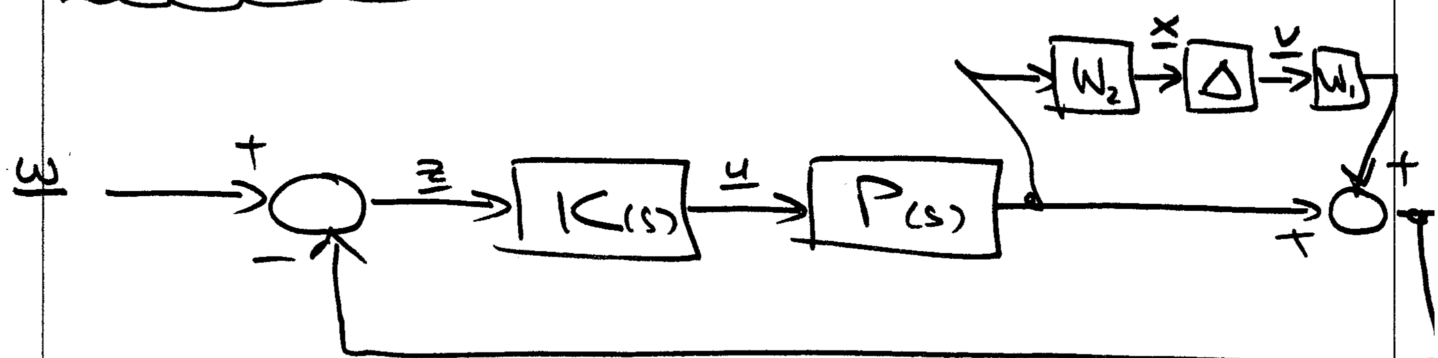
It is always possible to pull out the controllers from below and the normalized uncertainties from above.

Example:



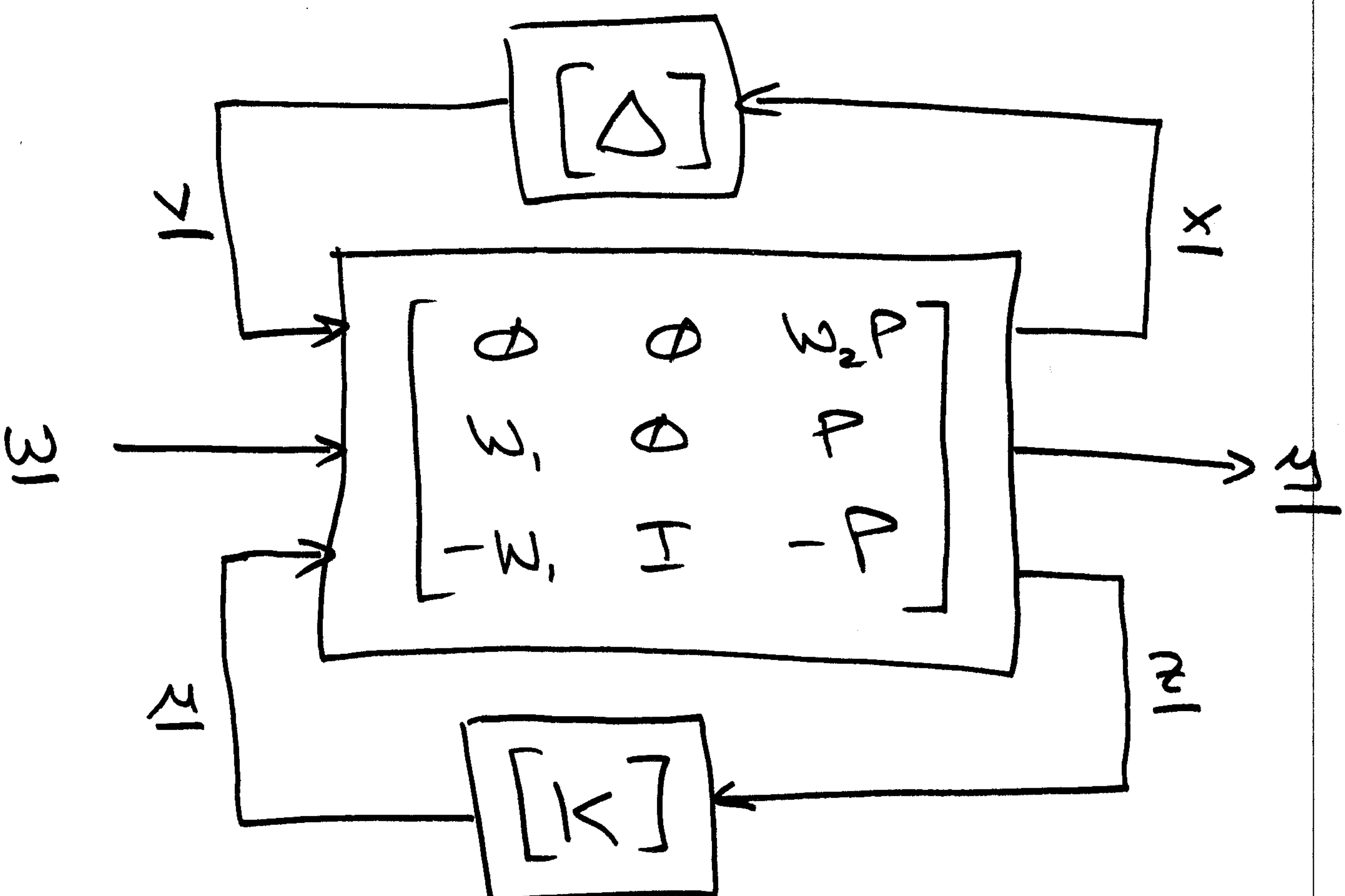
i.e., $\Pi(s) = \left(I + W_1(s)\cdot\Delta\cdot W_2(s)\right)P(s)$

