Numerical Simulation of Dynamic Systems XXVII

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Introduction

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- Also of importance are special solvers for dealing with *marginally stable systems*, and indeed, an *F-stable QSS-based solver* has also been developed since the book went to press.

Introduction II

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- Finally, a few remarks shall be offered on the use of QSS-based algorithms in *real-time simulations*.

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We need some mechanism to adjust the size of the quantum automatically during the simulation in accordance with the values that the state variable assumes over time.

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This idea can be applied to all QSS methods.

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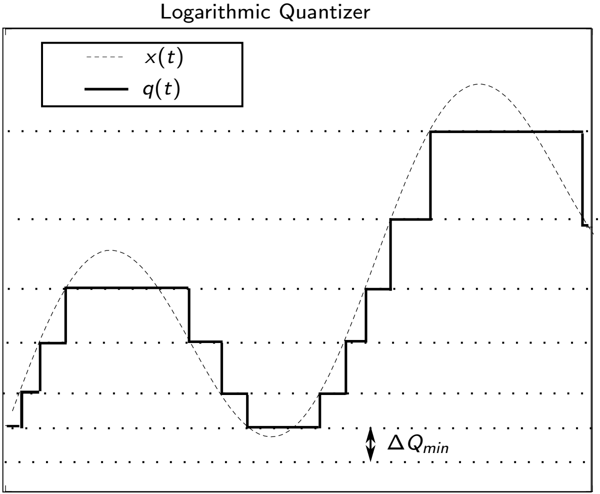

Perturbed Representation

Recall that the QSS approximation of a linear time-invariant ODE:

$$\dot{\mathbf{x}}_{\mathbf{a}}(t) = \mathbf{A} \cdot \mathbf{x}_{\mathbf{a}}(t) + \mathbf{B} \cdot \mathbf{u}(t)$$

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This perturbed representation does not depend on the type of quantization. However, with *logarithmic quantization*, the upper bound of $|\mathbf{\Delta x}(t))|$ depends on the state value $\mathbf{x}(t)$.

Stability and Error Bound

Assume that matrix $\mathbf{A} = \mathbf{V} \cdot \mathbf{\Lambda} \cdot \mathbf{V}^{-1}$ is Hurwitz, and define:

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Let $\mathbf{\Delta Q_{rel}}$ be a diagonal matrix with entries ΔQ_{rel_i}, and let $\mathbf{\Delta Q_{min}}$ be a column vector with components ΔQ_{min_i}. Then, provided that all eigenvalues of the matrix $\mathbf{R} \cdot \mathbf{\Delta Q_{rel}}$ lie inside the unit circle, it can be shown that:

$$|\mathbf{e}(t)| \leq (\mathbf{I^{(n)}} - \mathbf{R} \cdot \mathbf{\Delta Q_{rel}})^{-1} \cdot \mathbf{R} \cdot \max(\mathbf{\Delta Q_{rel}} \cdot \mathbf{x_{max}}, \mathbf{\Delta Q_{min}})$$

where $\mathbf{x_{max}}$ is the column vector of the maximum absolute values reached by each component of the analytical solution $\mathbf{x_a}(t)$.

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The use of logarithmic quantization yields an intrinsic control of the *relative error*.

QSS Methods and Stiff Systems

The linear time-invariant system:

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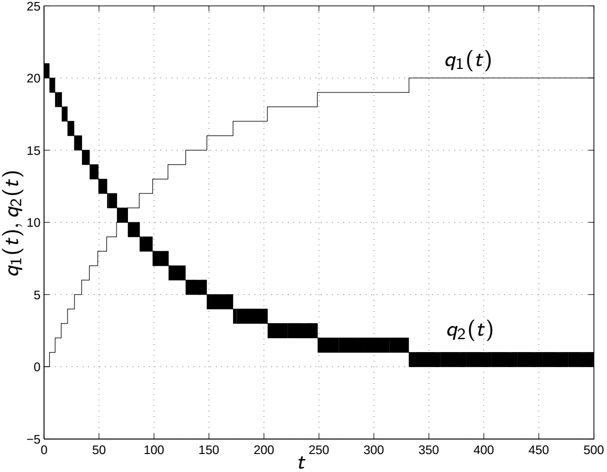
Let us simulate this system using the QSS solver with initial conditions $x_1(0) = 0$ and $x_2(0) = 20$ and with the quantization $\Delta Q_1 = \Delta Q_2 = 1$.

