Numerical Simulation of Dynamic Systems XVII

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The Solvability Issue

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- Yet, the problem is worse, because we don't know which root to choose. If we choose the positive root, $\dot{x}$ will also be positive, and x will keep growing. However, if we choose the negative root, $\dot{x}$ is negative, and x will decrease.
- Even worse, it could be that we should choose the positive root during some period of time, and the negative root during another. Thus, at any moment in time, we obtain a potential bifurcation in the solution depending on whether we choose the positive or the negative root.

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- Thus, if a DAE model contains solvability issues, this simply means that the DAE does not capture the physical phenomenon that it is supposed to describe in its full complexity. Some information is missing.

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- Thus, if a DAE model contains solvability issues, this simply means that the DAE does not capture the physical phenomenon that it is supposed to describe in its full complexity. Some information is missing.
- Unfortunately, solvability issues are encountered frequently in DAE models derived from object-oriented descriptions of physical systems, and consequently, we need to deal with the consequences.

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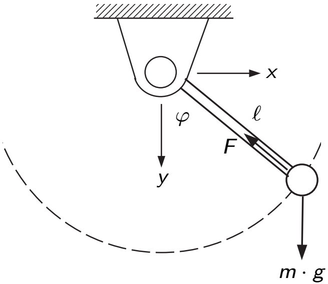

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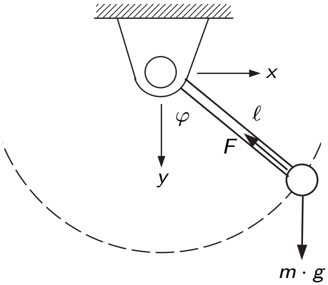

describable by the following set of DAEs:

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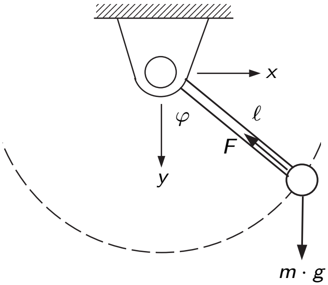

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- Since x, y, v_x, and v_y are known state variables, the last equation in the set is a *constraint equation*.

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- Evidently, the original problem was an index-3 problem, and the Pantelides algorithm needs to be applied a second time.

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- We now ended up with nine equations in the nine unknowns dv_x, x, F, $\frac{dv_y}{dt}$, dx, v_x, $d2x$, $\frac{dy}{dt}$, and $d2y$.

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- Finally, another integrator had to be eliminated. We let go of the one defining the variable v_x.
- This set of equations represents an index-1 DAE problem that can be causalized using the tearing method.

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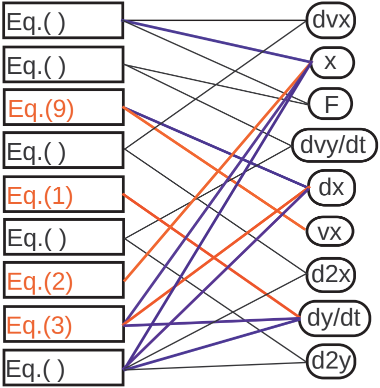

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- We chose $d2x$ as our tearing variable, and the last equation as the residual equation.

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- This selection allowed us to causalize all of the remaining equations.

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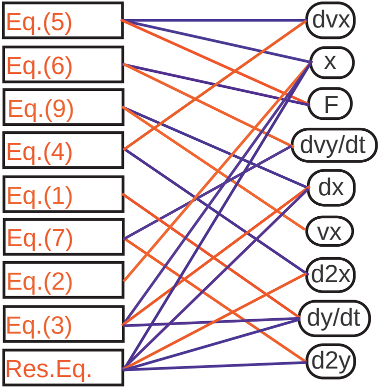

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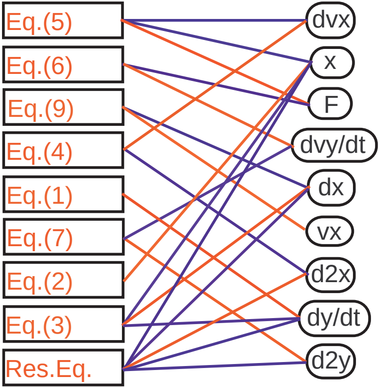

$$\frac{dy}{dt} = v_y$$

$$x = \pm\sqrt{\ell^2 - y^2}$$

$$dx = -\frac{y}{x} \cdot \frac{dy}{dt}$$

$$dv_x = d2x$$

$$F = -m \cdot \ell \cdot \frac{dv_x}{x}$$

$$\frac{dv_y}{dt} = g - \frac{F \cdot y}{m \cdot \ell}$$

$$d2y = \frac{dv_y}{dt}$$

$$d2x = -\frac{dx^2 + \left(\frac{dy}{dt}\right)^2 + y \cdot d2y}{x}$$

$$v_x = dx$$

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- Yet, we are encountering new problems.
- First, the simulation will blow up with a division by zero, as soon as $x = 0$.
- Second, we seem to have a *solvability issue*, as we don't know which of the two roots to choose.

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- From our physical understanding, we know that we must choose the positive root for $x > 0$ and the negative root for $x < 0$. We also know that the pendulum will swing through $x = 0$, i.e., as we pass through zero, we need to switch to the other root. Yet, our mathematical description doesn't contain that information.

