

# How Many Conflicts Does It Need to be Unsatisfiable?

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Applications of Satisfiability Testing

# What is a Conflict?

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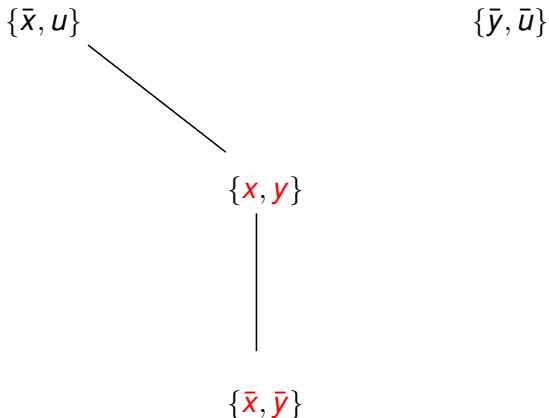
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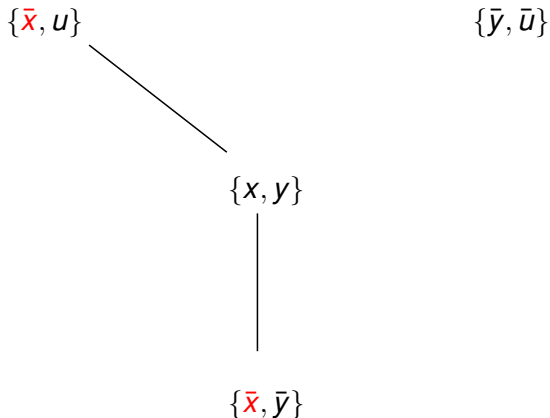
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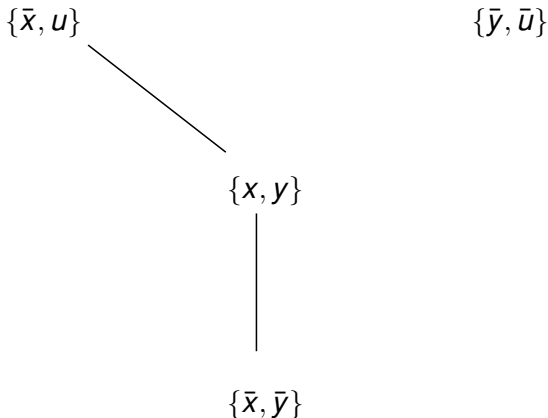
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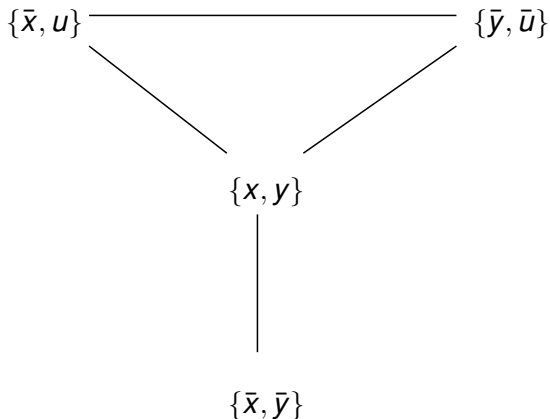
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- 4 *F has less than ??? conflicts.*

# Number of Conflicts—A Lower Bound

## Lemma

*Let  $F$  be a  $k$ -CNF formula. If  $F$  has  $< k2^{k-1}$  conflicts, then  $F$  is satisfiable.*

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How big is possible?

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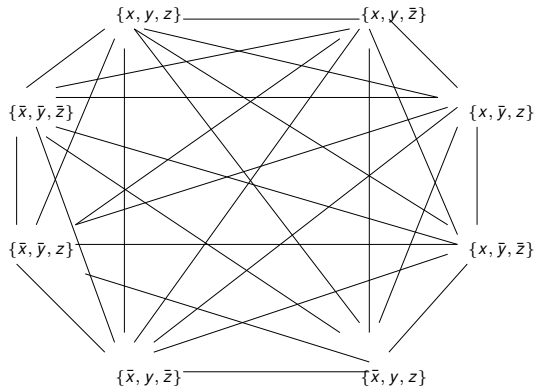
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- unsatisfiable  $k$ -CNF formula  $F$
- Every variable occurs in at most  $\mathcal{O}\left(\frac{\log(k)2^k}{k}\right)$  clauses
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Does there exist unsatisfiable  $k$ -CNF formulas with less conflicts?

## Theorem

*Any  $k$ -CNF formula with less than  $\mathcal{O}(2.69^k)$  conflicts is satisfiable.*

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**Proof.** Apply the Lovász Local Lemma. . .

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- If  $C$  has many neighbors, but every neighbor is extremely likely to be satisfied, that's okay too.

