Numerical Simulation of Dynamic Systems XXIV

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Introduction

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- A *model reference adaptive controller (MRAC)* makes use of a plant model for its control decisions. Evidently, the plant model must then be simulated under real-time constraints so that the control action can be taken in a timely manner.
- A *watchdog monitor* makes use of a plant model, simulated under real-time constraints, to compare the simulation trajectories with real measurements to identify anomalous behavior of the real plant when it occurs, i.e., it is being used for *fault detection*, *fault isolation*, and *fault identification*.

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- The *real-time clock* is responsible for the synchronization of real time and simulated time. The real-time clock is programmed to send a trigger impulse once every h time units of real time, where h is the current step size of the integration algorithm, and the simulation program is equipped with a *busy waiting* mechanism that is launched as soon as all computations associated with the current step have been completed, and that checks for arrival of the next trigger signal. The new step will not begin until the trigger signal has been received.

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- The *analog to digital (A/D) converters* are read at the beginning of each integration step to update the values of all external driving functions. This corresponds effectively to a *sample and hold (S/H)* mechanism. The inputs are updated once at the beginning of every integration step and are then kept constant during the entire step.

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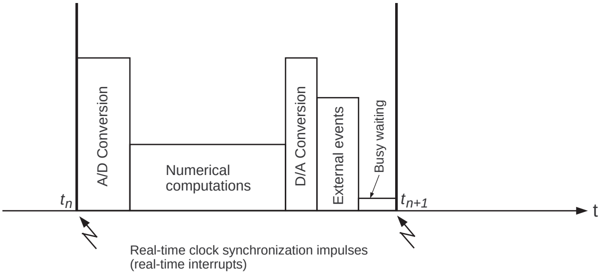
- *External events* are time events that are generated outside the simulation. External events are used for asynchronous communication with the simulation program, e.g. for the modification of parameter values, or for handling asynchronous readout requests, or for communication between several asynchronously running computer programs either on the same or different computers. External events are usually postponed to the end of the current step and replace a portion of the busy waiting period.

Introduction IV



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- In other cases, the same processor may be used for multiple real-time tasks using a *time-multiplexing* scheme. In that situation, the real-time clocks of the different tasks need to be synchronized with each other.

- When the *interrupt mechanism of the real-time operating system* is being used, we also need to make sure that tasks that are related to the real-time operation cannot be interrupted by other tasks.

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- How can we guarantee that all that needs to be accomplished during the integration step can be completed prior to the arrival of the next trigger impulse?
- How do we control the *computational load* of an integration algorithm during the execution of an integration step?
- What happens when we fail in this endeavor and cause an *over-run*?

Introduction VII

Previously, we introduced more and more bells and whistles that would help us in being able to *maximize user convenience*, to ensure the *robustness* of the modeling and simulation environment, and to guarantee the *correctness* of the simulation results obtained, but all these additional features were accompanied by a substantial amount of run-time overhead, and in many cases, the amount of time needed to bring these algorithms to completion was not fixed.

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- *Iteration on state events* is a great thing. Yet, can we afford it under real-time constraints?
- What happens if we do not iterate on the event? Can we still know something about the accuracy of the results obtained?

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Real-time simulation designers must know their simulation algorithms intimately, or else, they will surely fail in their endeavors.

The Race Against Time

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The fourth approach may sound like a measure of last resort, but in these times of cheap hardware and expensive software and manpower, it may often be the wisest thing to do.

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- Especially, when we are using an explicit integration algorithm, as we most likely will in order to avoid the need for iteration, the two bounds are interrelated. By increasing the step size to avoid over-runs, we are weakening the numerical stability of the algorithm.

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- If it happens that the lower bound is larger than the upper, then we are in real trouble.

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- Often the problems with spending too much time in function evaluations is related to needs for *interpolation in data tables*.
- An attractive answer to this problem may be the *parallelization of the function evaluation*, e.g. by *implementing the model as a neural network*.

The Race Against Time IV

The third approach is directly related to the integration algorithms themselves, and as these are the central focus point of this class, it is the third solution that we shall be pursuing in more detail in this presentation.