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- A new variable $d\varphi$ is introduced in the differentiation. Consequently, the equation defining φ, i.e., Eq.(5), needs to be differentiated as well.

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- We end up with 12 equations in 12 unknowns.

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- Thus, the Pantelides algorithm must be applied once more.
- This time around, we shall tell Dymola to treat $\dot{\varphi}$ as an additional preferred state.
- We end up with 12 equations in 12 unknowns.

$$
\begin{aligned}
m \cdot dv_x &= -\frac{F \cdot x}{\ell} \\
m \cdot dv_y &= m \cdot g - \frac{F \cdot y}{\ell} \\
dx &= v_x \\
d2x &= dv_x \\
dy &= v_y \\
d2y &= dv_y \\
x &= \ell \cdot \sin(\varphi) \\
dx &= \ell \cdot \cos(\varphi) \cdot \frac{d\varphi}{dt} \\
d2x &= \ell \cdot \cos(\varphi) \cdot \frac{d^2\varphi}{dt^2} - \ell \cdot \sin(\varphi) \cdot \left(\frac{d\varphi}{dt}\right)^2 \\
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$$

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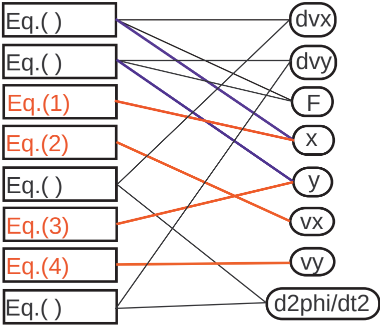

The Solvability Issue XVI

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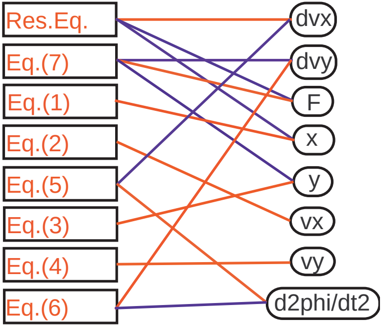

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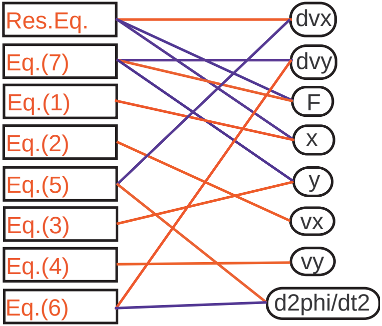

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$$\frac{d^2\varphi}{dt^2} = \frac{dv_x}{\ell \cdot \cos(\varphi)} + \frac{\sin(\varphi)}{\cos(\varphi)} \cdot \left(\frac{d\varphi}{dt}\right)^2$$

$$dv_y = -\ell \cdot \sin(\varphi) \cdot \frac{d^2\varphi}{dt^2} - \ell \cdot \cos(\varphi) \cdot \left(\frac{d\varphi}{dt}\right)^2$$

$$F = \frac{m \cdot g \cdot \ell}{y} - \frac{m \cdot \ell \cdot dv_y}{y}$$

$$dv_x = -\frac{F \cdot x}{m \cdot \ell}$$

The Solvability Issue XVII

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- The simulation will work fine, as long as we don't let the pendulum swing beyond the horizontal position.
- Unfortunately, as $y = 0$, the simulation will once again blow up with a division by zero.

The Solvability Issue XVIII

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- Many non-linear mechanical multi-body systems exhibit dynamic singularity issues when their dynamics are described in an object-oriented fashion by a set of DAEs.

- **Dymola** recognizes potential singularity issues during compilation. In such cases, Dymola will keep additional state variables in the set of equations and perform a *dynamic state selection*, i.e., Dymola will switch dynamically to another set of state variables on the fly as the current set approaches one of its singular points.

Conclusions

- In the previous three presentations on converting sets of DAEs to equivalent sets of ODEs, we concentrated for simplicity on *linear electric circuits* as examples, as these models are easily understandable.

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Conclusions

- In the previous three presentations on converting sets of DAEs to equivalent sets of ODEs, we concentrated for simplicity on *linear electric circuits* as examples, as these models are easily understandable.
- However, some issues don't show up in linear systems. In this final presentation on the symbolic preprocessing of DAE systems, we focused on precisely those remaining issues that can occur only in *non-linear systems*: the solvability issue and the dynamic singularity issue.
- **Dymola** recognizes *solvability issues* during compilation. To this end, Dymola avoids whenever possible to make use of state variables whose derivatives are to be solved from an equation, in which they appear in a non-linear form. Similarly, Dymola avoids to select tearing variables that appear in their residual equations in a non-linear form.

Conclusions II

- **Dymola** also recognizes *dynamic singularity issues* during compilation. To this end, Dymola avoids whenever possible to make use of state variables whose derivatives are multiplied by other variables in the equations from which they need to be solved, as these other variables would invariably turn up in the denominator after the symbolic manipulation. Similarly, Dymola avoids to select tearing variables that are multiplied by other variables in their residual equations. When this cannot be avoided, Dymola keeps additional state variables and/or additional tearing variables in the set of iteration variables.