QSS Methods and Stiff Systems II



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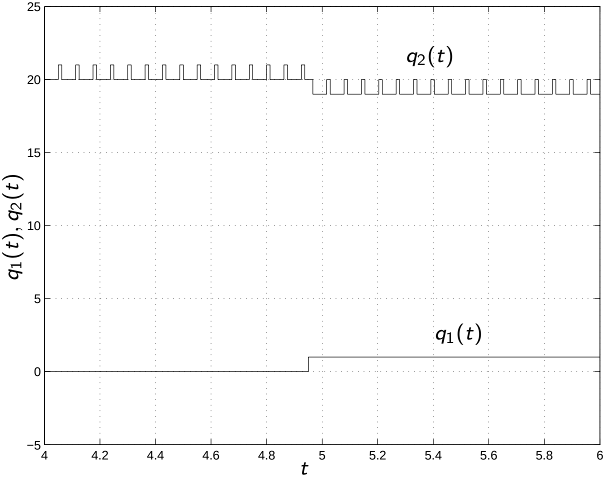

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Evidently, the QSS method is not appropriate for the simulation of stiff systems.

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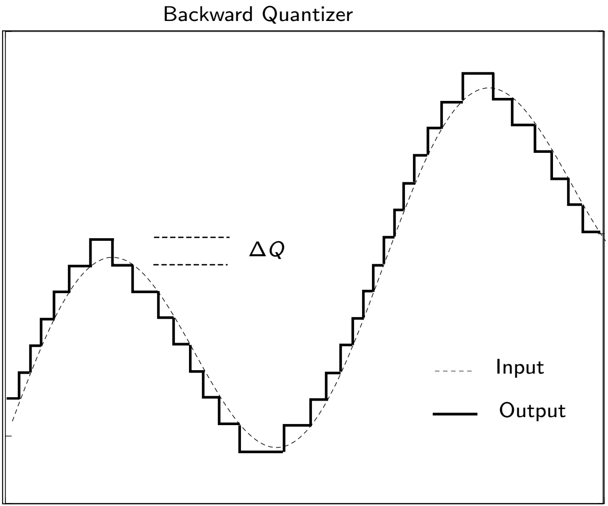
- If this happens, we keep $q_j(t) = x_j(t)$, but set $\dot{x}_j(t) = 0$, i.e., no further internal transition gets scheduled for the state variable in question.

This is an implicit algorithm, but it *does not require a Newton iteration*, as $q_j(t)$ can only assume one of two values at each step. The method is called *Backward QSS (BQSS) solver*.

Backward Quantization



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- In *non-stiff QSS solvers*, the trajectories of $q(t)$ and $x(t)$ at the beginning of the step always *move away from each other*. The next *internal transition* is scheduled to occur at time $\hat{t}$, where $|q(\hat{t}^-) - x(\hat{t})| = \Delta Q$. At that time, $q(t)$ is reset to $q(\hat{t}^+) = x(\hat{t})$.

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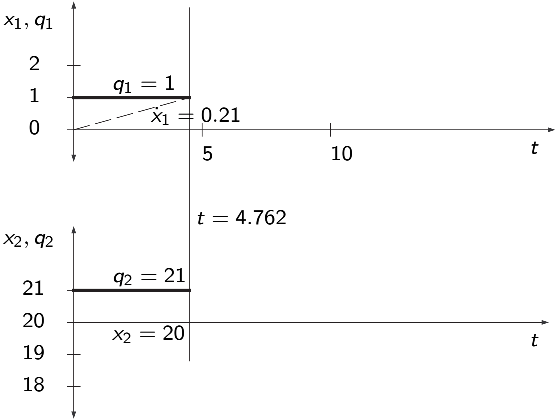
- *Higher-order stiff QSS solvers* need additional provisions that shall be discussed in due course.

Backward QSS: An Example

$\dot{x}_1 = 0.01 q_2 \qquad \dot{x}_2 = -100 q_1 - 100 q_2 + 2020$



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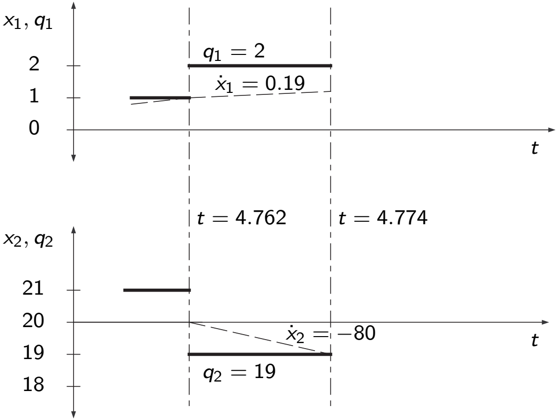

