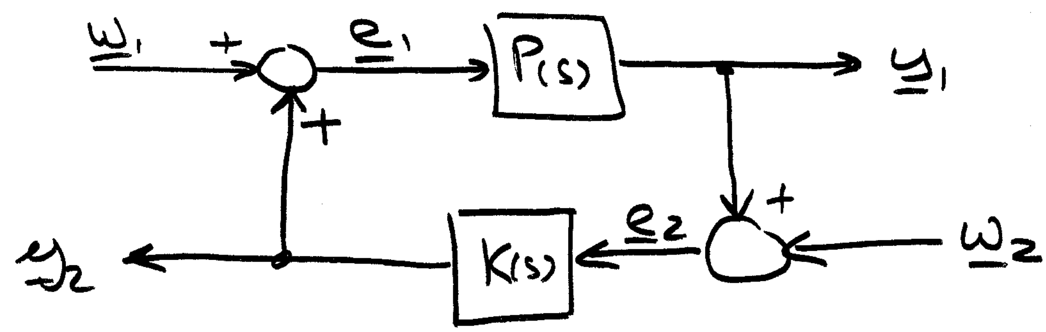
Stability and Performance

Given a plant $P(s)$ with a stabilizing controller $K(s)$:



The controller is <u>well-posed</u> if there are no "short circuits" in the feedback loop:

$$\left| \begin{array}{l} \underline{e}_1 = \underline{w}_1 + K\underline{e}_2 \\ \underline{e}_2 = \underline{w}_2 + P\underline{e}_1 \end{array} \right|$$

$$\Rightarrow \underline{e}_1 = \underline{w}_1 + K\underline{w}_2 + KP\underline{e}_1$$

$$\Rightarrow (I - KP)\,\underline{e}_1 = \underline{w}_1 + K\underline{w}_2$$

can be solved for $\underline{e}_1$ iff

$(I - KP)$ is invertible, <u>or</u>

$(I - K(\infty) \cdot P(\infty))$ is invertible.

or:

$$\left| \begin{array}{l} \underline{e}_1 - K\underline{e}_2 = \underline{w}_1 \\ \underline{e}_2 - P\underline{e}_1 = \underline{w}_2 \end{array} \right|$$

$$\begin{bmatrix} I & -K \\ -P & I \end{bmatrix} \cdot \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} = \begin{bmatrix} \underline{w}_1 \\ \underline{w}_2 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} I & -K \\ -P & I \end{bmatrix}$ must be invertible,

or (simpler to compute):

$\begin{bmatrix} I & -K(\infty) \\ -P(\infty) & I \end{bmatrix}$ must have an inverse.

In the time domain:

$$P = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array} \right] ; \quad K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right]$$

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}}_p = A_p \underline{x}_p + B_p \underline{e}_1 \\ \underline{e}_2 = C_p \underline{x}_p + D_p \underline{e}_1 + \underline{w}_2 \\ \dot{\underline{x}}_K = A_K \underline{x}_K + B_K \underline{e}_2 \\ \underline{e}_1 = C_K \underline{x}_K + D_K \underline{e}_2 + \underline{w}_1 \end{array} \right|$$

$$\Rightarrow \left| \begin{array}{l} \underline{e}_1 - D_K \underline{e}_2 = C_K \underline{x}_K + \underline{\omega}_1 \\ -D_P \underline{e}_1 + \underline{e}_2 = C_P \underline{x}_P + \underline{\omega}_2 \end{array} \right|$$

$$\Rightarrow \begin{bmatrix} I & -D_K \\ -D_P & I \end{bmatrix} \cdot \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} = \begin{bmatrix} \emptyset & C_K \\ C_P & \emptyset \end{bmatrix} \cdot \begin{bmatrix} \underline{x}_P \\ \underline{x}_K \end{bmatrix} + \begin{bmatrix} \underline{\omega}_1 \\ \underline{\omega}_2 \end{bmatrix}$$

is <u>well posed</u>, iff

$$\begin{bmatrix} I & -D_K \\ -D_P & I \end{bmatrix} \text{ is invertible.}$$

We shall now <u>assume</u> the well-posedness of a system.

$$\begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} = \begin{bmatrix} I & -K \\ -P & I \end{bmatrix}^{-1} \cdot \begin{bmatrix} \underline{\omega}_1 \\ \underline{\omega}_2 \end{bmatrix}$$

is <u>stable</u> if the transfer function matrix:

$$\begin{bmatrix} I & -K(s) \\ -P(s) & I \end{bmatrix}^{-1}$$

has all poles in the LHP, i.e., belongs to $\mathcal{H}_\infty$.