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Explicit linear multi-step methods are well suited for real-time simulation, as long as the model to be simulated is neither stiff nor discontinuous.

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Explicit ODE solvers are preferred for real-time simulation, as long as the model to be simulated is non-stiff.

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. . . but at least, they do it very fast.

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- Or are we not?

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where:

$$\mathcal{J}_{\mathbf{x_k}, t_k} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x_k}, t_k}$$

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We can truncate the Taylor series after the linear term, and plug the expression into the solver equation:

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The new method *approximates backward Euler*, since the Taylor series was truncated after the linear term. As the Taylor series gets multiplied by h, the approximation is second-order accurate, and since the entire method is only first-order accurate, the approximation is acceptable.

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The numerical stability domain of the new method is the same as that of BE.

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- Despite the fact that those calculations may turn out to be quite expensive, the *computational effort is predictable*, which makes the method suited for real-time simulation.
- Low-order linearly implicit methods may indeed often be the best choice for real-time simulation of stiff systems.
- However, they share one drawback with implicit methods: if the size of the problem is large, then the solution of the resulting linear equation system is computationally expensive.

Linearly Implicit Methods VI

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- Since the equation system to be solved is linear, the iteration will converge within a single iteration step. Thus, the computational load can be anticipated accurately.

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In those cases, it may make sense to split the simulation into a fast and a slow part and integrate these parts using different step sizes.

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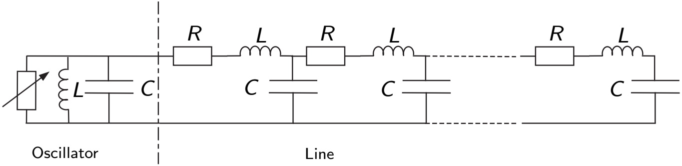

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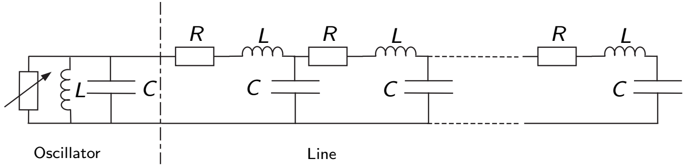

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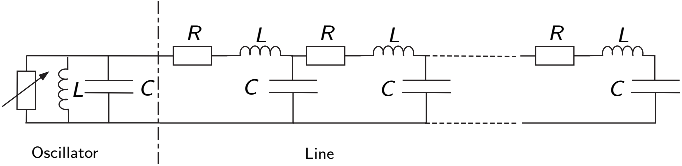

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The example is clearly non-physical, as the "resistor" doesn't operate strictly in the first and third quadrant of the plane spanned by u_R and i_R. Consequently, there are times when the resistor converts thermal energy to electrical energy, something no respectable resistor will ever do.

Multi-rate Integration III

The system can be described by the following set of state equations:

$$
\begin{aligned}
\frac{di_L}{dt} &= \frac{1}{L} u_C \\
\frac{du_C}{dt} &= \frac{1}{C}(u_C - k \cdot u_C^3 - i_L - i_1) \\
\frac{di_1}{dt} &= \frac{1}{L} u_C - \frac{R}{L} i_1 - \frac{1}{L} u_1 \\
\frac{du_1}{dt} &= \frac{1}{C} i_1 - \frac{1}{C} i_2 \\
\frac{di_2}{dt} &= \frac{1}{L} u_1 - \frac{R}{L} i_2 - \frac{1}{L} u_2 \\
\frac{du_2}{dt} &= \frac{1}{C} i_2 - \frac{1}{C} i_3 \\
&\vdots \\
\frac{di_n}{dt} &= \frac{1}{L} u_{n-1} - \frac{R}{L} i_n - \frac{1}{L} u_n \\
\frac{du_n}{dt} &= \frac{1}{C} i_n
\end{aligned}
$$

Multi-rate Integration IV

- Let us assume that the transmission line has 5 stages (i.e., $n = 5$), and the parameters are $L = 10\ \mathrm{mH}$, $C = 1\ \mathrm{mF}$, $R = 10\Omega$, and $k = 0.04$.

