) .$$

Thus if $k = o(n^{3/16})$ then *almost all* k-element subsets of P determine distinct mutual distances as n goes to infinity. This proves the following result.

Theorem 4.34 *Let n tend to infinity, $k = o(n^{3/16})$. Then almost all k-sets of a set of n points in the plane determine $\binom{k}{2}$ distinct distances.*

This improves a former result of [13], where the bound $k = o(n^{1/7})$ has been shown. With respect to an upper bound for this problem consider the following threshold result for the $[n]^2$-grid.

Theorem 4.35 *Let n tend to infinity. Then the following holds.*

(i) If $k = o\left(\frac{n^{1/2}}{\ln^{1/4} n}\right)$, then almost all k-sets of the $[n]^2$-grid determine $\binom{k}{2}$ distinct distances.

(ii) If $k = \omega\left(\frac{n^{1/2}}{\ln^{1/4} n}\right)$, then almost all k-sets of the $[n]^2$-grid determine less than $\binom{k}{2}$ distinct distances.

Remark. By considering the $[\sqrt{n}]^2$-grid, Theorem 4.35 (ii) implies that the bound $k = o(n^{3/16})$ in Theorem 4.34 cannot be improved beyond $k = o((n/\log n)^{1/4})$.

Proof. Consider the edge coloring of the complete graph with vertex set $[n]^2$ given by the euclidean distance of the endpoints. Note that a rainbow with respect to this coloring corresponds to a subset of the grid with distinct distances. Our aim is to apply Theorem 3.12. Let Δ be the maximum number of monochromatic edges incident to a common vertex and let t denote the number of 4-sets of the $[n]^2$-grid that determine two disjoint monochromatic edges.

By Theorem 4.6, the number of points equidistant to a fixed lattice point is bounded by $n^{c/\ln\ln n}$, where $c > 0$ is a constant. Thus, $\Delta \leq n^{c/\ln\ln n}$. Furthermore, by Theorem 4.8, we have $t = \Theta(n^6 \log n)$. Therefore, the condition $\Delta = O(t^{3/4}/(n^2)^2)$ is satisfied and Theorem 3.12 implies that

$$\tau(n) = n^2 \cdot t^{-1/4} \sim \frac{n^{1/2}}{\log^{1/4} n}$$

is a threshold function for the rainbow property, which completes the proof. ❑

Theorem 4.36 *Let $d \geq 3$ be a fixed integer and let n tend to infinity. Then the following holds.*

(i) If $k = o(\sqrt{n})$, then almost all k-sets of the $[n]^d$-grid determine $\binom{k}{2}$ distinct distances.

(ii) If $k = \omega(\sqrt{n})$, then almost all k-sets of the $[n]^d$-grid determine less than $\binom{k}{2}$ distinct distances.

Proof. Consider the edge coloring of the complete graph with vertex set $[n]^d$ given by the euclidean distance of the endpoints. Let Δ and t be defined as in the proof of Theorem 4.35.

By (4.3), the number of points equidistant to a fixed lattice point is bounded by $O(n^{d/2-1+\epsilon})$, for every constant $\epsilon > 0$. Thus, $\Delta = O(n^{d/2-1+\epsilon})$. Furthermore, by Theorem 4.15, we have $t = \Theta(n^{4d-2})$. Therefore, the condition $\Delta = O(t^{3/4}/(n^d)^2)$ is satisfied ($d \geq 3$ by assumption) and Theorem 3.12 implies that

$$\tau(n) = n^d \cdot t^{-1/4} \sim \sqrt{n}$$

is a threshold function for the rainbow property, which completes the proof. ❑

Considerations similar (but simpler) as in the proof of Theorems 4.35 and 4.36 yield the following two results. The easy proofs are left to the reader.

Proposition 4.37 *For every $n \in \mathbb{N}$ let P_n be a set of n equidistant points on a line. Then, as n goes to infinity, the following holds.*

(i) If $k = o(n^{1/4})$, then almost all k-sets of P_n determine $\binom{k}{2}$ distinct distances.

(ii) If $k = \omega(n^{1/4})$, then almost all k-sets of P_n determine less than $\binom{k}{2}$ distinct distances.

