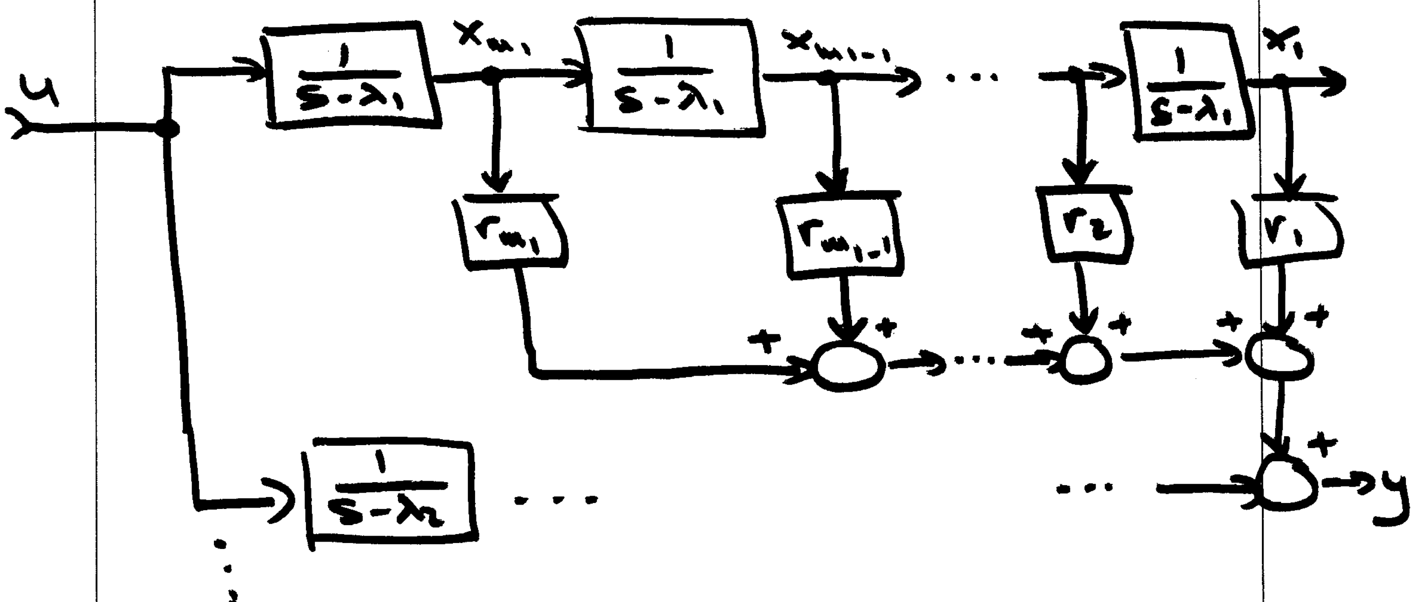
We apply Partial Fraction Expansion:

$$G(s) = \frac{r_{m_1}}{s-\lambda_1} + \frac{r_{m_1-1}}{(s-\lambda_1)^2} + \cdots + \frac{r_1}{(s-\lambda_1)^{m_1}} + \cdots$$



$$\Rightarrow \left| \begin{array}{l} \dot{x}_1 = \lambda_1 x_1 + x_2 \\ \dot{x}_2 = \lambda_1 x_2 + x_3 \\ \quad \vdots \\ \dot{x}_{m_1-1} = \lambda_1 x_{m_1-1} + x_{m_1} \\ \dot{x}_{m_1} = \lambda_1 x_{m_1} + u \\ \quad \vdots \qquad\qquad\qquad \vdots \end{array} \right|$$

$$A = \left[\begin{array}{cccc|ccc} \lambda_1 & 1 & & & & & \\ & \lambda_1 & 1 & & & \emptyset & \\ & & \ddots & 1 & & & \\ & & & \lambda_1 & & & \\ \hline & & & & \lambda_2 & & \\ & \emptyset & & & & \ddots & \\ & & & & & & \lambda_n \end{array}\right] ; \quad \underline{b} = \left[\begin{array}{c} \emptyset \\ \vdots \\ \emptyset \\ 1 \\ \hline 1 \\ \vdots \\ 1 \end{array}\right]$$