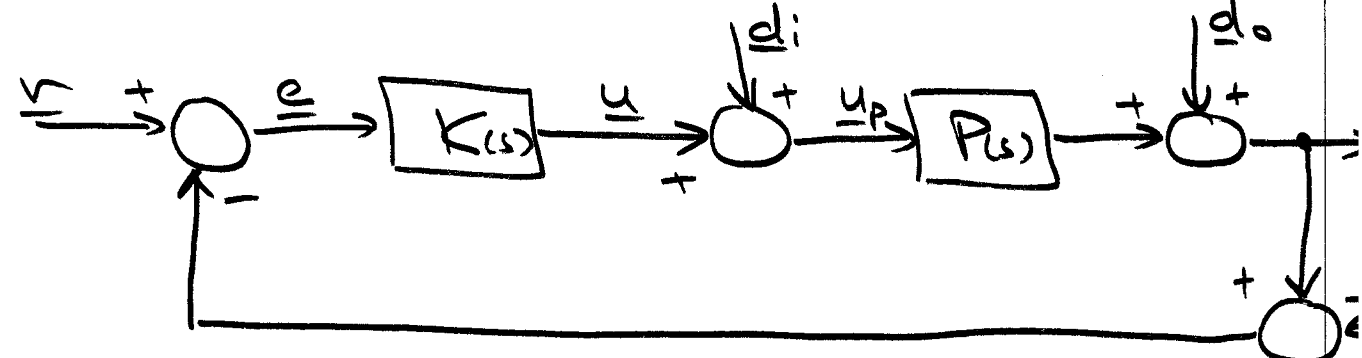
For reference:

$$\begin{bmatrix} I & -K \\ -P & I \end{bmatrix}^{-1} \equiv \begin{bmatrix} (I-KP)^{-1} & K(I-PK)^{-1} \\ P(I-KP)^{-1} & (I-PK)^{-1} \end{bmatrix}$$

$$\equiv \begin{bmatrix} I+K(I-PK)^{-1}P & K(I-PK)^{-1} \\ (I-PK)^{-1}P & (I-PK)^{-1} \end{bmatrix}$$

$$\equiv \begin{bmatrix} (I-KP)^{-1} & (I-KP)^{-1}K \\ P(I-KP)^{-1} & I+P(I-KP)^{-1}K \end{bmatrix}$$

Given the system:



We define:

$L_i = KP$: input loop transfer matrix

$L_o = PK$: output loop transfer matrix

$S_i = (I+L_i)^{-1}$: input sensitivity matrix

$S_o = (I+L_o)^{-1}$: output sensitivity matrix

$T_i = I - S_i = L_i(I + L_i)^{-1}$: input complementary sensitivity matrix

$T_o = I - S_o = L_o(I + L_o)^{-1}$: output complementary sensitivity matrix

$$\underline{y} = T_o(\underline{r} - \underline{n}) + S_o P \underline{d}_i + S_o \underline{d}_o$$

$$(\underline{r} - \underline{y}) = S_o(\underline{r} - \underline{d}) + T_o \underline{n} - S_o P \underline{d}_i$$

$$\underline{u} = K S_o(\underline{r} - \underline{n}) - K S_o \underline{d}_o - T_i \underline{d}_i$$

$$\underline{u}_p = K S_o(\underline{r} - \underline{n}) - K S_o \underline{d}_o - S_i \underline{d}_i$$

⇒ All internal signals can be computed from transfer functions that either belong to:

$$P, K, L_i, L_o, S_i, S_o, T_i, T_o$$

or are products of combinations of the above.

⇒ These transfer functions characterize the system completely.

There is some redundancy, because:

$$P(I+KP)^{-1} = P(I+L_i)^{-1} = P \cdot S_i$$

$$\equiv (I+PK)^{-1}P = (I+L_o)^{-1}P = S_o \cdot P$$

etc.

$$\underline{y} = T_o(\underline{r}-\underline{n}) + S_o P \underline{d}_i + S_o \underline{d}_o$$

⇒ Influence of output disturbance on system output:

$$\underline{y} = S_o \underline{d}_o$$

$$\Rightarrow \|\underline{y}\|_2 \leq \|S_o\|_\infty \cdot \|\underline{d}_o\|_2$$