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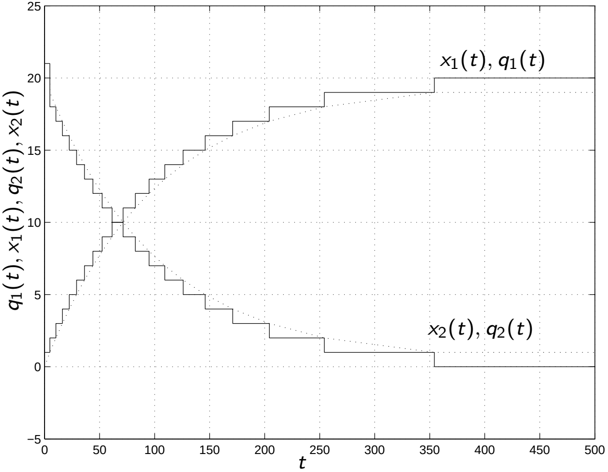
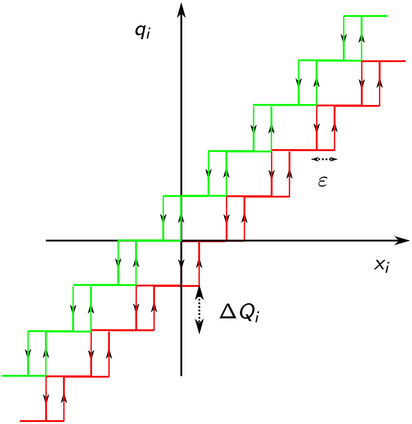


Quantization Functions in BQSS



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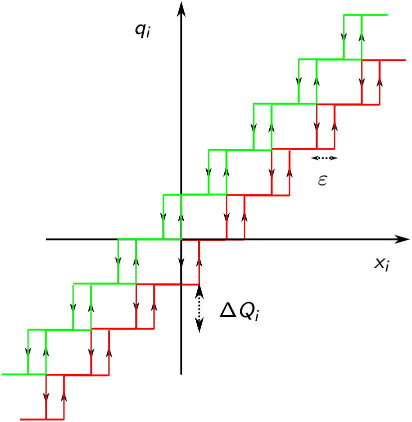

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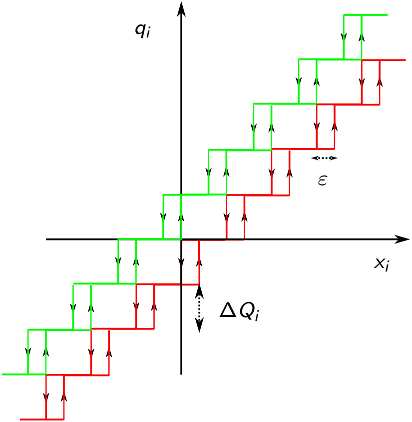

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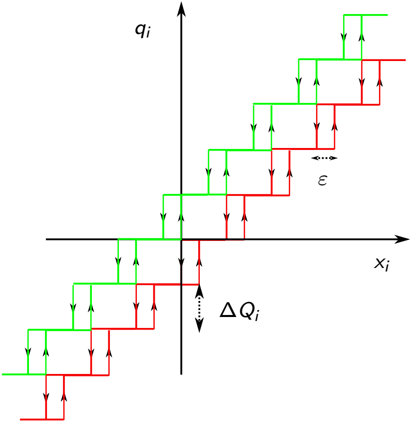

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Given the system:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$$

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where the components q_j of $\mathbf{q}$ take the values of either the lower quantized state $\underline{q}_j(t)$ or the upper quantized state $\overline{q}_j(t)$, such that:

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with $\hat{\mathbf{q}}^{(j)}(t) = [q_1, \ldots, \hat{q}_j, \ldots, q_n]^T$.

BQSS Definition II

In the BQSS approximation:

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the terms Δf_j will satisfy:

$$\Delta f_j = \begin{cases} 0, \text{ if } f_j(\mathbf{q}(t), \mathbf{u}(t)) \cdot (q_j - x_j) > 0 \\ -f_j(\mathbf{q}(t), \mathbf{u}(t)), \quad \text{otherwise} \end{cases}$$

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Algorithm:

1. We update $\underline{q}_i$ and $\overline{q}_i$ and choose $q_i = \overline{q}_i$ or $q_i = \underline{q}_i$ depending on the sign of $\dot{x}_i(t^-)$.
2. We evaluate the functions f_j depending on q_i.
3. **If** some f_j changes its sign:
   - We change q_j according to the new sign of $\dot{x}_j$.

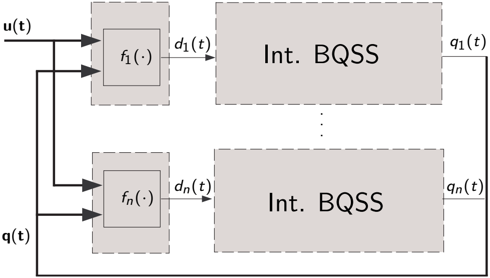
   Otherwise

   - End of the Algorithm.
4. We evaluate the functions f_j depending on the quantized states q_i that changed and we return to step 3.