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- Thus, we decided to split the system into two subsystems, the oscillator circuit and the transmission line, using two different step sizes: $10^{-4}\ \mathrm{seconds}$ for the former, and $10^{-3}\ \mathrm{seconds}$ for the latter.
- In this way, we integrate the fast but small (2^{nd}-order) sub-system using a small step size, whereas we integrate the slow and large (10^{th}-order) sub-system using a ten times larger step size.

Multi-rate Integration V

As a consequence, during each millisecond of real time, the computer has to evaluate ten times the two scalar functions corresponding to the two first state equations, whereas it only needs to evaluate once the remaining ten functions. Thus, the number of floating-point operations is reduced by about a factor of four compared with a regular simulation using a single step size throughout.

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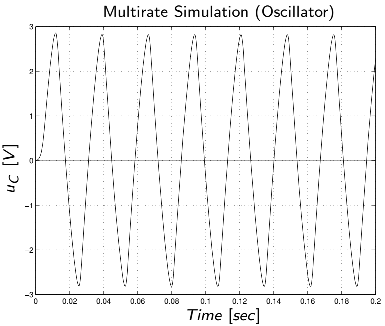

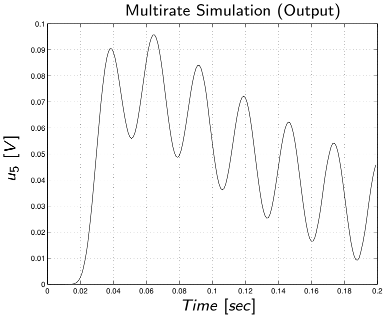

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We can generalize this procedure to systems of the form:

$$\begin{aligned} \dot{\mathbf{x}}_\mathbf{f}(t) &= \mathbf{f}_\mathbf{f}(\mathbf{x}_\mathbf{f}, \mathbf{x}_\mathbf{s}, t) \\ \dot{\mathbf{x}}_\mathbf{s}(t) &= \mathbf{f}_\mathbf{s}(\mathbf{x}_\mathbf{f}, \mathbf{x}_\mathbf{s}, t) \end{aligned}$$

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Then, the use of the *multi-rate version of Forward Euler with inlining* results in a set of difference equations of the form:

$$\begin{aligned}\mathbf{x}_\mathbf{f}(t_i + (j+1)\cdot h) &= \mathbf{x}_\mathbf{f}(t_i + j\cdot h) + h\cdot \mathbf{f}_\mathbf{f}(\mathbf{x}_\mathbf{f}(t_i + j\cdot h), \\ &\quad \mathbf{x}_\mathbf{s}(t_i + j\cdot h), t_i + j\cdot h) \\ \mathbf{x}_\mathbf{s}(t_i + k\cdot h) &= \mathbf{x}_\mathbf{s}(t_i) + k\cdot h\cdot \mathbf{f}_\mathbf{s}(\mathbf{x}_\mathbf{f}(t_i), \mathbf{x}_\mathbf{s}(t_i), t_i)\end{aligned}$$

where k is the (integer) ratio of the two step sizes, $j = 0 \ldots k-1$, and h is the step-size of the fast sub-system.

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- In our example, we used **FE**. Similar approaches have been reported in the literature based on the **AB2**, including some *improvements for parallel implementation*.

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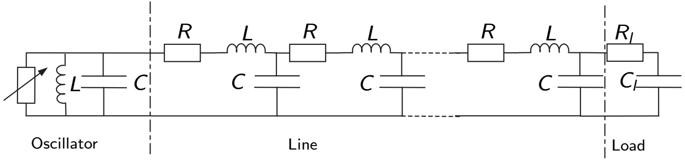

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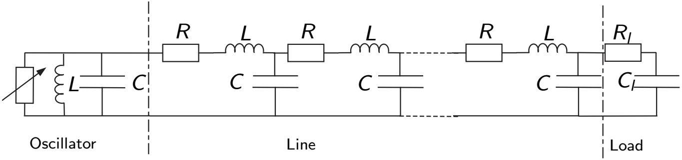

Let us assume that the load parameters are $R_l = 1\ \mathrm{k\Omega}$ and $C_l = 1\ \mathrm{nF}$.

