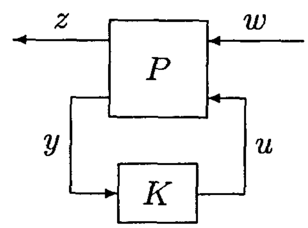
Remark 10.2 A simple interpretation of the result (c) is given by considering the signals in the feedback systems,



assuming this structure is well-posed. And we have

$$\begin{aligned} \begin{bmatrix} z \\ y \end{bmatrix} &= P \begin{bmatrix} w \\ u \end{bmatrix}, \qquad u = Ky \\ \Rightarrow z &= \mathcal{F}_\ell(P, K)w = Gw; \end{aligned}$$

hence

$$\begin{aligned} \begin{bmatrix} w \\ u \end{bmatrix} &= P^{-1} \begin{bmatrix} z \\ y \end{bmatrix}, \qquad z = Gw \\ \Rightarrow u &= \mathcal{F}_u(P^{-1}, G)y, \quad \text{or} \quad K = \mathcal{F}_u(P^{-1}, G). \end{aligned}$$

♡

10.2 Examples of LFTs

LFT is a very convenient tool to formulate many mathematical objects. In this section and the sections to follow, some commonly encountered control or mathematical objects are given new perspectives, i.e., they will be examined from the LFT point of view.

Polynomials

A very commonly encountered object in control and mathematics is a polynomial function. For example,

$$p(\delta) = a_0 + a_1\delta + \cdots + a_n\delta^n$$

with indeterminate δ. It is easy to verify that $p(\delta)$ can be written in the following LFT form:

$$p(\delta) = \mathcal{F}_\ell(M, \delta I_n)$$

with

$$M = \left[\begin{array}{c|cccc} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ \hline 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right]$$

Hence every polynomial is a linear fraction of its indeterminates. More generally, any multidimensional (matrix) polynomials are also LFTs in their indeterminates; for example,

$$p(\delta_1, \delta_2) = a_1\delta_1^2 + a_2\delta_2^2 + a_3\delta_1\delta_2 + a_4\delta_1 + a_5\delta_2 + a_6.$$

Then

$$p(\delta_1, \delta_2) = \mathcal{F}_\ell(N, \Delta)$$

with

$$N = \left[\begin{array}{c|cccc} a_6 & 1 & 0 & 1 & 0 \\ \hline a_4 & 0 & a_1 & 0 & a_3 \\ 1 & 0 & 0 & 0 & 0 \\ a_5 & 0 & 0 & 0 & a_2 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right], \qquad \Delta = \begin{bmatrix} \delta_1 I_2 & \\ & \delta_2 I_2 \end{bmatrix}.$$

It should be noted that these representations or realizations of polynomials are neither unique nor necessarily minimal. Here a minimal realization refers to a realization with the smallest possible dimension of Δ. As commonly known, in multidimensional systems and filter theory, it is usually very hard, if not impossible, to find a minimal realization for even a two variable polynomial. In fact, the minimal dimension of Δ depends also on the field (real, complex, etc.) of the realization. More detailed discussion of this issue is beyond the scope of this book, the interested readers should consult the references in 2-d or n-d systems or filter theory.

Rational Functions

As another example of LFT representation, we consider a rational matrix function (not necessarily proper), $F(\delta_1, \delta_2, \cdots, \delta_m)$, with a finite value at the origin: $F(0, 0, \cdots, 0)$ is finite. Then $F(\delta_1, \delta_2, \cdots, \delta_m)$ can be written as an LFT in $(\delta_1, \delta_2, \cdots, \delta_m)$ (some δ_i may be repeated). To see that, write

$$F(\delta_1, \delta_2, \cdots, \delta_m) = \frac{N(\delta_1, \delta_2, \cdots, \delta_m)}{d(\delta_1, \delta_2, \cdots, \delta_m)} = N(\delta_1, \delta_2, \cdots, \delta_m) \;\; (d(\delta_1, \delta_2, \cdots, \delta_m)I)^{-1}$$

where $N(\delta_1, \delta_2, \cdots, \delta_m)$ is a multidimensional matrix polynomial and $d(\delta_1, \delta_2, \cdots, \delta_m)$ is a scalar multidimensional polynomial with $d(0, 0, \cdots, 0) \neq 0$. Both N and dI can be represented as LFTs, and, furthermore, since $d(0, 0, \cdots, 0) \neq 0$, the inverse of dI is also an LFT as shown in Lemma 10.3. Now the conclusion follows by the fact that the product of LFTs is also an LFT. (Of course, the above LFT representation problem is exactly the problem of state space realization for a multidimensional transfer matrix.) However, this is usually not a nice way to get an LFT representation for a rational matrix since this approach usually results in a much higher dimensioned Δ than required. For example,

$$f(\delta) = \frac{\alpha + \beta\delta}{1 + \gamma\delta} = \mathcal{F}_\ell(M, \delta)$$

with

$$M = \left[\begin{array}{c|c} \alpha & \beta - \alpha\gamma \\ \hline 1 & -\gamma \end{array}\right].$$

By using the above approach, we would end up with

$$f(\delta) = \mathcal{F}_\ell(N, \delta I_2)$$

and

$$N = \left[\begin{array}{c|cc} \alpha & \beta & -\alpha\gamma \\ \hline 1 & 0 & -\gamma \\ 1 & 0 & -\gamma \end{array}\right].$$

Although the latter can be reduced to the former, it is not easy to see how to carry out such reduction for a complicated problem, even if it is possible.