Restriction: We do not allow any q_j to change more than once in the process.

BQSS and PowerDEVS

- The *block diagram structure* of BQSS is the same as that of QSS.



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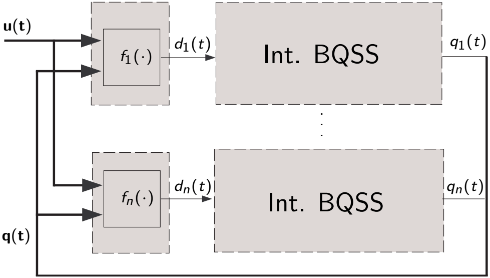

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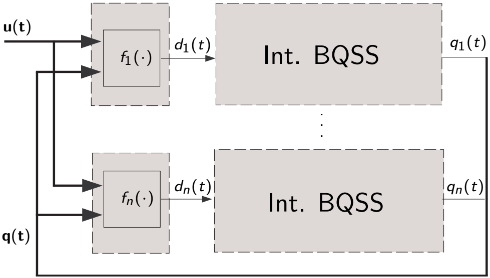

- The *static functions* are the same as for QSS.
- The *hysteretic quantized integrators* are similar, but:
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  - when the calculated value of q_i does not agree with the sign of $\dot{x}_i$, they act as if $\dot{x}_i = 0$.

Stability and Error Bound of BQSS

BQSS is very similar to **QSS**, but it has an additional *perturbation* term that enforces that $\dot{x}_i = 0$ when a consistent value for q_i cannot be found.

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{q}(t), t] + \mathbf{\Delta f}(t) = \mathbf{f}[\mathbf{x}(t) + \mathbf{\Delta x}(t), t] + \mathbf{\Delta f}(t)$$

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- The simulation trajectories are numerically stable for any quantum adopted.

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- Eigenvalues

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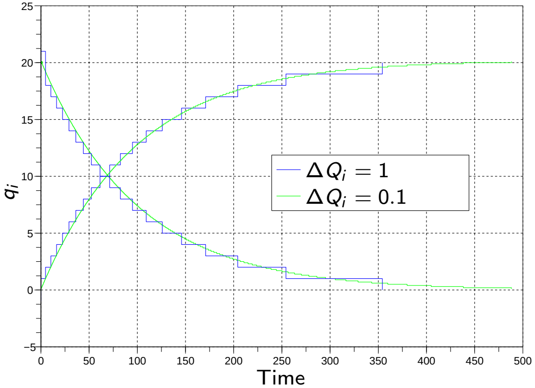

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- The search for $\hat{q}_j$ is done in a linear manner.
- Contrary to BQSS, once we have computed a new value for q_j, no further internal transition will be scheduled, until x_j has advanced by $\pm\Delta Q_j$, even if the sign of $\dot{x}_j$ changes during the step.

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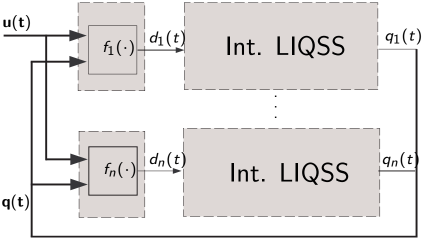

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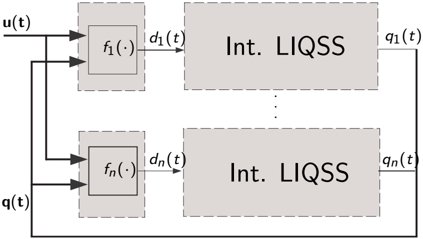

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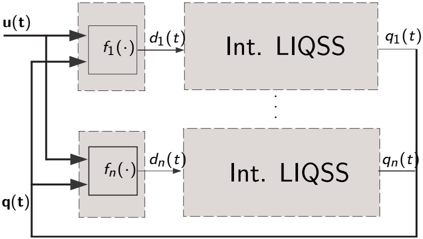

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  - before they output a new value for q_i, they estimate the value of the derivative $\dot{x}_i$ that each of the two options would produce;
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The properties of LIQSS are almost identical to those of QSS, but the error bound is twice the error bound of QSS.