# The Lovász Local Lemma

## Lemma

*Let  $F$  be any CNF formula. Set every variable  $x$  of  $F$  to true with some probability  $p(x)$ , independently. If for every clause  $C$ , it holds that*

$$\sum_{D \in N(C)} \Pr[D \text{ not satisfied}] \leq \frac{1}{4}$$

*then  $F$  is satisfiable.*

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- By the Lovász Local Lemma,  $F$  is satisfiable. □

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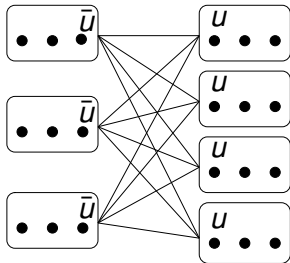
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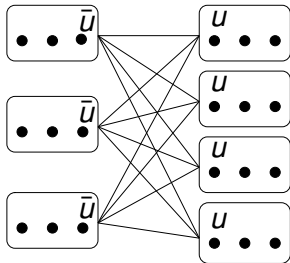
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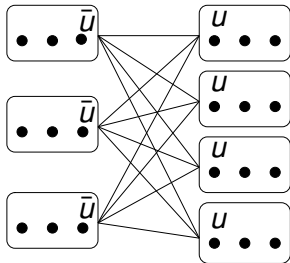


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- If  $F$  has  $\ll 4^k$  conflicts,  $\text{occ}(\bar{u}) \ll 2^k$ .

# Exploiting the Unbalancedness of Variables

- Number of conflicts  $\geq \text{occ}(u)\text{occ}(\bar{u})$ .
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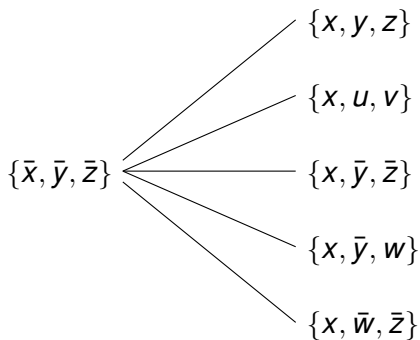
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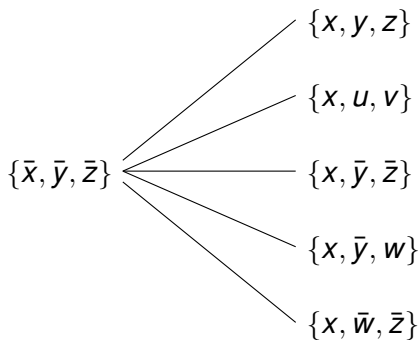
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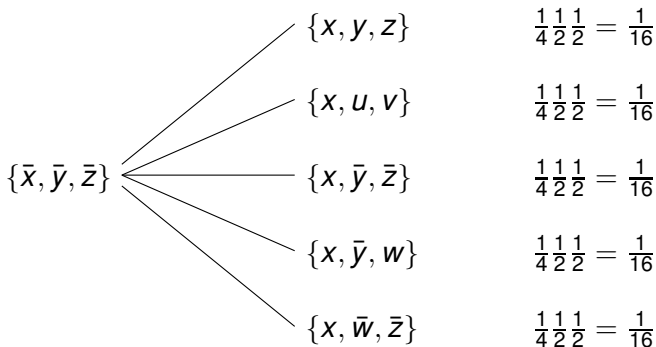


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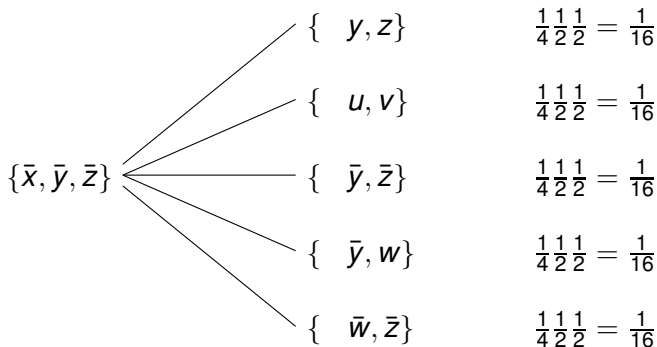
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- Remove certain literals  $\Rightarrow$  new formula  $F'$ .
- This “sparsifies” the conflict structure of  $F$ .

# Summary of Proof

- Assume  $F$  has few conflicts.
- Define an appropriate probability distribution, exploiting the unbalancedness of certain variables.
- Remove certain literals  $\Rightarrow$  new formula  $F'$ .
- This “sparsifies” the conflict structure of  $F$ .
- Use Lovász Local Lemma to show that  $F'$  is satisfiable.

## Theorem (Lower Bound)

*If a  $k$ -CNF  $F$  has at most  $\mathcal{O}(2.69^k)$  conflicts, it is satisfiable.*

## Theorem (Upper Bound)

*For every  $k$ , there is an unsatisfiable  $k$ -CNF formula with  $\mathcal{O}\left(\frac{\log^2(k)4^k}{k}\right)$  conflicts.*

Thank You For Your Attention!