State Space Realizations

We can use the LFT formulae to establish the relationship between transfer matrices and their state space realizations. A system with a state space realization as

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

has a transfer matrix of

$$G(s) = D + C(sI - A)^{-1}B = \mathcal{F}_u(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \frac{1}{s}I).$$

Now take $\Delta = \frac{1}{s}I$, the transfer matrix can be written as

$$G(s) = \mathcal{F}_u(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta).$$

More generally, consider a discrete time 2-D (or MD) system realized by the first-order state space equation

$$\begin{aligned} x_1(k_1+1, k_2) &= A_{11}x_1(k_1, k_2) + A_{12}x_2(k_1, k_2) + B_1u(k_1, k_2) \\ x_2(k_1, k_2+1) &= A_{21}x_1(k_1, k_2) + A_{22}x_2(k_1, k_2) + B_2u(k_1, k_2) \\ y(k_1, k_2) &= C_1x_1(k_1, k_2) + C_2x_2(k_1, k_2) + Du(k_1, k_2). \end{aligned}$$

In the same way, take

$$\Delta = \begin{bmatrix} z_1^{-1}I & 0 \\ 0 & z_2^{-1}I \end{bmatrix} =: \begin{bmatrix} \delta_1 I & 0 \\ 0 & \delta_2 I \end{bmatrix}$$

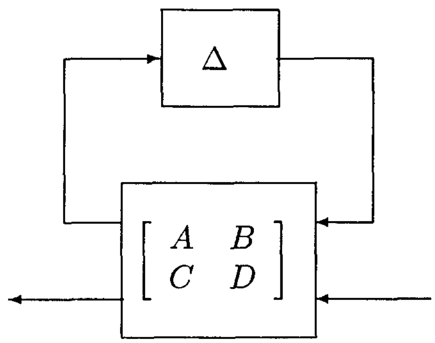
where z_i denotes the forward shift operator, and let

$$A \triangleq \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B \triangleq \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C \triangleq \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

then its transfer matrix is

$$\begin{aligned} G(z_1, z_2) &= D + C(\begin{bmatrix} z_1 I & 0 \\ 0 & z_2 I \end{bmatrix} - A)^{-1} B = D + C\Delta(I - \Delta A)^{-1} B \\ &=: \mathcal{F}_u(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta). \end{aligned}$$

Both formulations can correspond to the following diagram:



The following notation for a transfer matrix has already been adopted in the previous chapters:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \mathcal{F}_u(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta).$$

It is easy to see that this notation can be adopted for general dynamical systems, e.g., multidimensional systems, as far as the structure Δ is specified. This notation means that the transfer matrix can be expressed as an LFT of Δ with the coefficient matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. In this special case, we say the parameter matrix Δ is the *frequency structure* of the system state space realization. This notation is deliberately somewhat ambiguous and can be viewed as both a transfer matrix and one of its realizations. The ambiguity is benign and convenient and can always be resolved from the context.

Frequency Transformation

The bilinear transformation between the z-domain and s-domain

$$s = \frac{z+1}{z-1}$$

transforms the unit disk to the left-half plane and is the simplest example of an LFT. We may rewrite it in our standard form as

$$\frac{1}{s} I = I - \sqrt{2} I \ \ z^{-1} I \ \ (I + z^{-1} I)^{-1} \ \sqrt{2} I = \mathcal{F}_u(N, z^{-1} I)$$

— 108 —

where

$$N = \begin{bmatrix} I & \sqrt{2}I \\ -\sqrt{2}I & -I \end{bmatrix}.$$

Now consider a continuous system

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \mathcal{F}_u(M, \frac{1}{s}I)$$

where

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix};$$

then the corresponding discrete time system realization is given by

$$\tilde{G}(z) = \mathcal{F}_u(M, \frac{z-1}{z+1}I) = \mathcal{F}_u(M, \mathcal{F}_u(N, z^{-1}I)) = \mathcal{F}_u(\tilde{M}, z^{-1}I)$$