$\Rightarrow$ The Jordan-form of a system with multiple poles takes the form:

$$A = \begin{bmatrix} \Lambda_1 & & & \emptyset \\ & \Lambda_2 & & \\ & & \ddots & \\ \emptyset & & & \Lambda_k \end{bmatrix} ; \quad \underline{b} = \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \\ \underline{b}_k \end{bmatrix}$$

where: $\Lambda_i = \begin{bmatrix} \lambda_i & 1 & & \emptyset \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ \emptyset & & & \lambda_i \end{bmatrix} ; \quad \underline{b}_i = \begin{bmatrix} \emptyset \\ \vdots \\ \emptyset \\ 1 \end{bmatrix}$

- The Jordan-form is block-diagonal, and the Λ_i are called the Jordan blocks.

- A system that is completely controllable and observable (no pole/zero cancellation) has exactly one Jordan block associated with each eigenvalue.

- For a system to be totally controllable and observable, each eigenvalue can have and must have exactly one eigenvector associated with it.

- If the nullity of the matrix $(\lambda_i I - A)$ is > 1 for <u>any</u> $\lambda_i \Rightarrow$ there is at least one uncontrollable and unobservable mode in the system.

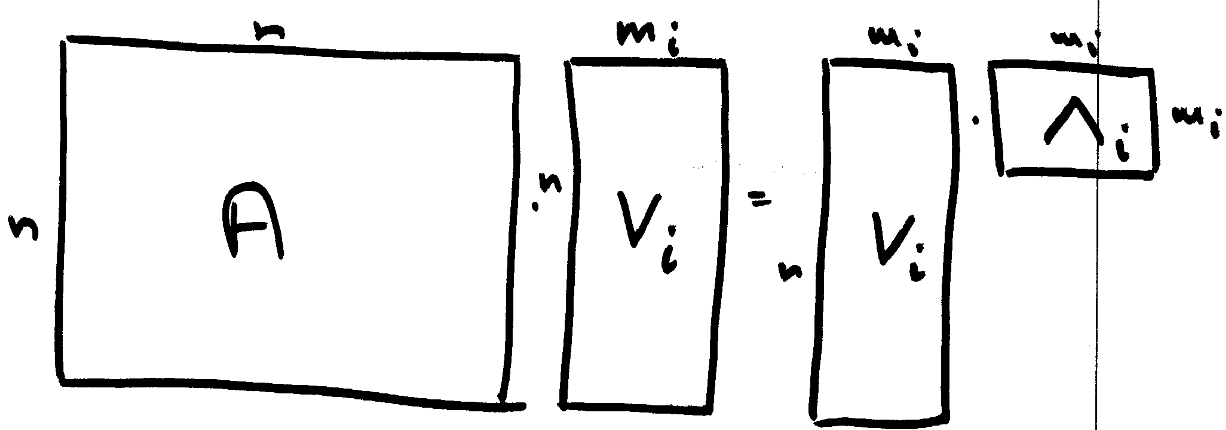
Question: Which similarity transformation will get us into Jordan-canonical form?

- From: $AV = V\Lambda$

  $\iff \quad A\underline{v}_i = \underline{v}_i \lambda_i$,

  we realize that, due to the complete input/output decoupling, we need to look at one Jordan block at a time only:

$$\underline{\underline{A V_i = V_i \cdot \Lambda_i}}$$



will get us into the desired form. However, we don't know yet what V_i is.