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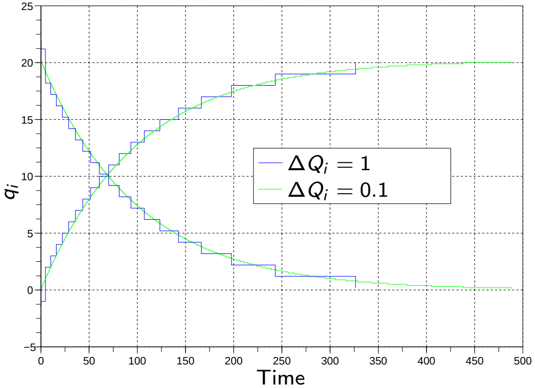

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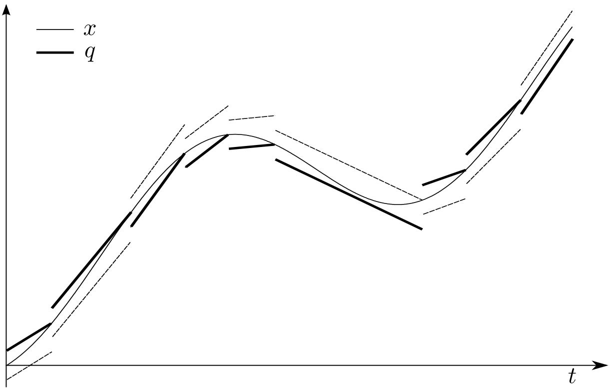

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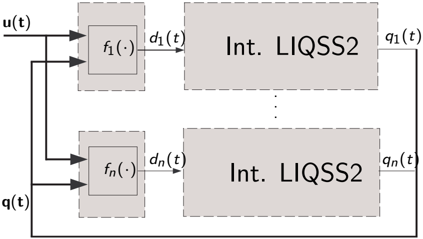

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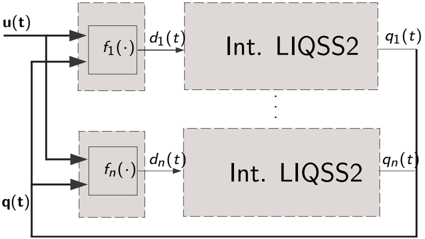

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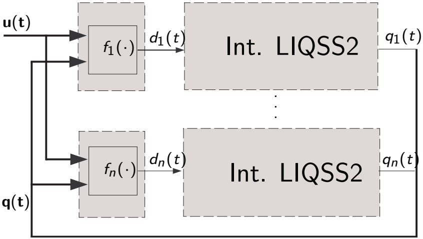

- The *static functions* are the same as for QSS2.
- The *hysteretic quantized integrators* are similar, but:
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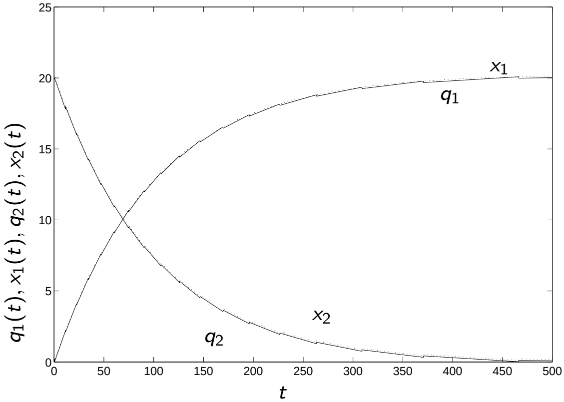

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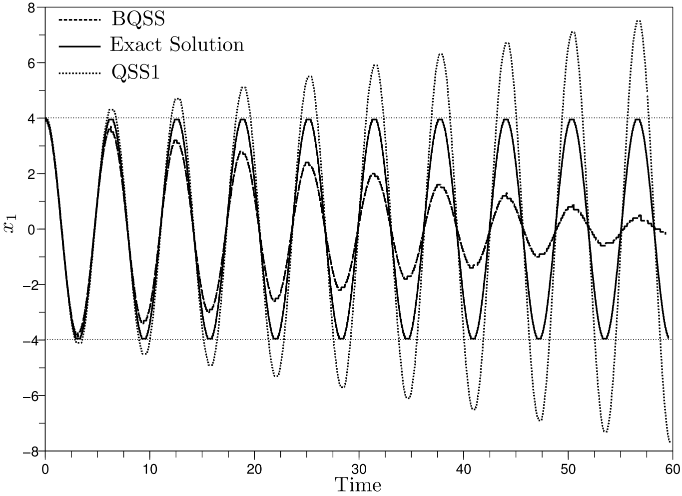
Let us check what happen with QSS and BQSS.

QSS, BQSS and Marginally Stable Systems



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Evidently, neither of the two methods is suited for the simulation of marginally stable systems.