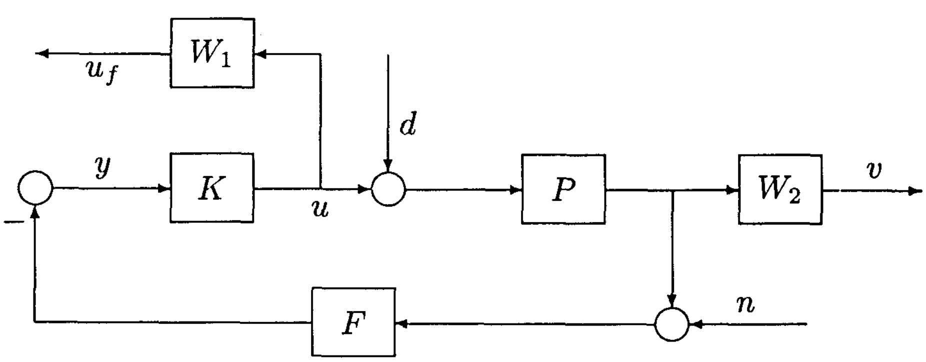
with

$$\tilde{M} = \begin{bmatrix} -(I-A)^{-1}(I+A) & -\sqrt{2}(I-A)^{-1}B \\ \sqrt{2}C(I-A)^{-1} & C(I-A)^{-1}B + D \end{bmatrix}.$$

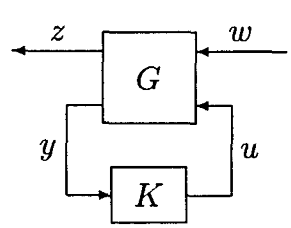
The transformation from the z-domain to the s-domain can be obtained similarly.

Simple Block Diagrams

A feedback system with the following block diagram



can be rearranged as an LFT:



with

$$w = \begin{pmatrix} d \\ n \end{pmatrix} \qquad z = \begin{pmatrix} v \\ u_f \end{pmatrix}$$

and

$$G = \left[\begin{array}{cc:c} W_2P & 0 & W_2P \\ 0 & 0 & W_1 \\ \hdashline -FP & -F & -FP \end{array}\right]$$

Constrained Structure Synthesis

Using the properties of LFTs, we can show that constrained structure control synthesis problems can be converted to constrained structure *constant* output feedback problems. Consider the synthesis structure in the last example and assume

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \qquad K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

Then it is easy to show that

$$\mathcal{F}_\ell(G, K) = \mathcal{F}_\ell(M(s), F)$$

where

$$M(s) = \left[\begin{array}{cc|c:cc} A & 0 & B_1 & 0 & B_2 \\ 0 & 0 & 0 & I & 0 \\ \hline C_1 & 0 & D_{11} & 0 & D_{12} \\ \hdashline 0 & I & 0 & 0 & 0 \\ C_2 & 0 & D_{21} & 0 & D_{22} \end{array}\right]$$

and

$$F = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$$

Note that F is a *constant matrix*, not a system matrix. Hence if the controller structure is fixed (or constrained), then the corresponding control problem becomes a constant (constrained) output feedback problem.

Parametric Uncertainty: A Mass/Spring/Damper System

One natural type of uncertainty is unknown coefficients in a state space model. To motivate this type of uncertainty description, we will begin with a familiar mass/spring/damper system, shown below in Figure 10.1.

The dynamical equation of the system motion can be described by

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F}{m}.$$

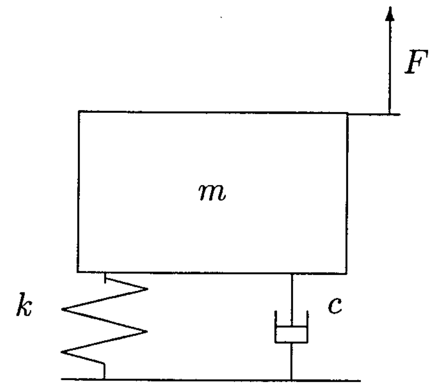


Figure 10.1: Mass/Spring/Damper System

Suppose that the 3 physical parameters m, c, and k are not known exactly, but are believed to lie in known intervals. In particular, the actual mass m is within 10% of a nominal mass, $\bar{m}$, the actual damping value c is within 20% of a nominal value of $\bar{c}$, and the spring stiffness is within 30% of its nominal value of $\bar{k}$. Now introducing perturbations δ_m, δ_c, and δ_k, which are assumed to be unknown but lie in the interval $[-1, 1]$, the block diagram for the dynamical system can be shown in Figure 10.2.

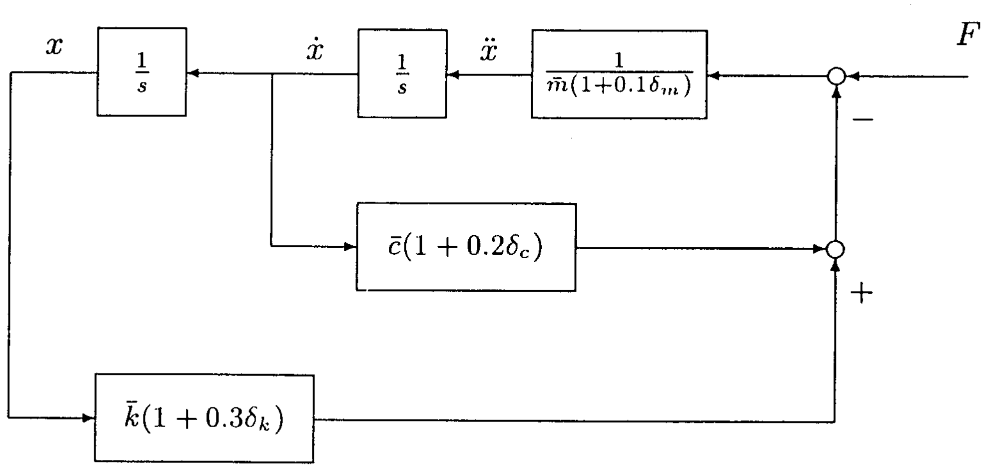


Figure 10.2: Block Diagram of Mass/Spring/Damper Equation

It is easy to check that $\frac{1}{m}$ can be represented as an LFT in δ_m:

$$\frac{1}{m} = \frac{1}{\bar{m}(1+0.1\delta_m)} = \frac{1}{\bar{m}} - \frac{0.1}{\bar{m}}\delta_m(1+0.1\delta_m)^{-1} = \mathcal{F}_\ell\left(M_1, \delta_m\right).$$

with $M_1 = \begin{bmatrix} \frac{1}{\bar{m}} & -\frac{0.1}{\bar{m}} \\ 1 & -0.1 \end{bmatrix}$. Suppose that the input signals of the dynamical system are selected as $x_1 = x, x_2 = \dot{x}, F$, and the output signals are selected as $\dot{x}_1$ and $\dot{x}_2$. To represent the system model as an LFT of the natural uncertainty parameters δ_m, δ_c and δ_k, we shall first isolate the uncertainty parameters and denote the inputs and outputs of δ_k, δ_c and δ_m as y_k, y_c, y_m and u_k, u_c, u_m, respectively, as shown in Figure 10.3. Then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ y_k \\ y_c \\ y_m \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\bar{k}}{\bar{m}} & -\frac{\bar{c}}{\bar{m}} & \frac{1}{\bar{m}} & -\frac{1}{\bar{m}} & -\frac{1}{\bar{m}} & -\frac{0.1}{\bar{m}} \\ 0.3\bar{k} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2\bar{c} & 0 & 0 & 0 & 0 \\ -\bar{k} & -\bar{c} & 1 & -1 & -1 & -0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ F \\ u_k \\ u_c \\ u_m \end{bmatrix}, \quad \begin{bmatrix} u_k \\ u_c \\ u_m \end{bmatrix} = \Delta \begin{bmatrix} y_k \\ y_c \\ y_m \end{bmatrix}$$

i.e.,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \mathcal{F}_\ell(M, \Delta) \begin{bmatrix} x_1 \\ x_2 \\ F \end{bmatrix}$$

where

$$M = \left[\begin{array}{ccc:ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\bar{k}}{\bar{m}} & -\frac{\bar{c}}{\bar{m}} & \frac{1}{\bar{m}} & -\frac{1}{\bar{m}} & -\frac{1}{\bar{m}} & -\frac{0.1}{\bar{m}} \\ \hdashline 0.3\bar{k} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2\bar{c} & 0 & 0 & 0 & 0 \\ -\bar{k} & -\bar{c} & 1 & -1 & -1 & -0.1 \end{array}\right], \quad \Delta = \begin{bmatrix} \delta_k & 0 & 0 \\ 0 & \delta_c & 0 \\ 0 & 0 & \delta_m \end{bmatrix}$$

General Affine State-Space Uncertainty

We will consider a special class of state space models with unknown coefficients and show how this type of uncertainty can be represented via the LFT formulae with respect to an uncertain parameter matrix so that the perturbations enter the system in a feedback form. This type of modeling will form the basic building blocks for components with *parametric* uncertainty.

Consider a linear system $G_\delta(s)$ that is parameterized by k uncertain parameters, $\delta_1, \ldots, \delta_k$, and has the realization

$$G_\delta(s) = \left[\begin{array}{c|c} A + \sum_{i=1}^{k} \delta_i \hat{A}_i & B + \sum_{i=1}^{k} \delta_i \hat{B}_i \\ \hline C + \sum_{i=1}^{k} \delta_i \hat{C}_i & D + \sum_{i=1}^{k} \delta_i \hat{D}_i \end{array}\right]$$