$$A \begin{bmatrix} \vdots & & \vdots \\ \underline{v}_{i_1} & \cdots & \underline{v}_{i_{m_i}} \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & & \vdots \\ \underline{v}_{i_1} & \cdots & \underline{v}_{i_{m_i}} \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \lambda_i & 1 & & \phi \\ & \lambda_i & 1 & \\ & & \lambda_i & 1 \\ \phi & & & \lambda_i \end{bmatrix}$$

$$\Rightarrow \left| \begin{array}{l} A\underline{v}_{i_1} = \lambda_i \underline{v}_{i_1} \\ A\underline{v}_{i_2} = \lambda_i \underline{v}_{i_2} + \underline{v}_{i_1} \\ \quad \vdots \\ A\cdot\underline{v}_{i_{m_i}} = \lambda_i \underline{v}_{i_{m_i}} + \underline{v}_{i_{m_i-1}} \end{array} \right|$$

$$\Rightarrow \left| \begin{array}{l} (A-\lambda_i I)\,\underline{v}_{i_1} = \phi \\ (A-\lambda_i I)\,\underline{v}_{i_2} = \underline{v}_{i_1} \\ \quad \vdots \\ (A-\lambda_i I)\,\underline{v}_{i_{m_i}} = \underline{v}_{i_{m_i-1}} \end{array} \right|$$

$\Rightarrow$ $\underline{v}_{i_1}$ is an eigenvector associated with λ_i.

The other vectors are so-called <u>generalized eigenvectors</u>.

- To simplify, we can multiply the second equation with $(A-\lambda_i I)$:

$$(A-\lambda_i I)^2 \underline{v}_{i_2} = (A-\lambda_i I)\underline{v}_{i_1} = \phi$$

- The next equation is multiplied with $(A-\lambda_i I)^2$:

$$(A-\lambda_i I)^3 \underline{v}_{i_3} = (A-\lambda_i I)^2 \underline{v}_{i_2} = \phi$$

<u>etc</u>.

- Thus, the set of equations can also be written as:

$$\left|\begin{array}{c} (A-\lambda_i I)\underline{v}_{i_1} = \phi \\ (A-\lambda_i I)^2 \underline{v}_{i_2} = \phi \\ \vdots \\ (A-\lambda_i I)^{m_i} \underline{v}_{i_{m_i}} = \phi \end{array}\right|$$

Of course:

$$\left| \begin{array}{l} (A-\lambda_i I)^{m_i} \; \underline{v}_{i_1} = \phi \\ (A-\lambda_i I)^{m_i} \; \underline{v}_{i_2} = \phi \\ \quad \vdots \\ (A-\lambda_i I)^{m_i} \; \underline{v}_{i_{m_i}} = \phi \end{array} \right|$$

is also correct.

- As a transformation into Jordan form must exist $\Longrightarrow$ $\boxed{\text{Nullity}\{(A-\lambda_i I)^{m_i}\} \equiv m_i}$.

- Among all these, we are interested to find the one generalized eigenvector of grade m_i, which satisfies the conditions:

$$\left| \begin{array}{l} (A-\lambda_i I)^{m_i} \; \underline{v}_{i_{m_i}} = \phi \\ (A-\lambda_i I)^{m_i-1} \cdot \underline{v}_{i_{m_i}} \neq \phi \end{array} \right|$$

Once, this generalized eigenvector is found, the <u>chain</u> of related eigenvectors of lower grade can be computed immediately:

$$\left|\begin{aligned}
\underline{v}_{i_{m_i-1}} &= (A-\lambda_i I)\cdot \underline{v}_{i_{m_i}} \\
\underline{v}_{i_{m_i-2}} &= (A-\lambda_i I)\cdot \underline{v}_{i_{m_i-1}} \\
&\;\vdots \\
\underline{v}_{i_1} &= (A-\lambda_i I)\cdot \underline{v}_{i_2}
\end{aligned}\right|$$

The general algorithm will be demonstrated at hand of an example:

$$\Lambda = \begin{bmatrix}
\begin{bmatrix} \lambda_1 & 1 & \phi & \phi \\ \phi & \lambda_1 & 1 & \phi \\ \phi & \phi & \lambda_1 & 1 \\ \phi & \phi & \phi & \lambda_1 \end{bmatrix} & \phi & & & \phi \\
 & \begin{bmatrix} \lambda_1 & 1 \\ \phi & \lambda_1 \end{bmatrix} & \phi & & \\
 & & \begin{bmatrix} \lambda_1 & 1 \\ \phi & \lambda_1 \end{bmatrix} & \phi & \\
\phi & & & \boxed{\lambda_2} & \phi \\
 & & & & \boxed{\lambda_3}
\end{bmatrix}$$