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- This means that we would have to reduce the step size by about a factor of 1000 with respect to the previous example, in order to obtain a numerically stable result.

Inline Integration II

- Since the load resistor is much bigger than the line resistors, the newly introduced term in the state-space model won't influence the dynamics of the transmission line significantly, and we can expect the load not to influence the behavior of the oscillator and the transmission line significantly.

- However, the added state equation introduces a fast pole. The position of this pole is approximately located at:

$$\lambda_l \approx -\frac{1}{R_l \cdot C_l} = -10^6 \text{ sec}^{-1}$$

- This means that we would have to reduce the step size by about a factor of 1000 with respect to the previous example, in order to obtain a numerically stable result.

- Unfortunately, such a solution is totally unacceptable in the context of a real-time simulation.

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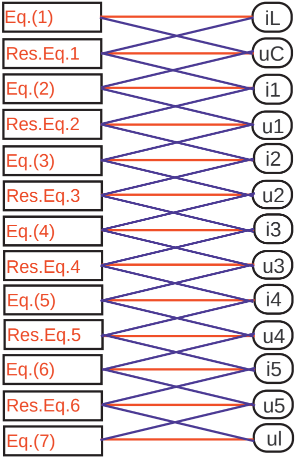

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- Unfortunately, we are now once again using an implicit algorithm, iterating on non-linear equations. This means that we have no guarantee that the iteration on the tearing variables will converge in time.
- If the Newton iteration converges after three steps, we may still be ahead of the game, but implicit algorithms are problematic for use in real-time simulation.

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In our example circuit, we inlined the equations once more, this time using the *explicit Forward Euler algorithm* everywhere except for the last equation, where we still used the *implicit Backward Euler method*.

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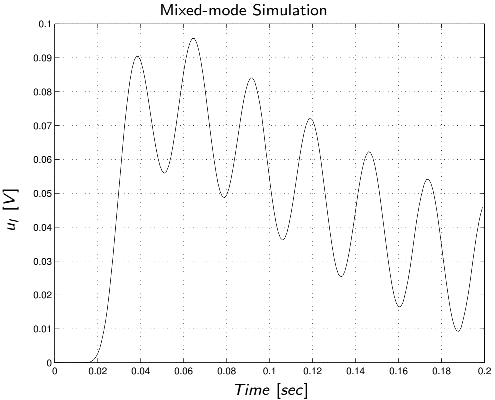
- All equations are now explicit, except for the very last equation, which is implicit in the variable u_l, but can be solved symbolically for u_l.
- We simulated the system using the same approach as before, i.e., we applied a step size of 10^{-4} seconds to the two oscillator equation, whereas we used a step size of 10^{-3} seconds to all other equations, including the implicit load equation.

Mixed-mode Integration III



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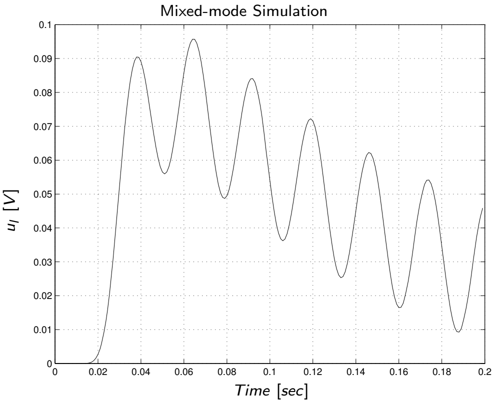

The simulation executed now very fast, while the simulation results are adequately accurate.

Mixed-mode Integration IV

In more general terms, the *Backward-Forward Euler Mixed-Mode integration scheme* can be written as:

$$\begin{aligned}
\mathbf{x_s}(t_{k+1}) &= \mathbf{x_s}(t_k) + h \cdot \mathbf{f_s}(\mathbf{x_f}(t_k), \mathbf{x_s}(t_k), t_k) \\
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- The algorithm starts by computing explicitly the value of $\mathbf{x_s}(t_{k+1})$.
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Discontinuous Systems

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We need to distinguish between external and internal events.