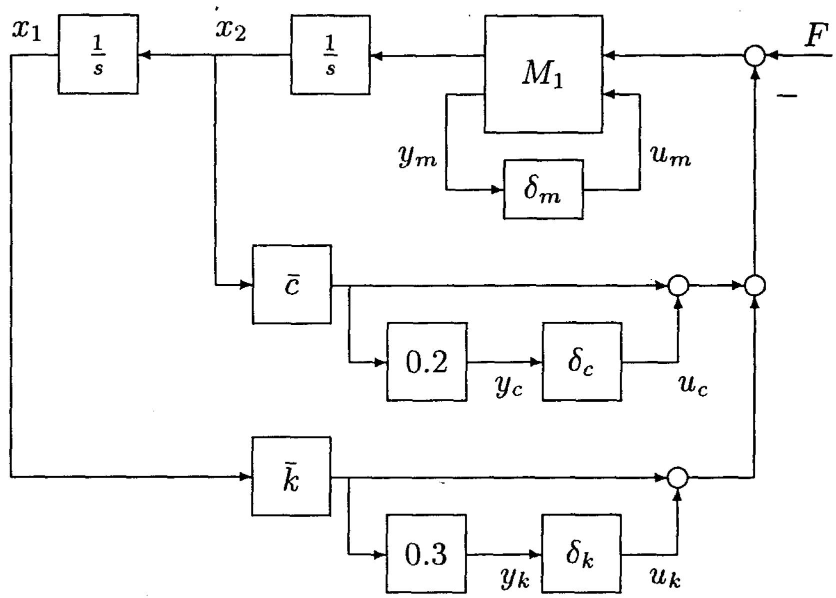


Figure 10.3: Mass/Spring/Damper System

Here $A, \hat{A}_i \in \mathbb{R}^{n \times n}, B, \hat{B}_i \in \mathbb{R}^{n \times n_u}, C, \hat{C}_i \in \mathbb{R}^{n_y \times n}$, and $D, \hat{D}_i \in \mathbb{R}^{n_y \times n_u}$.

The various terms in these state equations are interpreted as follows: the nominal system description $G(s)$, given by known matrices A, B, C, and D, is (A, B, C, D) and the parametric uncertainty in the nominal system is reflected by the k scalar uncertain parameters $\delta_1, \ldots, \delta_k$, and we can specify them, say by $\delta_i \in [-1, 1]$. The structural knowledge about the uncertainty is contained in the matrices $\hat{A}_i, \hat{B}_i, \hat{C}_i$, and $\hat{D}_i$. They reflect how the i'th uncertainty, δ_i, affects the state space model.

Now, we consider the problem of describing the perturbed system via the LFT formulae so that all the uncertainty can be represented as a nominal system with the unknown parameters entering it as the feedback gains. This is shown in Figure 10.4.

Since $G_\delta(s) = \mathcal{F}_u(M_\delta, \frac{1}{s}I)$ where

$$M_\delta \triangleq \begin{bmatrix} A + \sum_{i=1}^{k} \delta_i \hat{A}_i & B + \sum_{i=1}^{k} \delta_i \hat{B}_i \\ C + \sum_{i=1}^{k} \delta_i \hat{C}_i & D + \sum_{i=1}^{k} \delta_i \hat{D}_i \end{bmatrix},$$

we need to find an LFT representation for the matrix M_δ with respect to

$$\Delta_p = \operatorname{diag}\{\delta_1 I, \delta_2 I, \ldots \delta_k I\}.$$

To achieve this with the smallest possible size of repeated blocks, let q_i denote the rank

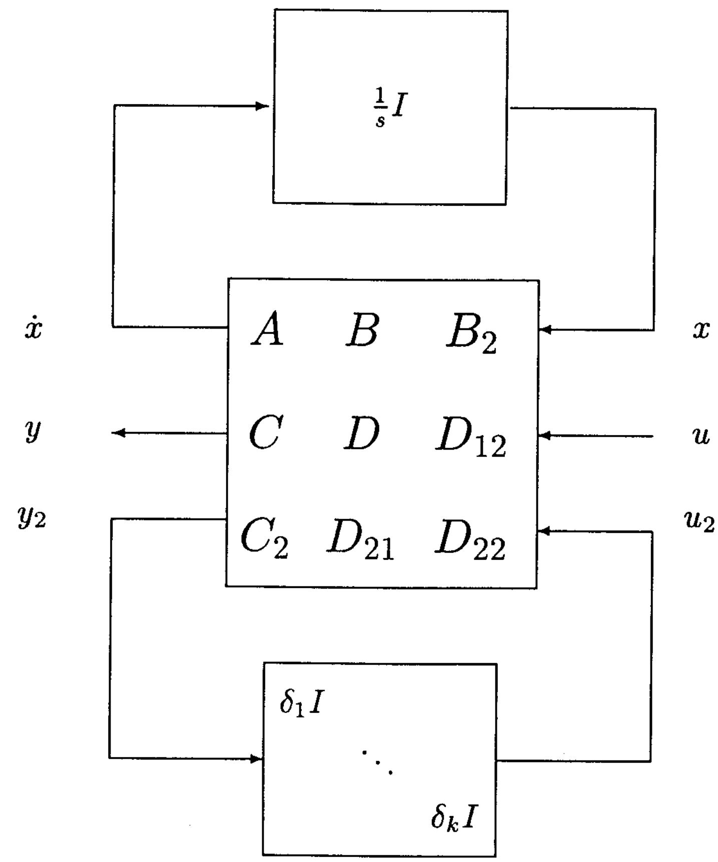


Figure 10.4: LFT Representation of State Space Uncertainty

of the matrix

$$P_i \triangleq \begin{bmatrix} \hat{A}_i & \hat{B}_i \\ \hat{C}_i & \hat{D}_i \end{bmatrix} \in \mathbb{R}^{(n+n_y)\times(n+n_u)}$$

for each i. Then P_i can be written as

$$P_i = \begin{bmatrix} L_i \\ W_i \end{bmatrix} \begin{bmatrix} R_i \\ Z_i \end{bmatrix}^*$$