$\Rightarrow$ There are three Jordan blocks associated with λ_1:

$$\Rightarrow \Lambda = \begin{bmatrix} \Lambda_1^{(4)} & & & & \\ & \Lambda_1^{(2)} & & \phi & \\ & & \Lambda_1^{(2)} & & \\ & \phi & & \lambda_2 & \\ & & & & \lambda_3 \end{bmatrix}$$

$\Rightarrow$ There are uncontrollable / unobservable modes.

$$\Rightarrow \qquad \nu_1 = \text{Nullity}\{(A-\lambda_1 I)\} = 3$$

$\Leftrightarrow$ There exist three eigenvectors for λ_1 $\Leftrightarrow$ there exist three Jordan blocks for λ_1.

- There must exist one generalized eigenvector of grade 4 leading to a chain: $4 \to 3 \to 2 \to 1$. There exist two more generalized eigenvectors of grade 2, leading each to a chain $2 \to 1$.

$\Rightarrow \quad n = 10 \quad ; \quad m_1 = 8$

Algorithm:

- We start by computing the nullities of $(A - \lambda_i I)^k$ for increasing k until $\nu\{(A-\lambda_i I)^k\} = m$

Example:

$$\rho\{(A-\lambda_1 I)^0\} = 10 \quad ; \quad \nu\{(A-\lambda_1 I)^0\} = 0$$
$$\rho\{(A-\lambda_1 I)^1\} = 7 \quad ; \quad \nu\{(A-\lambda_1 I)^1\} = 3$$
$$\rho\{(A-\lambda_1 I)^2\} = 4 \quad ; \quad \nu\{(A-\lambda_1 I)^2\} = 6$$
$$\rho\{(A-\lambda_1 I)^3\} = 3 \quad ; \quad \nu\{(A-\lambda_1 I)^3\} = 7$$
$$\rho\{(A-\lambda_1 I)^4\} = 2 \quad ; \quad \nu\{(A-\lambda_1 I)^4\} = 8$$

$$\Rightarrow \underline{k = 4}$$

Abbreviations: $\rho\{(A-\lambda_i I)^k\} \equiv \rho_i^{(k)}$

$\nu\{(A-\lambda_i I)^k\} \equiv \nu_i^{(k)}$

Obviously, the following rules apply always:

(1) $\quad \rho_i^{(\phi)} \equiv n \; ; \; \nu_i^{(\phi)} \equiv \phi \; ; \; \forall i$

(2) $\quad \rho_i^{(j)} + \nu_i^{(j)} \equiv n \; ; \; \forall i, j$

(3) $\quad \nu_i^{(k)} \equiv m_i$

- After we have determined k, we look for:

$$\left| \begin{array}{l} (A - \lambda_1 I)^4 \underline{v}_4 = \phi \\ (A - \lambda_1 I)^3 \underline{v}_4 \neq \phi \end{array} \right|$$

$$\Longrightarrow \underline{\underline{v_4}}$$

Then: $\left| \begin{array}{l} \underline{v}_3 = (A - \lambda_1 I) \underline{v}_4 \\ \underline{v}_2 = (A - \lambda_1 I) \underline{v}_3 \\ \underline{v}_1 = (A - \lambda_1 I) \underline{v}_2 \end{array} \right|$

As $\nu_1^{(4)} - \nu_1^{(3)} = 1 \Rightarrow$ there exists exactly <u>one</u> generalized eigenvector of grade 4 ($\underline{v}_4$)

As $\gamma_1^{(3)} - \gamma_1^{(2)} = 1 \Rightarrow$ there exists exactly one generalized eigenvector of grade 3 ($\underline{v}_3$) which has already been found.