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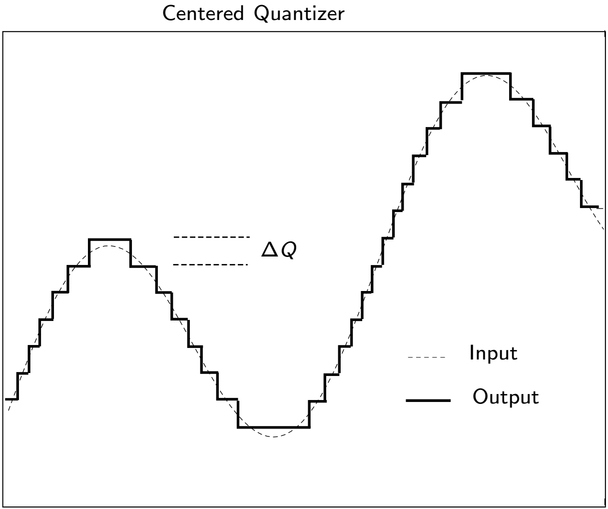
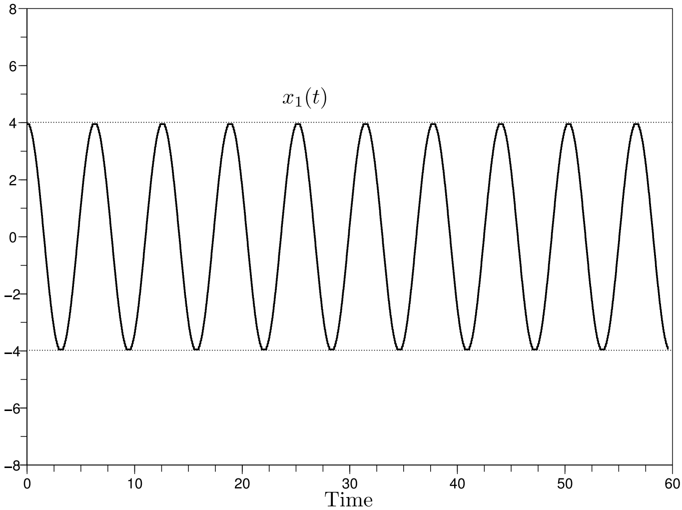
Let us see what happens if we take q_j as the *mean* value between the last and the next quantized values, i.e., if we choose q_j in the middle of the quantization interval. We shall call this new method *Centered QSS (CQSS) solver*.

Centered Quantization



CQSS Simulation of the Harmonic Oscillator



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- Consequently, **CQSS** is well suited for *real-time simulation of marginally stable systems* with low to modest accuracy requirements.
- Unfortunately, **CQSS** is only first-order accurate. There are reasons to believe that *higher-order CQSS methods* cannot be constructed.
- It is certainly possible to design higher-order QSS-based solvers that are decently well suited for the simulation of marginally stable systems, e.g., by creating a *cyclic method*, in which steps of $\textbf{QSS}_i$ toggle with steps of $\textbf{LIQSS}_i$. Unfortunately, these methods will not preserve the geometric properties of **CQSS** in a strict sense, and they will not be truly F-stable.

Delay Differential Equations

Delay differential equations (DDEs) are similar to ODEs, but their right-hand functions also depend explicitly on past state values.

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{x}(t - \tau_1(\mathbf{x}, t)), \ldots, \mathbf{x}(t - \tau_m(\mathbf{x}, t)), \mathbf{u}(t))$$

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- The algorithms are however more complex, as they must look backward in time in order to compute the state derivatives.
- From the point of view of the numerical integration, DDEs have some features in common with discontinuous ODEs.

DDEs and QSS Methods

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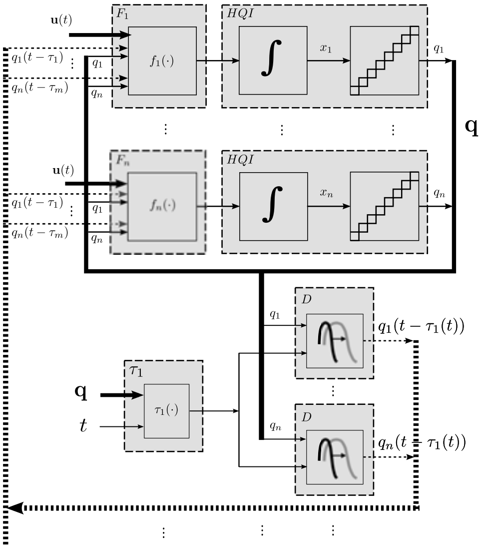
The so-called *DQSS methods (DQSS1, DQSS2, or DQSS3)* approximate it as:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{q}(t), \mathbf{q}(t - \tau_1(\mathbf{q}, t)), \ldots, \mathbf{q}(t - \tau_m(\mathbf{q}, t)), \mathbf{u}(t))$$

where the components of $\mathbf{q}$ and $\mathbf{x}$ are related by quantization functions of the given approximation order.