where $L_i \in \mathbb{R}^{n\times q_i}$, $W_i \in \mathbb{R}^{n_y\times q_i}$, $R_i \in \mathbb{R}^{n\times q_i}$ and $Z_i \in \mathbb{R}^{n_u\times q_i}$. Hence, we have

$$\delta_i P_i = \begin{bmatrix} L_i \\ W_i \end{bmatrix} [\delta_i I_{q_i}] \begin{bmatrix} R_i \\ Z_i \end{bmatrix}^*,$$

and M_δ can be written as

$$M_\delta = \overbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}^{M_{11}} + \overbrace{\begin{bmatrix} L_1 & \cdots & L_k \\ W_1 & \cdots & W_k \end{bmatrix}}^{M_{12}} \overbrace{\begin{bmatrix} \delta_1 I_{q_1} & & \\ & \ddots & \\ & & \delta_k I_{q_k} \end{bmatrix}}^{\Delta_p} \overbrace{\begin{bmatrix} R_1^* & Z_1^* \\ \vdots & \vdots \\ R_k^* & Z_k^* \end{bmatrix}}^{M_{21}}$$

i.e.

$$M_\delta = \mathcal{F}_\ell(\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & 0 \end{bmatrix}, \Delta_p).$$

Therefore, the matrices B_2, C_2, D_{12}, D_{21}, and D_{22} in the diagram are

$$\begin{aligned} B_2 &= \begin{bmatrix} L_1 & L_2 & \cdots & L_k \end{bmatrix} \\ D_{12} &= \begin{bmatrix} W_1 & W_2 & \cdots & W_k \end{bmatrix} \\ C_2 &= \begin{bmatrix} R_1 & R_2 & \cdots & R_k \end{bmatrix}^* \\ D_{21} &= \begin{bmatrix} Z_1 & Z_2 & \cdots & Z_k \end{bmatrix}^* \\ D_{22} &= 0 \end{aligned}$$

and

$$G_\delta(\Delta) = \mathcal{F}_u(\mathcal{F}_\ell(\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & 0 \end{bmatrix}, \Delta_p), \frac{1}{s}I).$$

10.3 Basic Principle

We have studied several simple examples of the use of LFTs and, in particular, their role in modeling uncertainty. The basic principle at work here in writing a matrix LFT is often referred to as *"pulling out the* Δs*"*. We will try to illustrate this with another picture. Consider a structure with four substructures interconnected in some known way as shown in Figure 10.5.

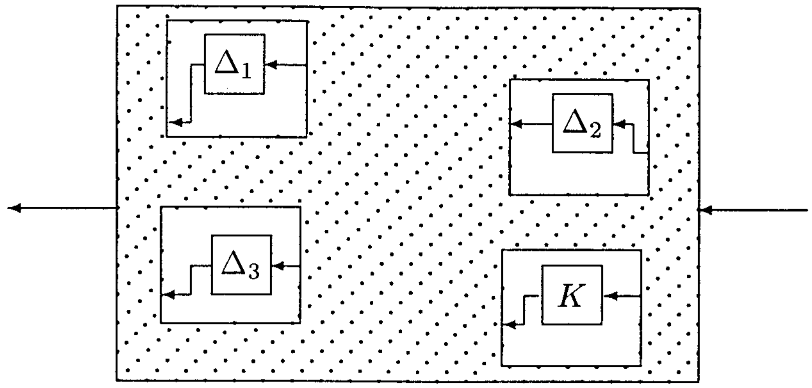


Figure 10.5: Multiple Source of Uncertain Structure

This diagram can be redrawn as a standard one via "pulling out the Δs" in Figure 10.6. Now the matrix "M" of the LFT can be obtained by computing the corresponding transfer matrix in the shadowed box.

We shall illustrate the above principle with an example. Consider an input/output relation

$$z = \frac{a + b\delta_2 + c\delta_1\delta_2^2}{1 + d\delta_1\delta_2 + e\delta_1^2} w =: Gw$$

where a, b, c, d and e are given constants or transfer functions. We would like to write G as an LFT in terms of δ_1 and δ_2. We shall do this in three steps:

1. Draw a block diagram for the input/output relation with each δ separated as shown in Figure 10.7.

2. Mark the inputs and outputs of the δ's as y's and u's, respectively. (This is essentially *pulling out the* δs).

3. Write z and y's in terms of w and u's with all δ's taken out. (This step is equivalent to computing the transformation in the shadowed box in Figure 10.6.)