In order to keep the influence small, we must make the infinity norm of S_o small, e.g.

$$\|S_o\|_\infty = \max_\omega \bar{\sigma}(S_o) < 1$$

Usually, we are interested to suppress disturbances in a certain frequency range:

$$\bar{\sigma}(S_o) < 1 \; ; \; \forall \omega \in [\omega_{min}, \omega_{m}$$

Similarly, to suppress the influence of the input disturbance on the plant:

$$\underline{u}_p = -S_i \underline{d}_i$$

$$\Rightarrow \|\underline{u}_p\|_2 < \|S_i\|_\infty \cdot \|\underline{d}_i\|_2$$

$$\Rightarrow \bar{\sigma}(S_i) < 1 \; ; \; \forall \omega \in [\omega_{min}, \omega_{max}]$$

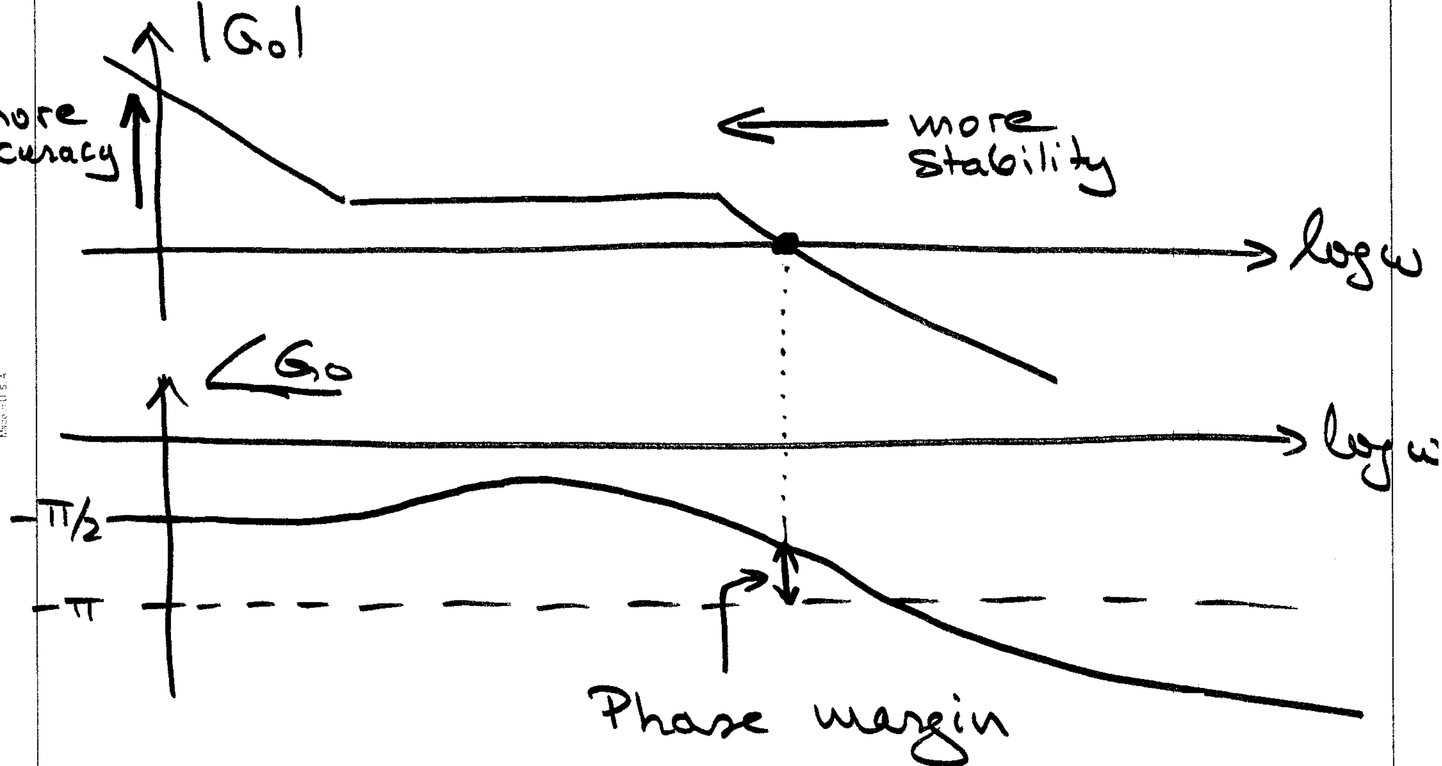
$$\bar{\sigma}(S_o) = \bar{\sigma}\left((I+PK)^{-1}\right) = \frac{1}{\underline{\sigma}(I+PK)}$$

$$\underline{\sigma}(PK) - 1 \leq \underline{\sigma}(I+PK) \leq \underline{\sigma}(PK) + 1$$