Proposition 4.38 *For every $n \in \mathbb{N}$ let P_n be the vertex set of the regular n-gon. Then, as n goes to infinity, the following holds.*

(i) If $k = o(n^{1/4})$, then almost all k-sets of P_n determine $\binom{k}{2}$ distinct distances.

(ii) If $k = \omega(n^{1/4})$, then almost all k-sets of P_n determine less than $\binom{k}{2}$ distinct distances.

By Theorem 4.24 and (4.26) it follows that for every n-point set P in $\mathbb{R}^2$ in general position the fraction of those k-element subsets of P which determine less than $\binom{k}{2}$ distinct distances is bounded from above by

$$\frac{\sum_{i=1}^{t} m_i^2 \cdot \binom{n-4}{k-4} + n^2 \cdot \binom{n-3}{k-3}}{\binom{n}{k}} = O\left(\frac{k^4}{n} + \frac{k^3}{n}\right) = O\left(\frac{k^4}{n}\right) .$$

Thus we have the following result.

Theorem 4.39 *Let n tend to infinity, $k = o(n^{1/4})$. Then almost all k-sets of a set of n points in the plane in general position determine $\binom{k}{2}$ distinct distances.*

By Proposition 4.38 this bound is tight since almost all k-sets of the points of the regular n-gon determine less than $\binom{k}{2}$ distinct distances for $k = \omega(n^{1/4})$.

4.5 $\mathbb{B}_2$-Sets of square numbers

For finite sets $X \subset \mathbb{N}$ a subset $S \subseteq X$ is called a $\mathbb{B}_2$-set (or Sidon set) if all pairwise sums $s + s'$, $s \neq s'$, are distinct. One is interested in the maximum size of S. For the case $X = \{1, 2, \ldots, n\}$ the maximum size of a $\mathbb{B}_2$-set $S \subseteq X$ is asymptotically well known by results from Erdős and Turán [37] to be $(1 + o(1)) \cdot n^{1/2}$. In [6] Alon and Erdős considered the maximum size of $\mathbb{B}_2$-subsets of the set $\{1^2, 2^2, \ldots, n^2\}$ consisting of the first n squares. Using an idea similar to the one given in the proof of Theorem 4.30 they showed the following:

Theorem 4.40 (Alon and Erdős [6]) *For every $\epsilon > 0$ there exists $c = c(\epsilon) > 0$ such that for every positive integer n there exists a $\mathbb{B}_2$-set $S \subset \{1^2, 2^2, \ldots, n^2\}$ with*

$$|S| \geq c \cdot n^{2/3-\epsilon} . \tag{4.28}$$

As already observed in [6], by a theorem of Landau [52] one has the upper bound $|S| \leq c' \cdot \frac{n}{(\ln n)^{1/4}}$. Here we will improve inequality (4.28), namely we will show:

Theorem 4.41 *For every integer $n \geq 1$ there exists a $\mathbb{B}_2$-set $S \subset \{1^2, 2^2, \ldots, n^2\}$ with*

$$|S| \geq c \cdot n^{2/3}, \tag{4.29}$$

where $c > 0$ is a constant.

The first idea to prove Theorem 4.41 might be to consider a complete graph with vertex set $V = \{1^2, 2^2, \ldots, n^2\}$ and a coloring of the edges, where the edge $\{i^2, j^2\}$ receives color $i^2 + j^2$. Then a totally multicolored complete subgraph on k vertices gives rise to a $\mathbb{B}_2$-subset of V of cardinality k. But Theorem 3.10 is not applicable to prove Theorem 4.41, as by condition (ii) we can only guarantee a totally multicolored complete subgraph of size less than $c \cdot n^{1/2}$. But it turns out, that with more refined counting arguments a similar strategy as used for the proof of Theorem 3.10 will show (4.29).