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ z \end{bmatrix} = M \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ w \end{bmatrix}$$

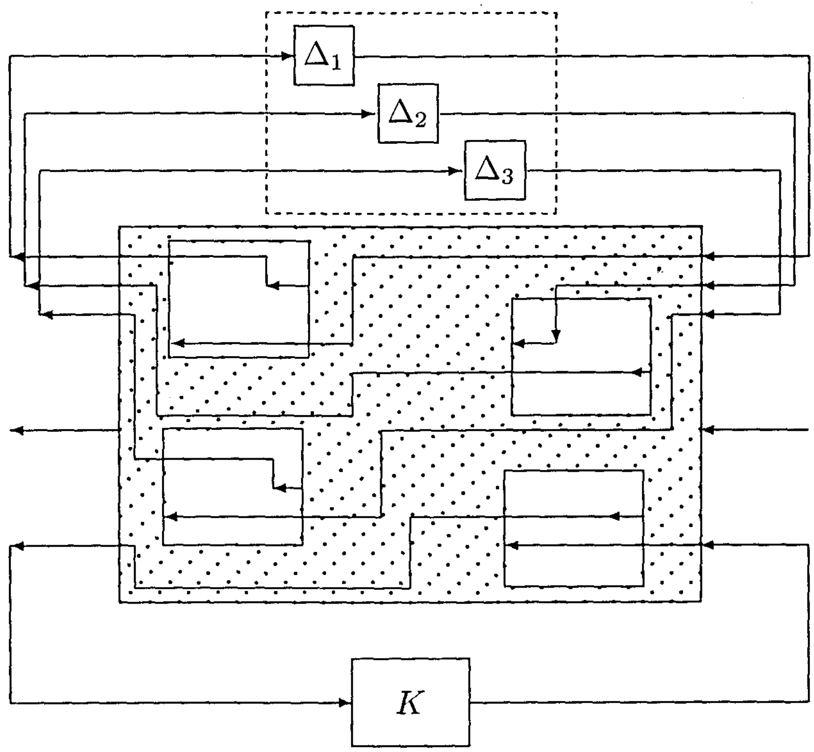


Figure 10.6: Pulling out the Δs

where

$$M = \left[\begin{array}{cccc:c} 0 & -e & -d & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -be & -bd+c & 0 & b \\ \hdashline 0 & -ae & -ad & 1 & a \end{array}\right].$$

Then

$$z = \mathcal{F}_u(M, \Delta)w, \quad \Delta = \begin{bmatrix} \delta_1 I_2 & 0 \\ 0 & \delta_2 I_2 \end{bmatrix}.$$

All LFT examples in the last section can be obtained following the above steps.

10.4 Redheffer Star-Products

The most important property of LFTs is that any interconnection of LFTs is again an LFT. This property is by far the most often used and is the heart of LFT machinery. Indeed, it is not hard to see that most of the interconnection structures discussed early, e.g., feedback and cascade, can be viewed as special cases of the so-called *star product*.

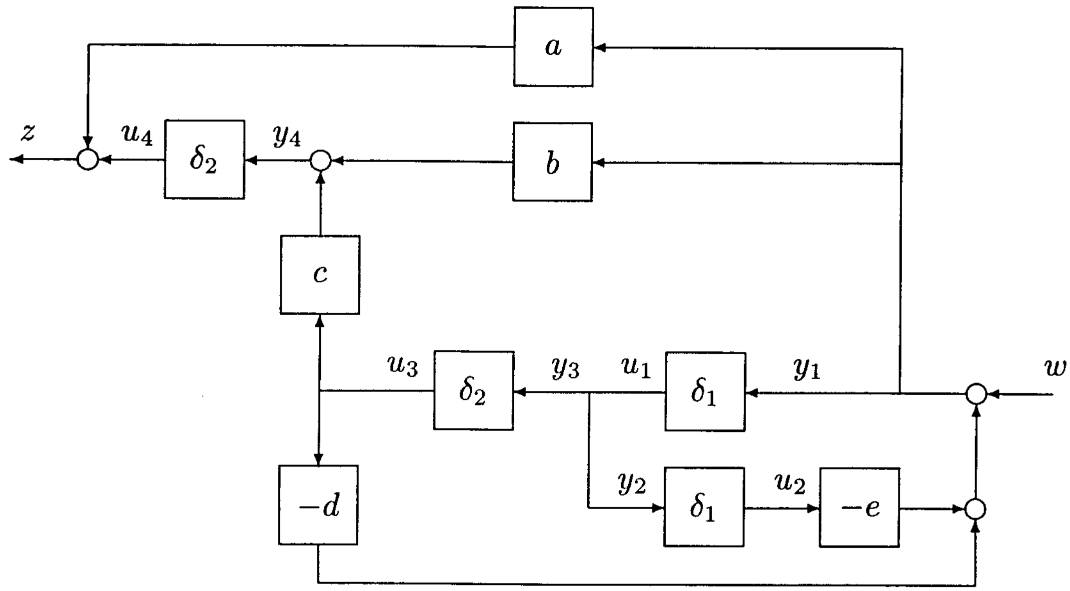


Figure 10.7: Block diagram for G

Suppose that P and K are compatibly partitioned matrices

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

such that the matrix product $P_{22}K_{11}$ is well defined and square, and assume further that $I - P_{22}K_{11}$ is invertible. Then the *star product of P and K with respect to this partition* is defined as

$$\mathcal{S}(P,K) := \begin{bmatrix} \mathcal{F}_l(P, K_{11}) & P_{12}(I - K_{11}P_{22})^{-1}K_{12} \\ K_{21}(I - P_{22}K_{11})^{-1}P_{21} & \mathcal{F}_u(K, P_{22}) \end{bmatrix}.$$

Note that this definition is dependent on the partitioning of the matrices P and K above. In fact, this star product may be well defined for one partition and not well defined for another; however, we will not explicitly show this dependence because it is always clear from the context. In a block diagram, this dependence appears, as shown in Figure 10.8.