$$\Rightarrow \left| \begin{aligned} \underline{x} &= W_2\cdot P\cdot \underline{u} \\ \underline{y} &= W_1\cdot \underline{v} + P\cdot \underline{u} \\ \underline{z} &= \underline{w} - W_1\cdot \underline{v} - P\cdot \underline{u} \end{aligned} \right|$$

$$\Rightarrow \quad \begin{bmatrix} \underline{x} \\ \underline{y} \\ \underline{z} \end{bmatrix} = \underbrace{\begin{bmatrix} \emptyset & \emptyset & W_2 \cdot P \\ W_1 & \emptyset & P \\ -W_1 & I & -P \end{bmatrix}}_{\hat{P}(s)} \cdot \begin{bmatrix} \underline{v} \\ \underline{w} \\ \underline{u} \end{bmatrix}$$

$$\underline{u} = [K]\,\underline{z}$$

$$\underline{v} = [\Delta]\,\underline{x}$$



is an LFT parametrization of the system.

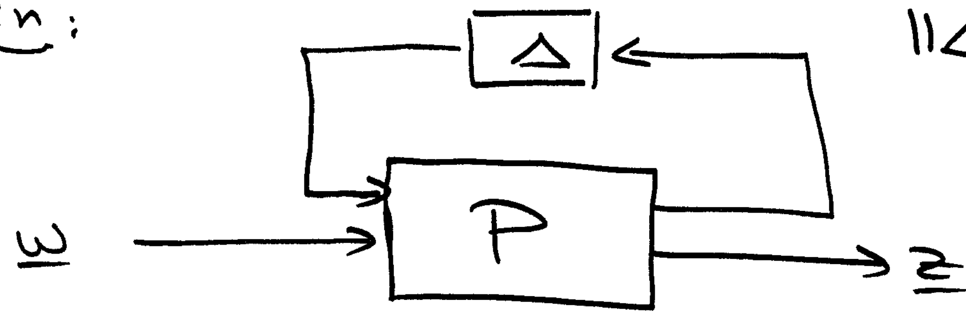
$$\Rightarrow T_{wy}(s) = \mathcal{F}_u(\mathcal{F}_\ell(\hat{P}, K), \Delta)$$

$$\equiv \mathcal{F}_\ell(\mathcal{F}_u(\hat{P}, \Delta), K)$$

- $\hat{P}(s)$ contains everything that is known about the plant.
- $K(s)$ contains all the controllers.
- $\Delta(s)$ contains all the normalized uncertainties about the plant: $\|\Delta\|_\infty = 1$.