- *External events* are time events that are received from outside the simulation. They are used by either humans or control agents to interfere with the real-time simulation, e.g. for changing a parameter value.
- External events are not time-critical, as they are always *asynchronous* to the simulation. Consequently, a small delay is acceptable, and therefore, external events are always delayed until the end of the current integration step and are handled during the waiting period, i.e., before the next synchronization impulse arrives from the real-time clock.

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State-event handling in real-time simulation is simplified, when comparing it to the techniques introduced earlier, by two factors:

1. As we are using low-order integration techniques, we can also use low-order event localization algorithms.
2. Since we use much smaller step sizes, the precise localization of state events becomes less critical, and there shouldn't occur as often multiple state events within a single integration step.

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Of course, it may be possible to reduce the residual on the zero-crossing function further by iteration, but this does not necessarily imply that we would thereby locate the event more accurately, as already the previous integration steps are contaminated by numerical errors.

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- Starting from the new initial state, we perform a partial step advancing the state vector from time t_{next} to time t_{n+1}.

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Internal time events are handled essentially in the same manner, except that their time of occurrence is known in advance, i.e., we only need two partial integration steps within the time allotted for one step instead of three.

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Even the sensor and actuator units contain their own hardware-built sample-and-hold equipment.

Simulation Architecture II

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This is usually accomplished by making read and write operations out of and into the mailboxes *non-interruptible*.

Simulation Architecture III

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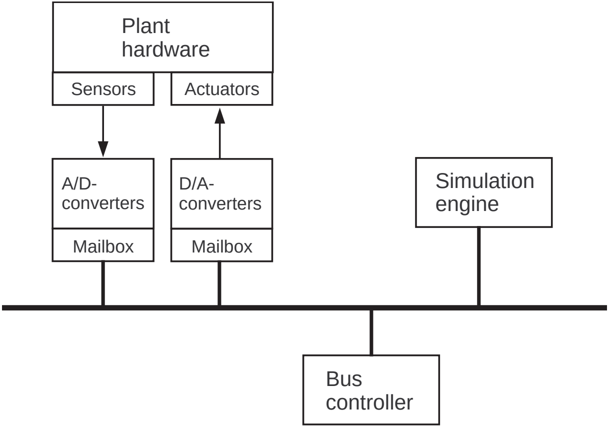

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- Since the total time needed for computing all activities associated with a single integration step must be known, both HLA and CORBA offer mechanisms for specifying the *maximum allowed latency* in answering requests for information transfer across the architecture using the established communication channels and protocols.

Overruns

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Most real-time simulations specify the *maximum percentage of overruns* as e.g. 1% or 2%.

Overruns II

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In this way, we allow an integration step to be computed less accurately once in a while in order to stay synchronous with the real-time clock.

Conclusions

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- As real-time simulation is not conceptually different from general-purpose simulations, many of the observations made were rather practical, and not highly mathematical.
- Consequently, this presentation contains much more text and much less formulae than any of the previous presentations.
- Yet, real-time simulation is a very important branch of simulation, and consequently, its special demands require often solutions that we wouldn't ever consider apart from a real-time simulation task.

Conclusions II

- We introduced a new (derived) class of integration algorithms, the *linearly implicit integration algorithms*. These can be designed as variants of practically all implicit integration schemes, although we concentrated on one algorithm only, the *linearly implicit Backward Euler (LIBE)* algorithm. Whereas such algorithms could also be used for general-purpose simulation, this is hardly ever done.

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- We then looked at *multi-rate integration schemes*. They have a role to play in real-time simulations of systems with clearly separate groups of eigenvalues, such as in the real-time simulation of physical systems including multiple energy domains. Once again, multi-rate integration schemes are hardly ever used outside the world of real-time simulation.

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- We then showed that *inline integration methods* and *mixed-mode integration schemes* can be very useful for speeding up real-time simulations.

- The presentation ended with some general remarks about *real-time simulation architectures*.

References

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