Now suppose that P and K are transfer matrices with state space representations:

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \qquad K = \left[\begin{array}{c|cc} A_K & B_{K1} & B_{K2} \\ \hline C_{K1} & D_{K11} & D_{K12} \\ C_{K2} & D_{K21} & D_{K22} \end{array}\right].$$

Then the transfer matrix

$$\mathcal{S}(P,K) : \begin{bmatrix} w \\ \hat{w} \end{bmatrix} \mapsto \begin{bmatrix} z \\ \hat{z} \end{bmatrix}$$

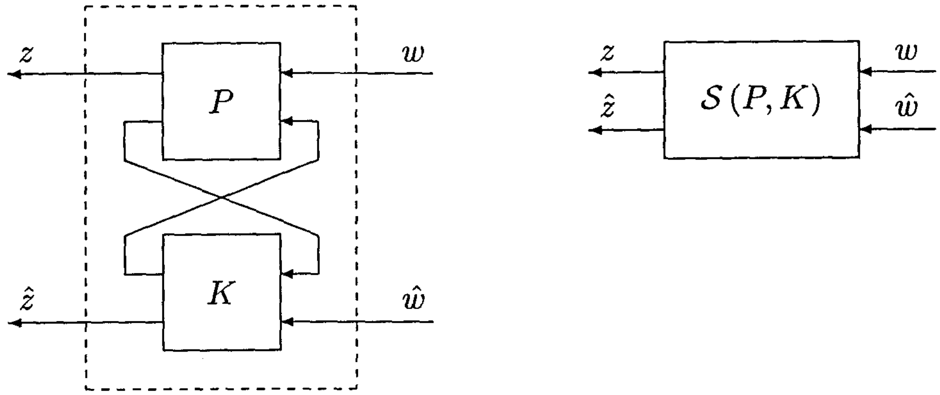


Figure 10.8: Interconnection of LFTs

has a representation

$$\mathcal{S}(P,K) = \left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & \bar{D}_{11} & \bar{D}_{12} \\ \bar{C}_2 & \bar{D}_{21} & \bar{D}_{22} \end{array}\right] = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$$

where

$$\begin{aligned} \bar{A} &= \begin{bmatrix} A + B_2\tilde{R}^{-1}D_{K11}C_2 & B_2\tilde{R}^{-1}C_{K1} \\ B_{K1}R^{-1}C_2 & A_K + B_{K1}R^{-1}D_{22}C_{K1} \end{bmatrix} \\ \bar{B} &= \begin{bmatrix} B_1 + B_2\tilde{R}^{-1}D_{K11}D_{21} & B_2\tilde{R}^{-1}D_{K12} \\ B_{K1}R^{-1}D_{21} & B_{K2} + B_{K1}R^{-1}D_{22}D_{K12} \end{bmatrix} \\ \bar{C} &= \begin{bmatrix} C_1 + D_{12}D_{K11}R^{-1}C_2 & D_{12}\tilde{R}^{-1}C_{K1} \\ D_{K21}R^{-1}C_2 & C_{K2} + D_{K21}R^{-1}D_{22}C_{K1} \end{bmatrix} \\ \bar{D} &= \begin{bmatrix} D_{11} + D_{12}D_{K11}R^{-1}D_{21} & D_{12}\tilde{R}^{-1}D_{K12} \\ D_{K21}R^{-1}D_{21} & D_{K22} + D_{K21}R^{-1}D_{22}D_{K12} \end{bmatrix} \end{aligned}$$

$$R = I - D_{22}D_{K11}, \quad \tilde{R} = I - D_{K11}D_{22}.$$

In fact, it is easy to show that

$$\begin{aligned} \bar{A} &= \mathcal{S}\left(\begin{bmatrix} A & B_2 \\ C_2 & D_{22} \end{bmatrix}, \begin{bmatrix} D_{K11} & C_{K1} \\ B_{K1} & A_K \end{bmatrix}\right), \\ \bar{B} &= \mathcal{S}\left(\begin{bmatrix} B_1 & B_2 \\ D_{21} & D_{22} \end{bmatrix}, \begin{bmatrix} D_{K11} & D_{K12} \\ B_{K1} & B_{K2} \end{bmatrix}\right), \end{aligned}$$

$$
\begin{aligned}
\bar{C} &= \mathcal{S}\left(\begin{bmatrix} C_1 & D_{12} \\ C_2 & D_{22} \end{bmatrix}, \begin{bmatrix} D_{K11} & C_{K1} \\ D_{K21} & C_{K2} \end{bmatrix}\right), \\
\bar{D} &= \mathcal{S}\left(\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}, \begin{bmatrix} D_{K11} & D_{K12} \\ D_{K21} & D_{K22} \end{bmatrix}\right).
\end{aligned}
$$

10.5 Notes and References

This chapter is based on the lecture notes by Packard [1991] and the paper by Doyle, Packard, and Zhou [1991].