Proof. As in the proof of Theorem 3.10 we can assume that n is sufficiently large. In the following $c_1, c_2, \ldots, c_{10}$ are positive constants. We construct a 4-uniform hypergraph $\mathcal{G} = (V, \mathcal{E})$ with vertex set $V = \{1^2, 2^2, \ldots, n^2\}$ and edges $\{i^2, j^2, k^2, l^2\} \in \mathcal{E} \subseteq [V]^4$ if and only if $i^2 + j^2 = k^2 + l^2$. As the number of representations of any positive integer x by a sum of two squares is given by $r_2(x)$, we have by Theorem 4.8 that

$$|\mathcal{E}| \;\leq\; \sum_{i=1}^{2n^2} \binom{r_2(i)}{2} \leq c_1 \cdot n^2 \cdot \ln n. \tag{4.30}$$

Next we will count the number of 2-cycles in $\mathcal{G}$. To count $c_{2,2}(\mathcal{G})$ choose an edge $E \in \mathcal{E}$, say $E = \{i^2, j^2, k^2, l^2\}$ where $i^2 + j^2 = k^2 + l^2$. There are six possibilities to choose a two-element subset of E, say we choose $\{i^2, j^2\}$. Then the number of pairs $\{x^2, y^2\}$ with $i^2 + j^2 = x^2 + y^2$ is bounded from above by $r_2(i^2 + j^2) \leq n^{\frac{c_2}{\ln \ln n}}$, which follows from Theorem 4.6. Now consider those pairs $\{x^2, y^2\}$ with $i^2 + x^2 = j^2 + y^2$. Assuming $j > i$, we have $j^2 - i^2 = (x + y) \cdot (x - y)$, i.e. $(x + y)$ divides $j^2 - i^2$. Fixing this divisor fixes both x and y. Hence the number of such pairs $\{x^2, y^2\}$ is bounded from above by the number of divisors of $(j^2 - i^2)$, which is at most $n^{\frac{c_3}{\ln \ln n}}$ (see [48]). Summarizing these considerations, we have

$$c_{2,2}(\mathcal{G}) \leq c_4 \cdot n^2 \cdot \ln n \cdot n^{\frac{c_5}{\ln \ln n}}. \tag{4.31}$$

Concerning $c_{2,3}(\mathcal{G})$, we choose an edge $E \in \mathcal{E}$ and a three-element subset $T \subset E$. Then T can be extended in at most two ways to an edge $E' \in \mathcal{E} \setminus \{E\}$, thus

$$c_{2,3}(\mathcal{G}) \leq c_6 \cdot n^2 \cdot \ln n. \tag{4.32}$$

As in the proof of Theorem 3.10 we choose a random subset of V by picking vertices $v \in V$, independently of the others, with probability

$$p = n^{-1/3+\epsilon} \cdot (\ln n)^{-1/3},$$

where $\epsilon < \frac{1}{18}$. Let R be the arising random subset of V. Then,

$$\mathbf{Prob}(|R| \geq 0.99pn) = 1 - o(1) \tag{4.33}$$

and by (4.30) we have

$$\mathbf{E}(|[R]^4 \cap \mathcal{E}|) = p^4 \cdot |\mathcal{E}| \leq c_1 \cdot \frac{n^{2/3+4\epsilon}}{(\ln n)^{1/3}} \,. \qquad (4.34)$$

The expected number $\mathbf{E}(c_2(R))$ of 2-cycles in the subhypergraph induced on R can be bounded from above by (4.31) and (4.32) as follows:

$$\mathbf{E}(c_2(R)) = p^6 \cdot c_{2,2}(\mathcal{G}) + p^5 \cdot c_{2,3}(\mathcal{G}) \leq c_4 \cdot \frac{n^{6\epsilon + \frac{c_5}{\ln \ln n}}}{\ln n} + c_6 \cdot \frac{n^{1/3+5\epsilon}}{(\ln n)^{2/3}} = o(pn) \qquad (4.35)$$

for $\epsilon < \frac{1}{18}$.

As in the proof of Theorem 3.10 we infer with (4.33), (4.34) and (4.35) by using Chernoff's and Markov's inequality, deleting one point from each 2-cycle that there exists a subset $Y \subset V$, $|Y| \geq c_7 \cdot p \cdot n$ such that the subhypergraph $\mathcal{G}'$ of $\mathcal{G}$ induced on Y has no 2-cycles and has average degree at most $t^3 = c_9 \cdot n^{3\epsilon}$. By Lemma 2.21 applied to $\mathcal{G}'$ we see that

$$\alpha(\mathcal{G}) \geq \alpha(\mathcal{G}') \geq c_{10} \cdot \frac{n^{2/3+\epsilon}}{t \cdot (\ln n)^{1/3}} \cdot (\ln t)^{1/3} \geq c \cdot n^{2/3} \,,$$

which finishes the proof. ❑

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