DQSS and PowerDEVS



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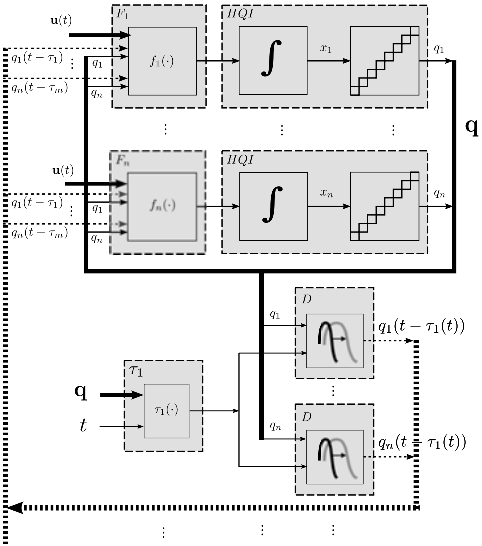

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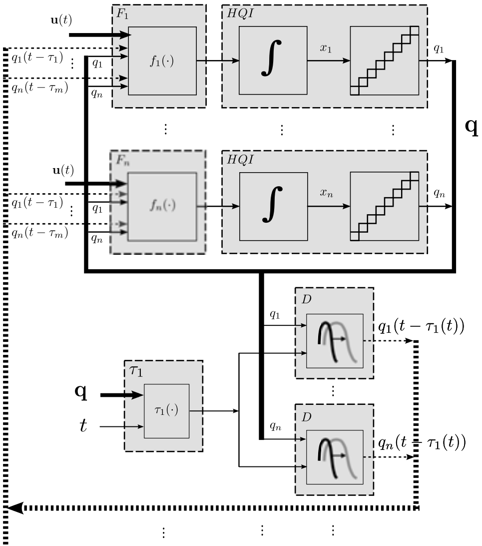

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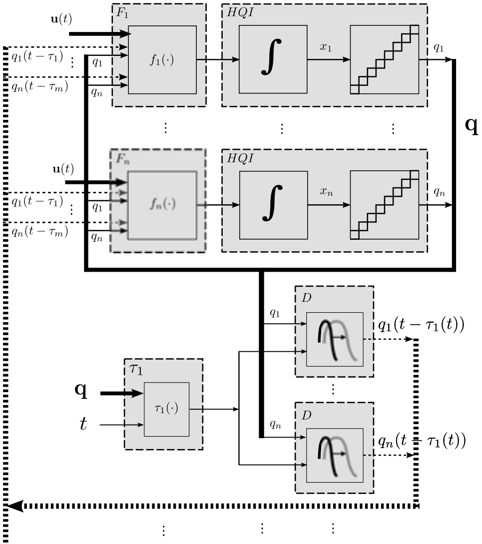

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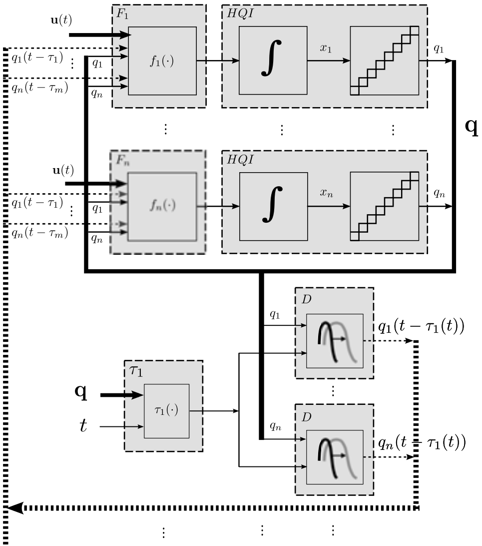

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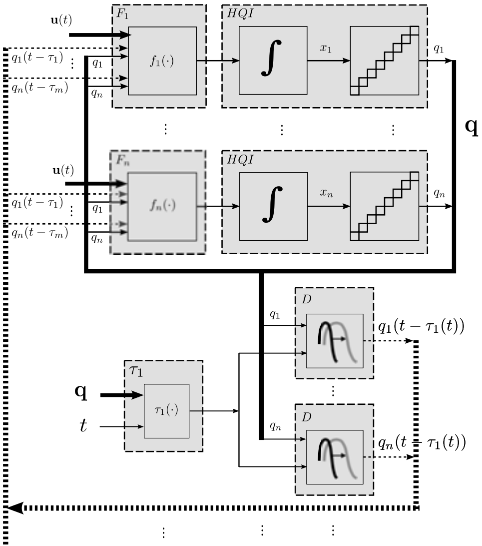