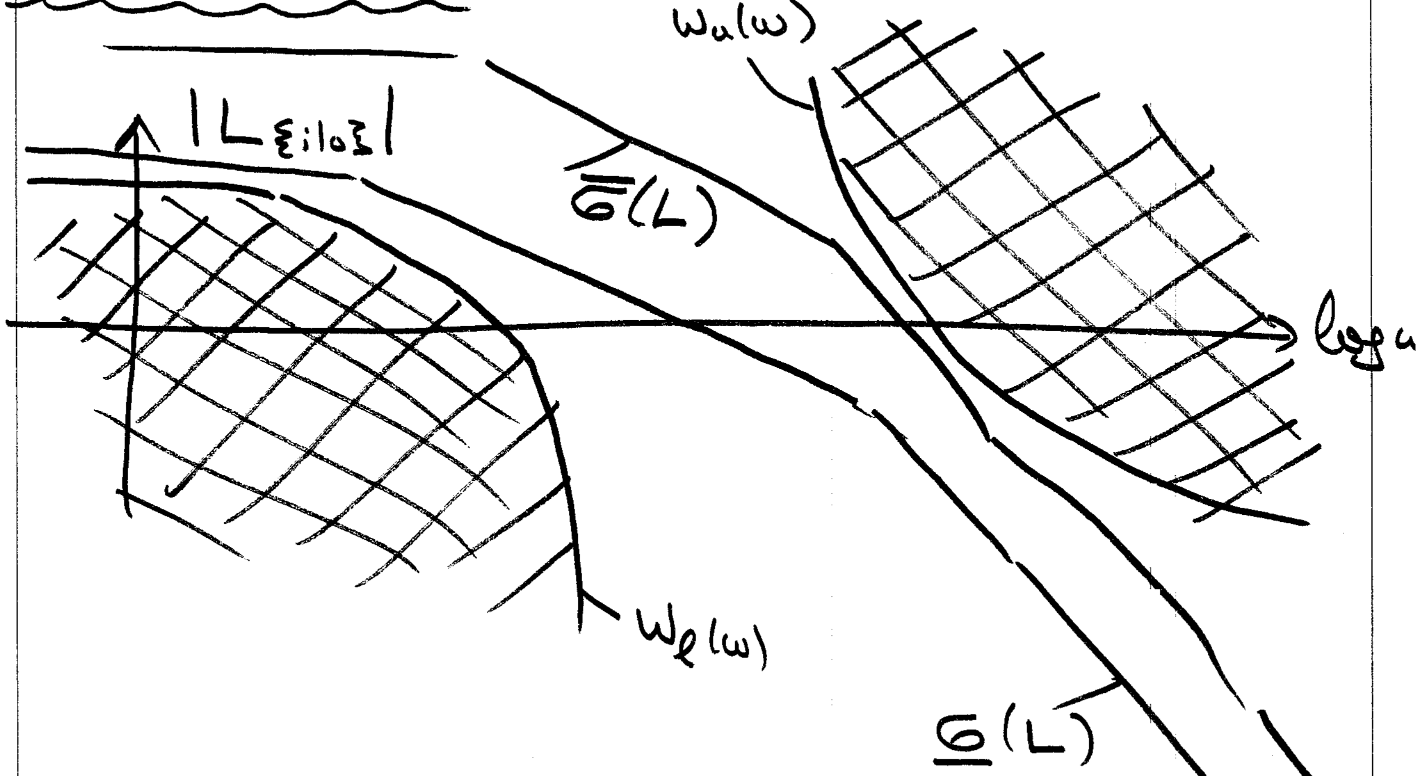
$$\Rightarrow \frac{1}{\underline{\sigma}(PK)+1} \leq \bar{\sigma}(S_o) \leq \frac{1}{\underline{\sigma}(PK)-1}$$

$$\Rightarrow \bar{\sigma}(S_o) \ll 1 \iff \underline{\sigma}(PK) \gg 1$$

From ECE 441:



Generalization:



