Numerical Computation of Eigenvalues

Given:

$$\underline{\dot{x}} = A\underline{x} + \underline{b}u$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad \underline{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This is:

→ a Jordan-canonical form with a multiple eigenvalue:

$$\lambda_1 = 0 \quad ; \quad m_1 = n$$

→ a controller-canonical form with the characteristic polynomial

$$\lambda^n = 0$$

Let us perturb the system with a small ε:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \varepsilon & 0 & 0 & 0 & 0 \end{bmatrix} ; \quad \underline{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This is no longer a Jordan-canonical form. However, it is still a controller-canonical form with the characteristic polynomial:

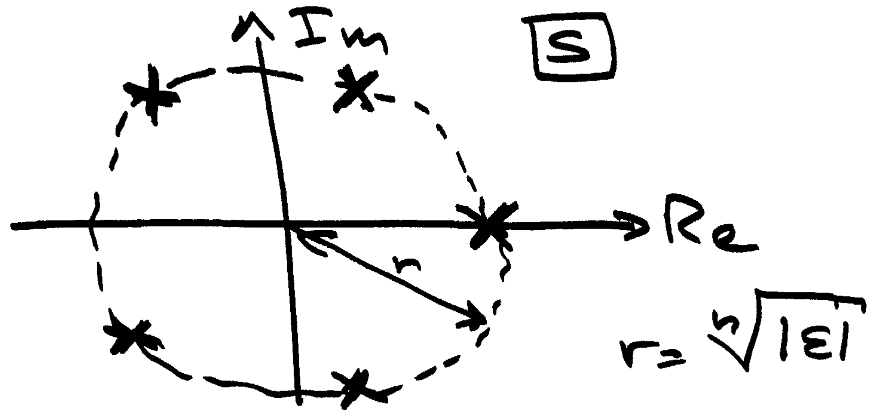
$$\lambda^n - \varepsilon = 0$$

$$\Rightarrow \lambda_i = \sqrt[n]{\varepsilon}$$

Let us write ε as a complex number in polar coordinates:

$$\varepsilon = |\varepsilon| \cdot e^{j2\pi}$$

$$\Rightarrow \lambda_i = \sqrt[n]{\varepsilon} = \sqrt[n]{|\varepsilon|} \cdot e^{j\frac{2k\pi}{n}}$$



Let: $\varepsilon = 10^{-10}$

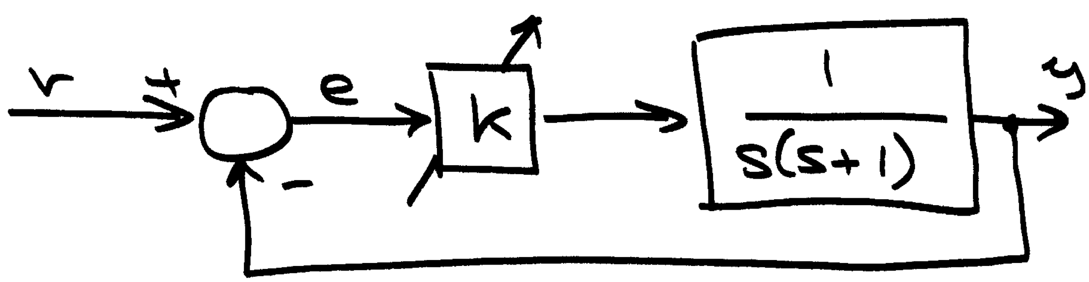
$n = 10$

$\Rightarrow \sqrt[n]{|\varepsilon|} = 0.1$

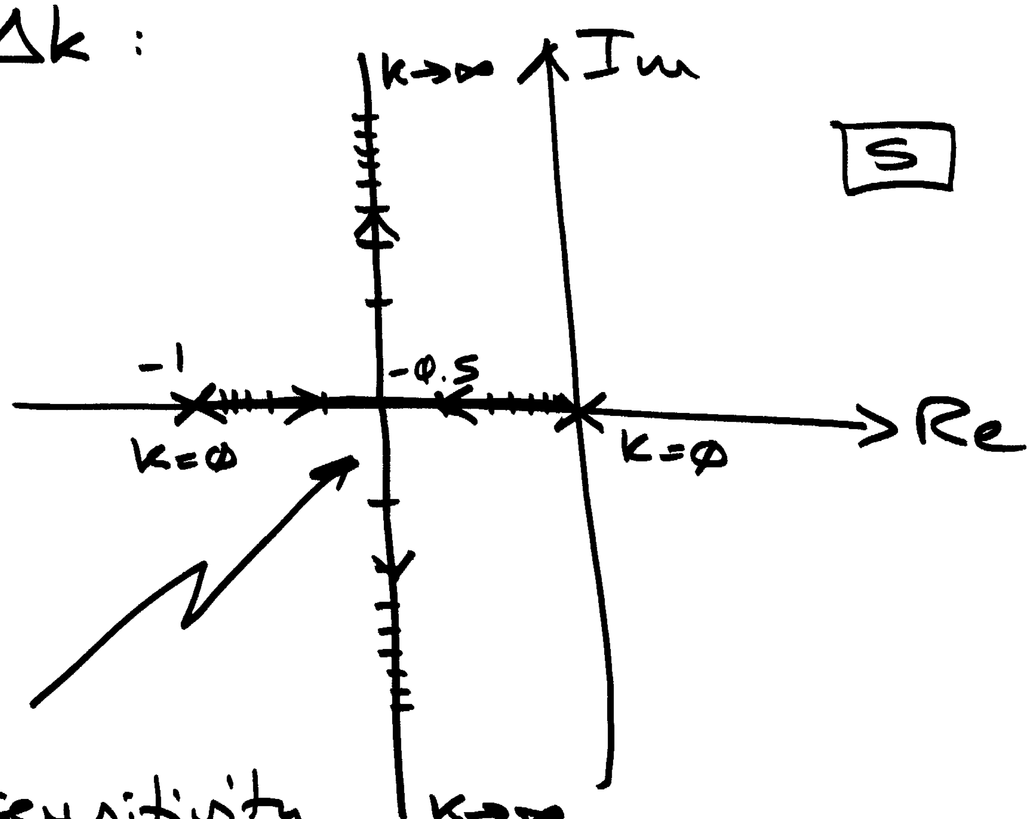
A small change in the parameters of the system matrix A had a huge effect on the location of the eigenvalues.

This is generally true in the vicinity of multiple poles.

Remember:



Let us draw a root locus for $k \in [0, \infty)$ in increments of Δk :



The sensitivity of the pole location is much bigger in the vicinity of -0.5, where the root locus shows a double pole.

- Once the eigenvalues are known precisely, computation of the corresponding eigen= vectors is comparatively benign from a numerical perspective.

- Computing the eigenvalue location accurately may not be possible.
- The worst case is the one illustrated before. Let $\sigma_1 = \|A\|_2$ be the largest singular value

$$\Rightarrow \underline{\underline{\operatorname{err}(\lambda_i) \leq \sigma_1 \cdot \varepsilon^{-1/n}}}$$

The eigenvalues cannot be computed more accurately than $\sqrt[n]{\varepsilon} \cdot \sigma_1$ in the worst case.

- We shall need to discuss algorithms to assess whether the matrix A has eigenvalues that are close to being multiple.

Recipe: Given A with

$$\lambda_i = \mathrm{eig}(A).$$

We compute the following perturbation:

$$\hat{A} = A + \epsilon \cdot \|A\|_2 \cdot \mathrm{rand}(\mathrm{size}(A))$$

$$\Rightarrow \hat{\lambda}_i = \mathrm{eig}(\hat{A})$$

Then: $$\mathrm{cond} = \frac{\|\lambda_i - \hat{\lambda}_i\|_\infty}{\epsilon}$$

can be used to estimate the sensitivity of the pole locations to perturbations of A. You may want to repeat a few times.

$\mathrm{Cond} \approx 1 \Rightarrow$ all eigenvalues are distinct

$\mathrm{Cond} \gg 1 \Rightarrow$ multiple or almost multiple eigenvalues are present.