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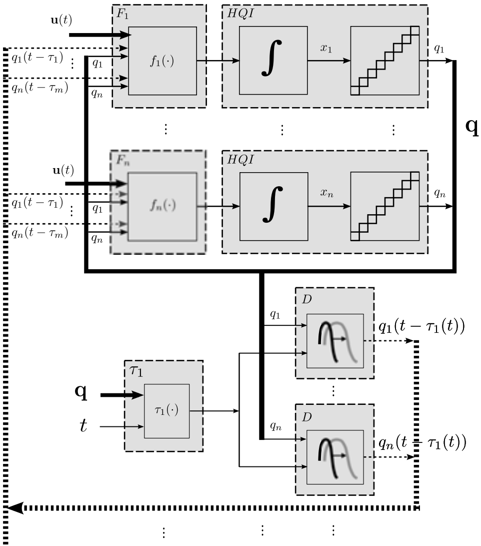

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- a new type of *delay blocks* that compute $q_i(t - \tau_j(t))$ given $q_i(t)$ and $\tau_j(t)$.

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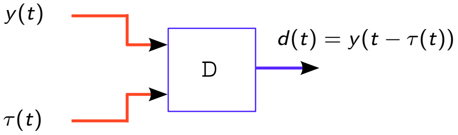

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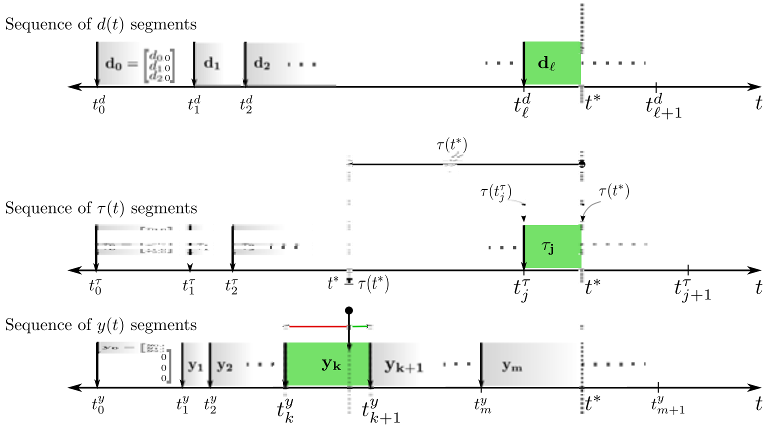
- The *DEVS delay model* computes the corresponding event sequences of $d(t) = y(t - \tau(t))$.

The Delay Algorithm



Theoretical Properties of DQSS

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There is a similar property for *general nonlinear DDE models with state and time dependent delays*. In that case, a stronger condition (*input-to-state stability*) is required for the original system, and the global error boundedness is only guaranteed for sufficiently small quanta.

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- These algorithms are *naturally asynchronous*. Hence distributing QSS simulations across a *parallel multi-processor architecture* is trivial.

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- **PowerDEVS-RT** has been successfully used in the *real-time simulation of power electronic converters*, working at frequencies where most real-time simulation tools fail.

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- QSS-based algorithms have meanwhile proven themselves to offer an *interesting and innovative new approach to numerical ODE, DAE, and DDE solutions* for many different problems.
- QSS-based algorithms can be *highly competitive* when used for applications with *low to average accuracy requirements*. Problems calling for high-order algorithms cannot be dealt with effectively using QSS-based solvers.

Conclusions

- A *broad palette of different QSS-based algorithms* has been presented in the last three presentations.
- QSS-based algorithms have meanwhile proven themselves to offer an *interesting and innovative new approach to numerical ODE, DAE, and DDE solutions* for many different problems.
- QSS-based algorithms can be *highly competitive* when used for applications with *low to average accuracy requirements*. Problems calling for high-order algorithms cannot be dealt with effectively using QSS-based solvers.
- Already QSS4 is rarely more cost-effective than QSS3, because the increased overhead eats into the benefit to be reaped from the larger step sizes. Already QSS5 will require iterations in every step, because a fifth-order polynomial will have to be solved, which can only be done iteratively.

Conclusions II

- QSS-based algorithms are of particular interest for the *simulation of systems exhibiting frequent discontinuities*, as state events can be handled much more efficiently by state-quantization algorithms than by time-slicing algorithms.

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- QSS-based solvers perform equally well as classical solvers when dealing with *small-scale models of systems without discontinuities*. Yet in those cases, there is nothing that the QSS solvers can exploit that would make them more attractive than the much simpler classical ODE solvers.

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- The communication band-width is furthermore minimized, because *only a single bit needs to be communicated* between processors to indicate a state change. A message of "1" means that the sending state incremented its level, whereas a message of "0" means that it decremented its level. As long as the state variable doesn't change its level, it doesn't send an output event through its output channel.

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