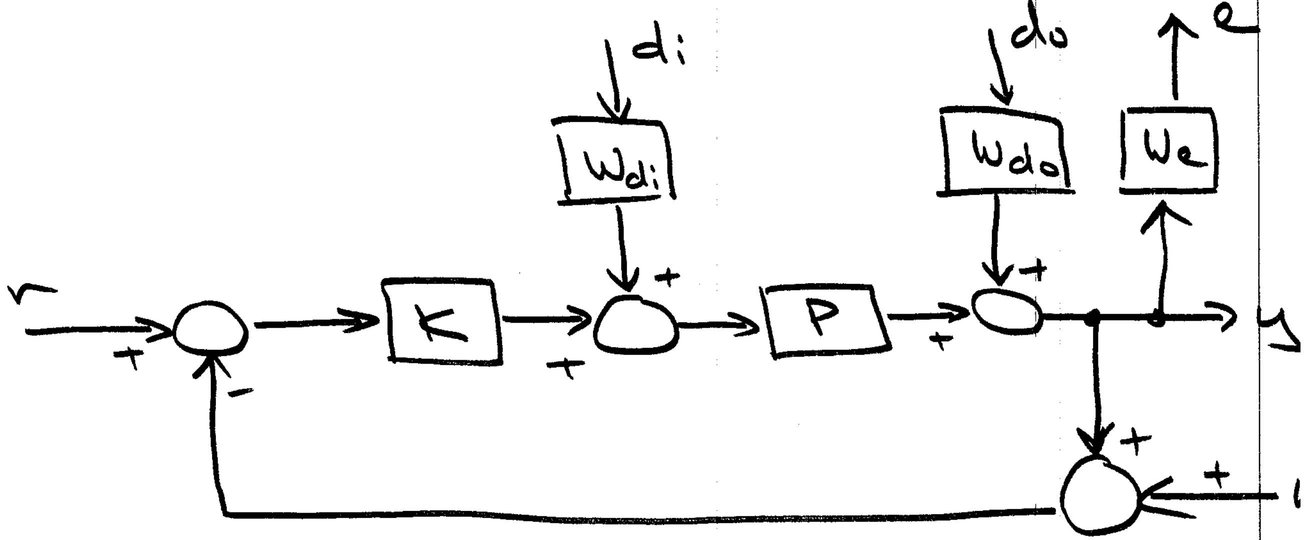
Performance (accuracy, attenuation of disturbances)

$$\underline{\sigma}(L_o) > W_e(\omega)$$
$$\underline{\sigma}(L_i) > W_e(\omega)$$

Stability:

$$\bar{\sigma}(L_o) < W_u(\omega)$$
$$\bar{\sigma}(L_i) < W_m(\omega)$$

Sometimes, more knowledge is available about a disturbance, which can be described by a transfer function $W_d(s)$, or not all frequencies are equally important. Also, the signal to be kept small may be another signal than the output, usually an error signal, e, that can be related to the output by $W_e(s)$.



In those cases, replace

$$\|S_o\|_\infty \longrightarrow \|W_e S_o W_{do}\|_\infty$$

$$\|S_i\|_\infty \longrightarrow \|W_e S_i W_{di}\|_\infty$$

etc.

Coprime Factorization:

We need a mathematical tool that will allow us to characterize sets of controllers, sets of plants, or sets of disturbances.

Two polynomials $m(s)$ and $n(s)$ are relative coprime, if there exist two other polynomials $x(s)$ and $y(s)$, such that:

$$x(s)\cdot m(s) + y(s)\cdot n(s) \equiv 1$$

This is called <u>Bezout identity</u>.

<u>Proof</u>: (trivial)

<u>Assume</u>: $m(s)$ and $n(s)$ have a common factor $t(s)$:

$$m(s) = \hat{m}(s)\cdot t(s)$$
$$n(s) = \hat{n}(s)\cdot t(s)$$

$$\Rightarrow x(s)\cdot m(s) + y(s)\cdot n(s)$$
$$= x(s)\cdot \hat{m}(s)\cdot t(s) + y(s)\cdot \hat{n}(s)\cdot t(s)$$
$$= \left[x(s)\cdot \hat{m}(s) + y(s)\cdot \hat{n}(s)\right]\cdot t(s) \neq 1$$

<u>Generalization</u>:

$M(s)$ and $N(s)$ are <u>relative right coprime</u>, if there exist matrices $X_r(s)$ and $Y_r(s)$, such that:

$$X_r(s) \cdot M(s) + Y_r(s) \cdot N(s) = I^{(u}$$

or:

$$\begin{bmatrix} X_r(s) & Y_r(s) \end{bmatrix} \cdot \begin{bmatrix} M(s) \\ N(s) \end{bmatrix} = I^{(u)}$$