As $\gamma_1^{(2)} - \gamma_1^{(1)} = 3 \Rightarrow$ there exist two more generalized eigenvectors of grade 2 beside from $\underline{v}_2$:

$$\left| \begin{array}{l} (A-\lambda_1 I)^2 \underline{v}_6 = \phi \\ (A-\lambda_1 I)^1 \underline{v}_6 \neq \phi \end{array} \right|$$

and: $\underline{v}_6$ lin. indep. from $\underline{v}_2$

$$\Rightarrow \underline{\underline{v_6}}$$

$$\Rightarrow \left| \underline{v}_5 = (A-\lambda_1 I)\underline{v}_6 \right|$$

Then: $\left| \begin{array}{l} (A-\lambda_1 I)^2 \underline{v}_8 = \phi \\ (A-\lambda_1 I)^1 \underline{v}_8 \neq \phi \end{array} \right|$

and: v_8 lin. indep. from $\underline{v}_2, \underline{v}_6$

$\Rightarrow$ $\underline{v}_8$ ✓

$$\Rightarrow \quad \boxed{\underline{v}_7 = (A - \lambda_1 I)\,\underline{v}_8}$$

Of course:

$$\left| \begin{array}{l} (A - \lambda_2 I)\,\underline{v}_9 = \phi \\ (A - \lambda_3 I)\,\underline{v}_{10} = \phi \end{array} \right|$$

as usual.

$\Rightarrow$ $V = [\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_{10}]$

is the generalized right modal matrix, and

$$T = V^{-1}$$

will get us into Jordan-canonical form.

Warning:

[V, Lambda] = eig(A)

will not give you a

Let:

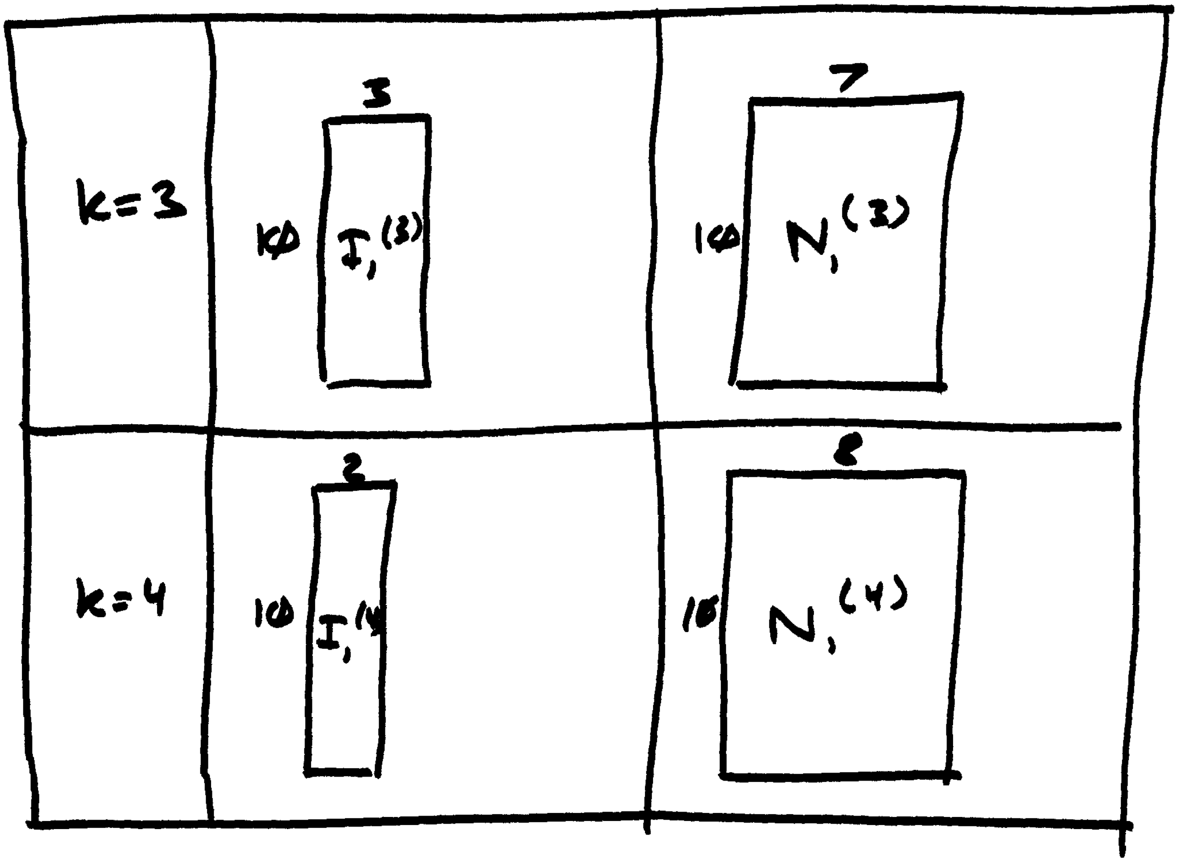
$$A_i = A - \lambda_i I$$

$$I_i^{(k)} = \text{Image}\left(A_i^{\,k}\right)$$

$$N_i^{(k)} = \text{Nullspace}\left(A_i^{\,k}\right)$$

Previous example:

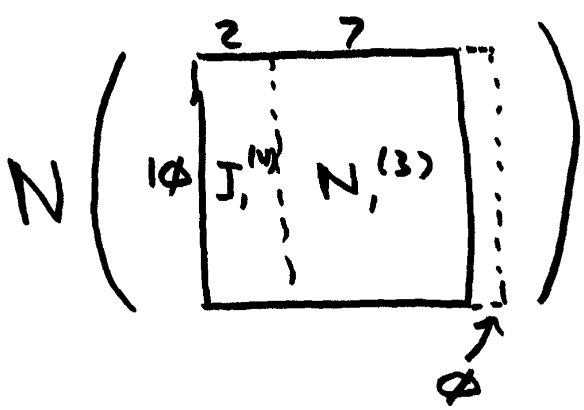
Grade	Image	Nullspace
$k=0$	$I_1^{(0)}$ (10×10)	$N_1^{(0)} = [\,]$
$k=1$	$I_1^{(1)}$ (10×7)	$N_1^{(1)}$ (10×3)
$k=2$	$I_1^{(2)}$ (10×4)	$N_1^{(2)}$ (10×6)



$$\underline{v}_4 \in \{I_1^{(3)}, N_1^{(4)}\}$$

$$\Rightarrow \underline{v}_4 \notin \{I_1^{(4)}, N_1^{(3)}\}$$

$$\Rightarrow \underline{v}_4^* = N([I_1^{(4)}, N_1^{(3)}])$$



$\longrightarrow \underline{v}_4$

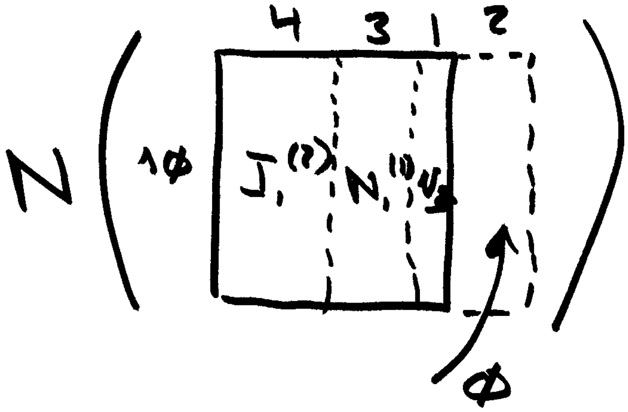
$\hookrightarrow \underline{v}_3, \underline{v}_2, \underline{v}_1$

(chain)

$$\underline{v}_6, \underline{v}_8 \in \{ I_1^{(1)}, N_1^{(2)} \}$$

$$\Rightarrow \underline{v}_6, \underline{v}_8 \notin \{ I_1^{(2)}, N_1^{(1)}, \underline{v}_2 \}$$

$$\Rightarrow \underline{v}_6, \underline{v}_8 = N\left([I_1^{(2)}, N_1^{(1)}, \underline{v}_2] \right)$$



$$\rightarrow \begin{cases} \underline{v}_6 \longrightarrow \underline{v}_5 \\ \underline{v}_8 \longrightarrow \underline{v}_7 \end{cases}$$