Small Gain Theorem:

Given:



$\|\Delta\|_\infty = 1$

$$\Rightarrow T_{\underline{w}\underline{z}} = \mathcal{F}_u(P, \Delta) = P_{22} + P_{21}\Delta\left[I - P_{11}\Delta\right]^{-1} P_{12}$$

Assume P is stable

$\Rightarrow T_{\underline{w}\underline{z}}$ is stable and well-posed

iff $\|P_{11}\Delta\|_\infty < 1$

$$\Rightarrow \|P_{11}\|_\infty < 1$$

Robust Stability:

Let us assume:

$$\Pi = (I + W_1 \Delta W_2) P$$

$$\Rightarrow P_{11} = W_2 T_o W_1$$

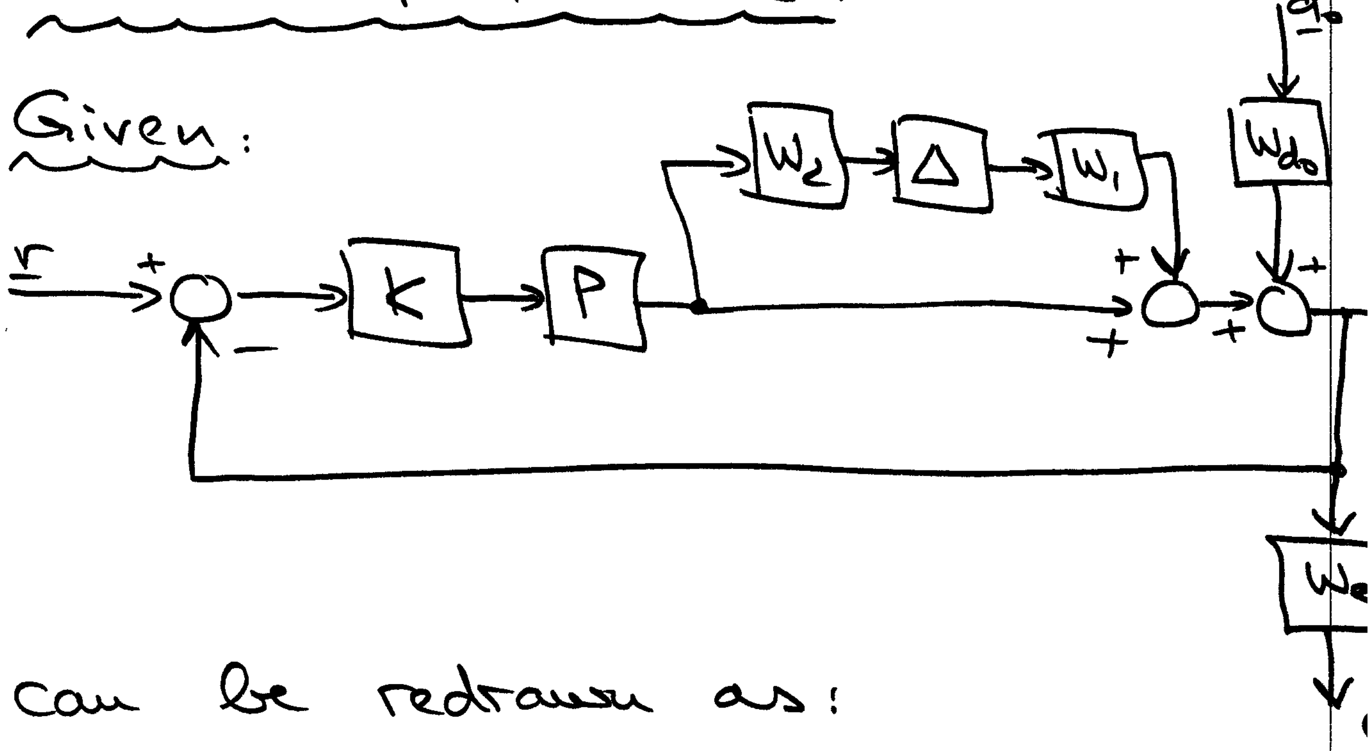
and robust stability implies:

$$\|P_{11}\|_\infty = \| W_2 T_o W_1 \|_\infty < 1$$

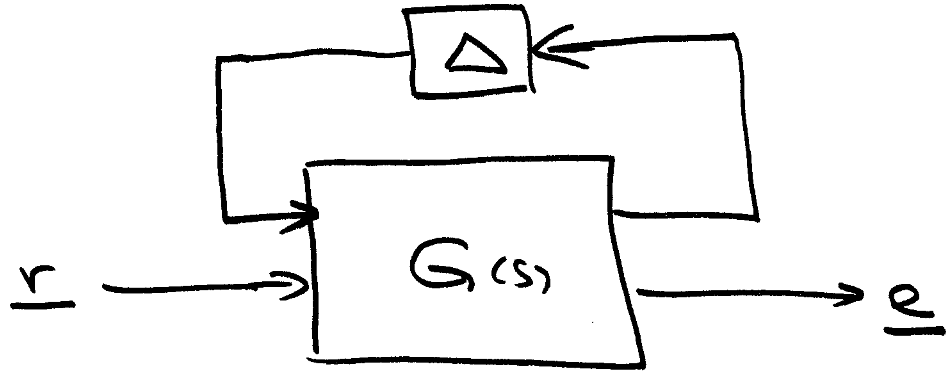
as shown earlier.

Robust performance:

Given:



can be redrawn as:



<u>with</u>:

$$G(s) = \begin{bmatrix} -W_2 T_o W_1 & -W_2 T_o W_{d_o} \\ W_e S_o W_1 & W_e S_o W_{d_o} \end{bmatrix}$$

Robust stability requires:

$$\| W_2 T_o W_1 \|_\infty < 1$$

Robust performance requires:

$$\| \mathcal{F}_u(G, \Delta) \|_\infty < 1$$

This can be rewritten as:

$$\|G\|_\mu = \inf_{d_\omega \in \mathbb{R}_+} \bar{\sigma} \left(\begin{bmatrix} -W_2 T_o W_1 & -d_\omega \cdot W_2 T_o W_{d_o} \\ \frac{1}{d_\omega} \cdot W_e S_o W_1 & W_e S_o W_{d_o} \end{bmatrix} \right)$$

⇒ The robust performance problem can be reinterpreted as a robust stability problem with a second Δ feed back from $\underline{e}$ to $\underline{d}_o$.

The $\|\cdot\|_\mu$ gives us less conservative results than the $\|\cdot\|_\infty$ norm, as shall be shown shortly.

Let us assume a system with S structured sources of uncertainty:

$$\Delta_i = \delta_i I \quad ; \quad i = 1, \ldots, S$$

and F unstructured sources of uncertainty:

$$\Delta_j \quad ; \quad j = 1, \ldots, F$$

Pulling out all the Δs, leaves us with:

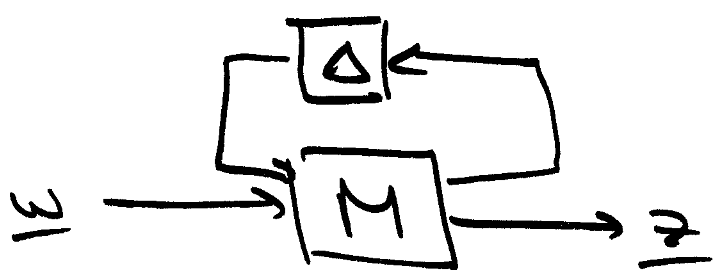
$$\Delta = \begin{bmatrix} \delta_1 I & & & & & \emptyset \\ & \ddots & & & & \\ & & \delta_s I & & & \\ & & & \Delta_1 & & \\ & \emptyset & & & \ddots & \\ & & & & & \Delta_F \end{bmatrix}$$

Δ is block-diagonal, with

$$\|\delta_i\|_\infty = 1 \; ; \; \|\Delta_j\|_\infty = 1$$

$$\Rightarrow \|\Delta\|_\infty = 1 \quad .$$

The small gain theorem suggests that:



$$\|M_{11}\|_\infty < 1$$

for robust stability. The problem is that there may not exist a M_{11} with $\|M_{11}\|_\infty < 1$ if all we assume is $\|\Delta\|_\infty = 1$.

⇒ The design may be too conservative.

We have thrown away <u>valuable</u> information about the internal structure of Δ, i.e., the fact that it is block-diagonal.

- We can <u>exploit</u> this knowledge be reducing our requirement to:

$$\|M_{11}\|_\mu < 1$$

which may have a solution, even if $\|M_{11}\|_\infty < 1$ does not.

For robust performance, we simply look at M as a whole, and request:

$$\|M\|_\mu < 1$$

I already gave you an algorithm to calculate $\|M\|_\mu$. However, we still may have a BAD search problem, because M is parametrized in the set of all stabilizing controllers, either using Youla-Kucera parametrization or the technique that follows now.

General decomposition theorem:

Given a nominal plant P:

$$\underline{\dot{x}} = A\underline{x} + B\underline{u}$$

We can find an optimal controller with:

$$\min_{\underline{u}} \int_0^\infty \begin{bmatrix} \underline{x} \\ \underline{u} \end{bmatrix}^* \cdot \begin{bmatrix} Q & N \\ N^* & R \end{bmatrix} \cdot \begin{bmatrix} \underline{x} \\ \underline{u} \end{bmatrix} dt$$

where:

$$\begin{bmatrix} Q & N \\ N^* & R \end{bmatrix} \geq 0$$

We must assume: $R > 0$.
We can always normalize R to I:

$$\Rightarrow \begin{bmatrix} Q & N \\ N^* & I \end{bmatrix} \geq \phi$$

If this matrix doesn't have full rank, we can decompose it as:

$$\begin{bmatrix} Q & N \\ N^* & I \end{bmatrix} = \begin{bmatrix} C_1^* \\ D_{12}^* \end{bmatrix} \cdot \begin{bmatrix} C_1 & D_{12} \end{bmatrix}$$

and reformulate the optimization problem as:

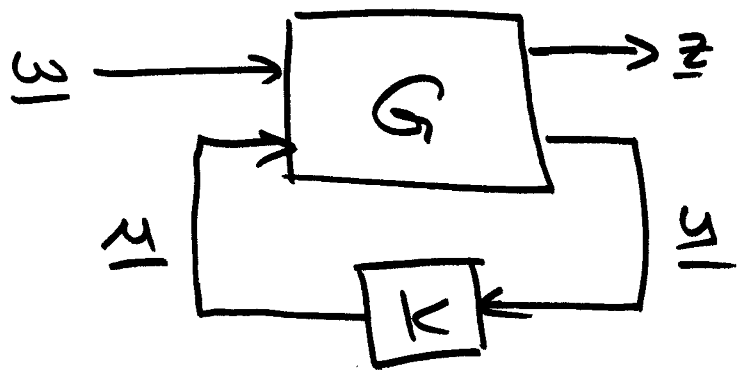
$$\min_{\underline{u}} \| C_1 \underline{x} + D_{12} \underline{u} \|_2$$

$\Rightarrow$ We can reinterpret this as an output equation:

$$\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + B_2 \underline{u} \\ \underline{y} = C_1 \underline{x} + D_{12} \underline{u} \\ \min_{\underline{u}} \| \underline{y} \|_2 \end{array} \right|$$

Can be solved by Riccati equations.

<u>Generalization</u>:



$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & \emptyset & D_{12} \\ C_2 & D_{21} & \emptyset \end{array}\right]$$

$\Rightarrow$ We formulate the two Hamiltonians:

$$H_2 = \begin{bmatrix} A & \emptyset \\ -C_1^* C_1 & -A^* \end{bmatrix} - \begin{bmatrix} B_2 \\ -C_1^* D_{12} \end{bmatrix} \cdot \begin{bmatrix} D_{12}^* C_1 & B_2^* \end{bmatrix}$$

$$J_2 = \begin{bmatrix} A^* & \emptyset \\ -B_1 B_1^* & -A \end{bmatrix} - \begin{bmatrix} C_2^* \\ -B_1 D_{21}^* \end{bmatrix} \cdot \begin{bmatrix} D_{21} B_1^* & C_2 \end{bmatrix}$$

We solve:

$$X_2 = Ric(H_2) \geq 0$$
$$Y_2 = Ric(J_2) \geq 0$$

$$\Rightarrow F_2 = -(B_2^* X_2 + D_{12}^* C_1)$$
$$L_2 = -(Y_2 C_2^* + B_1 D_{21}^*)$$

$$\Rightarrow A_{F_2} = A + B_2 F_2$$
$$C_{1F_2} = C_1 + D_{12} F_2$$

$$A_{L_2} = A + L_2 C_2$$
$$B_{1L_2} = B_1 + L_2 D_{21}$$

$$A_K = A + B_2 F_2 + L_2 C_2$$

$$\Rightarrow G_c(s) = \left[\begin{array}{c|c} A_{F_2} & I \\ \hline C_{1F_2} & \emptyset \end{array}\right];$$

$$G_L(s) = \left[\begin{array}{c|c} A_{L_2} & B_{1L_2} \\ \hline I & \emptyset \end{array}\right];$$

$$K_{nom}(s) = \left[\begin{array}{c|c} A_K & -L_2 \\ \hline F_2 & \emptyset \end{array}\right]$$

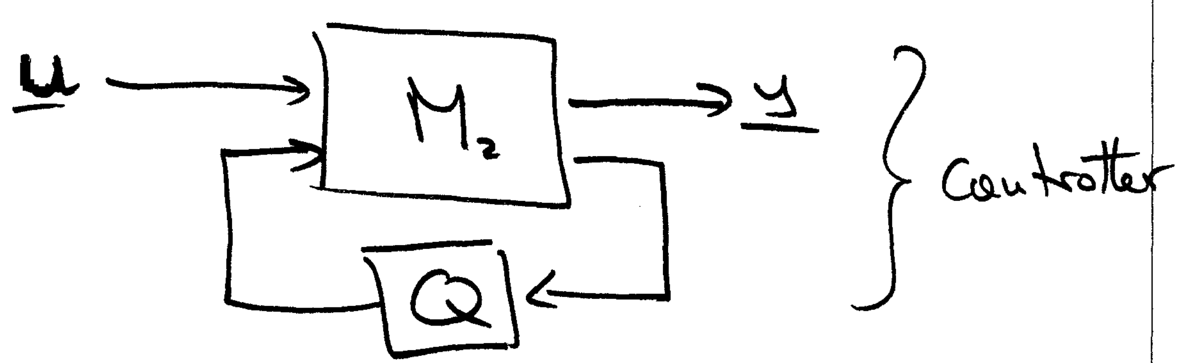
Also:

$$\|T_{\underline{w}\underline{z}}\|_2^2 = \|G_c B_1\|_2^2 + \|F_2 G_L\|_2^2$$
$$= \|G_c L_2\|_2^2 + \|C_1 G_L\|_2^2$$

Parametrization of all stabilizing controllers:

We are interested in finding all stabilizing controllers with:

$$\|T_{\underline{w}\underline{z}}\|_2 < \gamma :$$



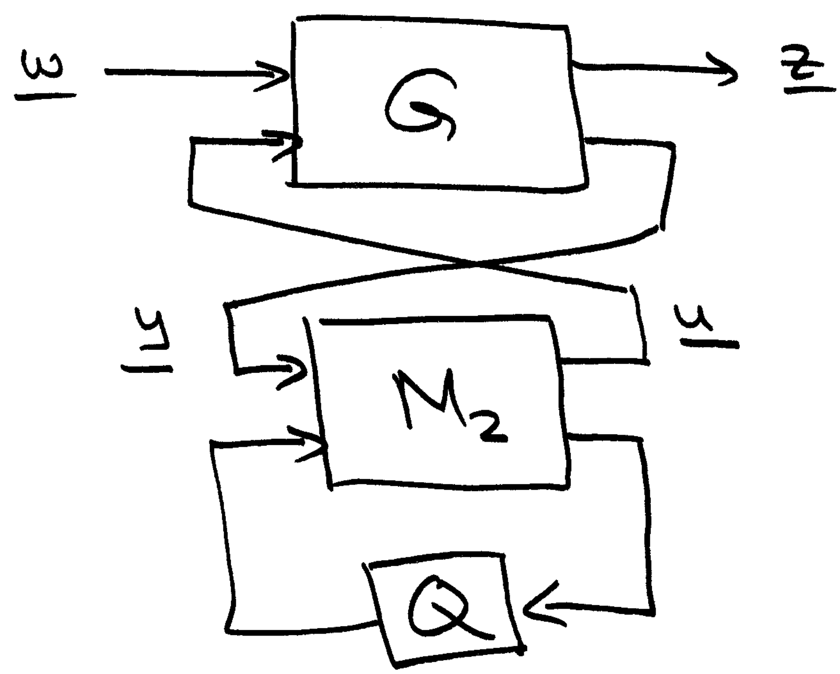
$$M_2(s) = \left[\begin{array}{c|cc} A_k & -L_2 & B_2 \\ \hline F_2 & \Phi & I \\ -C_2 & I & \Phi \end{array}\right]$$

$$Q > \Phi\ ;\quad \|Q\|_2^2 < \gamma^2 - \left(\|G_c B_1\|_2^2 + \|F_2 G_L\|_2^2\right.$$

is a set of suboptimal stabilizing controllers.

Clearly: $Q = \Phi \Leftrightarrow M_2(s) = K_{nom}(s)$

Adding the plant:



$$\Rightarrow \left| \begin{array}{l} K(s) = \mathcal{F}_l\left(M_2(s), Q(s)\right) \\ T_{\underline{w}\underline{z}}(s) = \mathcal{F}_l\left(G(s), K(s)\right) \end{array} \right|$$

<u>or</u>: $T_{\underline{w}\underline{z}}(s) = \mathcal{F}_l\left(N(s), Q(s)\right)$

<u>with</u>:

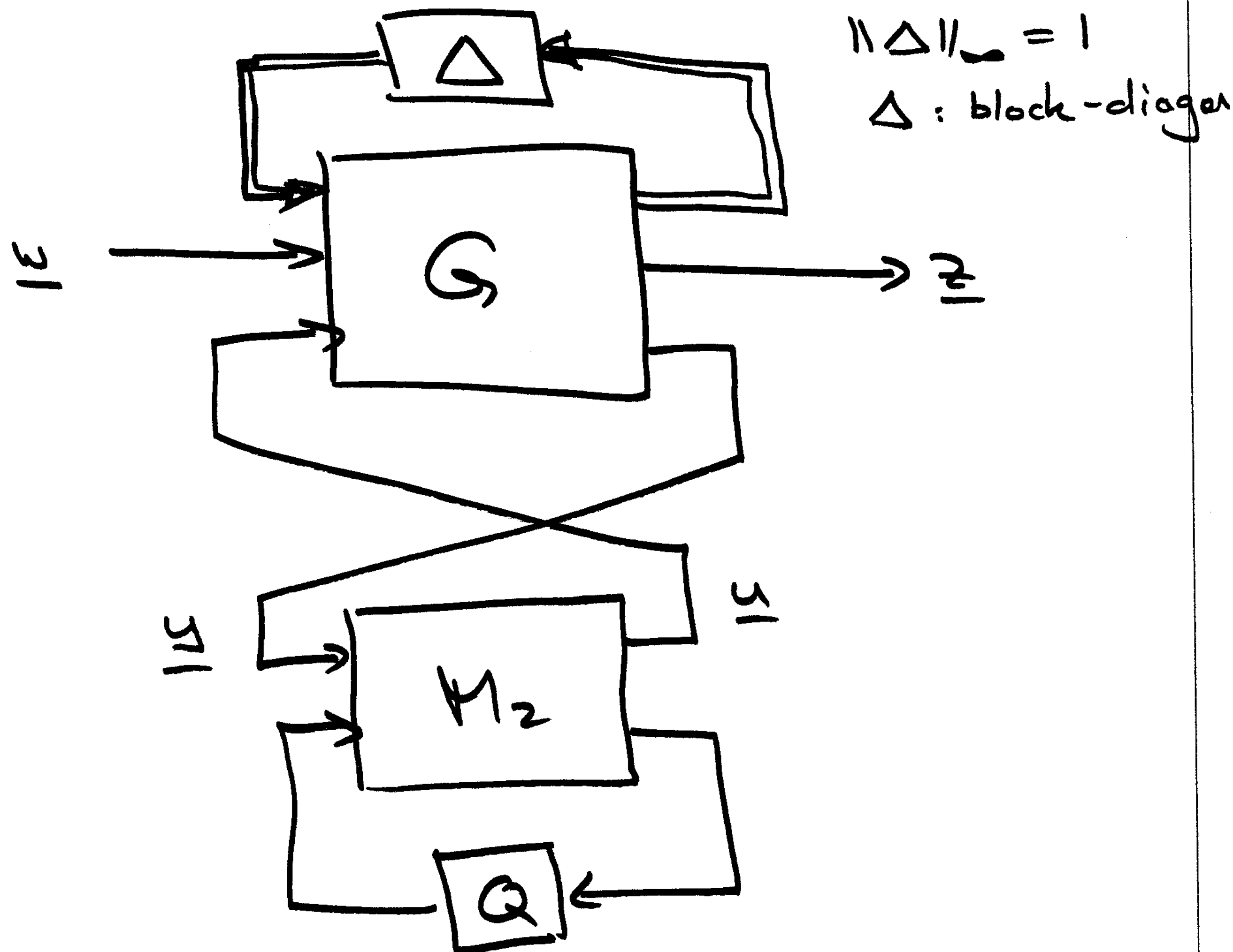
$$N(s) = \left[\begin{array}{cc|cc} A_{F_2} & -B_2 \cdot F_2 & B_1 & B_2 \\ \phi & A_{L_2} & B_{1L_2} & \phi \\ \hline C_{1F_2} & -D_{12} \cdot F_2 & \phi & D_{12} \\ \phi & C_2 & D_{21} & \phi \end{array}\right]$$

<u>or</u>: <u>let</u>:

$$U(s) = \left[\begin{array}{c|c} A_{F_2} & B_2 \\ \hline C_{1F_2} & D_{12} \end{array}\right]; \quad V(s) = \left[\begin{array}{c|c} A_{L_2} & B_{1L_2} \\ \hline C_2 & D_{21} \end{array}\right]$$

$$\Rightarrow T_{\underline{w}\underline{z}}(s) = G_c B_1 - U F_2 G_L + UQV$$

Adding the plant uncertainty:



$\Rightarrow K(s) = \mathcal{F}_\ell(M_2, Q)$

$M(s) = \mathcal{F}_\ell(G, K)$

Choose: Q such that

$$\|M\|_\mu < 1$$

or: $\min_Q \|M\|_\mu$