Similarly, $\tilde{M}(s)$ and $\tilde{N}(s)$ are <u>relative left coprime</u>, if there exist $\tilde{X}_\ell(s)$ and $\tilde{Y}_\ell(s)$, such that:

$$\tilde{M}(s) \cdot \tilde{X}_\ell(s) + \tilde{N}(s) \cdot \tilde{Y}_\ell(s) = I^{(n)}$$

or:

$$\begin{bmatrix} \tilde{M}(s) & \tilde{N}(s) \end{bmatrix} \cdot \begin{bmatrix} \tilde{X}_\ell(s) \\ \tilde{Y}_\ell(s) \end{bmatrix} = I^{(u)}$$

Generalization:

Given $P(s)$ a proper real-rational matrix. We call a right-coprime factorization of $P(s)$ a set of matrices $M(s)$ and $N(s)$, such that

$$P(s) = N(s) \cdot M^{-1}(s) .$$

Among all those factorizations, we are particularly interested in those where:

$N(s), M(s)$ are proper real-rational

and: $N(s), M(s)$ are stable.

Example:

$$p(s) = \frac{1}{s-1} \quad \text{is} \left\{ \begin{array}{l} \text{proper} \\ \text{real-rational} \\ \text{unstable.} \end{array} \right.$$

$$\Rightarrow \quad m(s) = \frac{1}{s+1}, \qquad n(s) = \frac{s-1}{s+1}$$

proper real-rational **stable**

$$m(s) \cdot n^{-1}(s) = \frac{1}{s+1} \cdot \frac{s+1}{s-1} = \frac{1}{s-1} = p(s) \quad \checkmark$$

$$x(s) = 2$$
$$y(s) = 1$$

$$\Rightarrow \quad x(s)m(s) + y(s)n(s) =$$

$$2 \cdot \frac{1}{s+1} + 1 \cdot \frac{s-1}{s+1} =$$

$$\frac{2}{s+1} + \frac{s-1}{s+1} = \frac{2+s-1}{s+1} = \frac{s+1}{s+1} = 1 \quad \checkmark$$

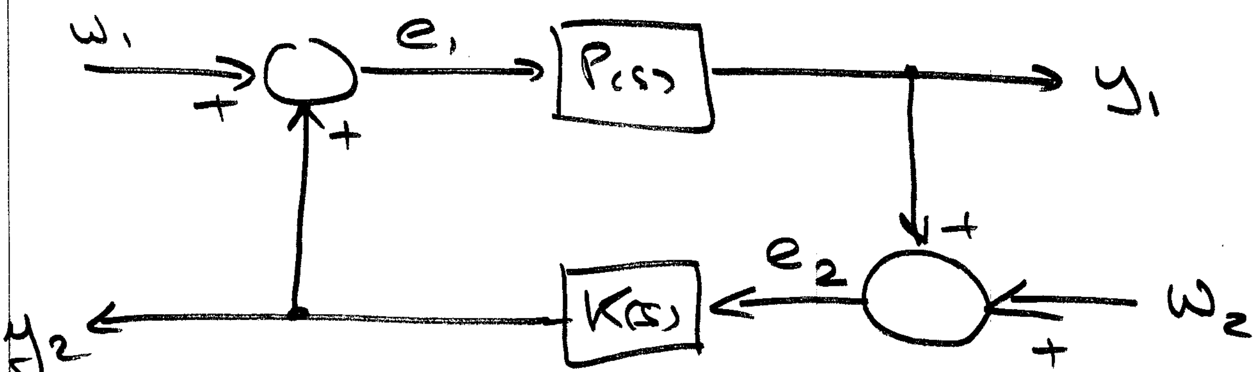
$\Rightarrow$ $m(s), n(s)$ are **coprime**.

Similarly:

$$P(s) = \tilde{M}^{-1}(s) \cdot \tilde{N}(s)$$

is a left coprime factorization.

Given:



Find:

$$P(s) = N(s) \cdot M^{-1}(s) = \tilde{M}^{-1}(s) \cdot \tilde{N}(s)$$

$$K(s) = U(s) \cdot V^{-1}(s) = \tilde{V}^{-1}(s) \cdot \tilde{U}(s)$$

We want:

- → All pairs are coprime
- → All individual decomposition matrices are stable
- → K stabilizing P

and: