Modeling / Control under Uncertainty (Robust Control)

<u>Reference</u>:

Kemin Zhou, John Doyle, Keith Glover: Robust & Optimal Control, Prentice Hall, 1996.

Until now, we always looked at <u>nominal behavior</u> of a plant and controller. On the few occasions, where we afterwards disturbed either the controller or the plant parameters, we noticed that it often takes very little, before the controlled perturbed plant no longer performs as desired. Often, it may even become unstable.

The question is:

Can we guarantee that the controlled system will still satisfy the design criteria or at least remain stable for <u>any</u> perturbation in a given range around the nominal behavior.

- How can we track the behavior of the perturbed plant under <u>all</u> perturbances? How can we mathematically describe the set of <u>all</u> perturbations?

- Clearly, we need a way to denote the <u>maximal</u> disturbance.

$\Rightarrow$ For this, the $\|\cdot\|_\infty$ is very useful. We need to introduce the concept of <u>norms</u>.

Norms & Inner Products:

(1) Vectors:

- Measure to denote "length" or "magnitude".

Necessary conditions:

(α) $\|\underline{x}\| \geq \phi$

(β) $\|\underline{x}\| = \phi \iff \underline{x} = \phi$

(γ) $\|\alpha \cdot \underline{x}\| = |\alpha| \cdot \|\underline{x}\|$; $\alpha \in \mathbb{C}^1$

(δ) $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$

Sometimes, we shall relax the condition (β). This is then called a seminorm.

p-Norm:

$$\|\underline{x}\|_p \stackrel{!}{=} \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} ; \; p \in [1, \infty) \quad [p \in \mathbb{R}^1]$$

In Matlab: n = norm(x, p)

Most books limit p to the real range $[1, \infty)$. Matlab allows any value $\in \mathbb{R}^1$.

Special cases:

$$\|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|\underline{x}\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

$$\|\underline{x}\|_\infty = \max_{1 \le i \le n} |x_i|$$

In Matlab:

$$\text{norm}(x) \overset{!}{\equiv} \underbrace{\text{norm}(x, 2)}_{}$$

default

norm(x, inf) = norm(x, 'inf')

is the infinity norm.

$$\text{norm}(x, -\text{inf}) = \min_{1 \le i \le n} |x_i|$$

$\|\underline{x}\|_{-\infty}$ is only a <u>seminorm</u>.

(2) <u>Matrices</u>:

Most norms are defined as <u>induced</u> norms from the vector norms.

$$\|A\|_p \stackrel{!}{=} \sup_{\underline{x} \neq \underline{0}} \frac{\|A\underline{x}\|_p}{\|\underline{x}\|_p}$$

Without loss of generality, we can normalize the length of $\|\underline{x}\|$ to 1 (because of condition (γ)):

$$\|A\|_p = \sup_{\|\underline{x}\|=1} \|A\underline{x}\|_p$$

In general, these norms may be difficult to compute. Only special cases are easy:

$$\|A\|_1 = \max_{1\le j\le m} \sum_{i=1}^{n} |a_{ij}| \quad ; A \in \mathbb{C}^{n\times m}$$

$$\|A\|_2 = \sqrt{\lambda_{max}(A^* A)} \begin{cases} = \sigma_{max}(A) \\ = \overline{\sigma}(A) \end{cases}$$

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{m} |a_{ij}|$$

In Matlab:

$$\|A\|_1 = \text{norm}(A, 1)$$
$$\|A\|_2 = \text{norm}(A, 2) = \text{norm}(A)$$
$$\|A\|_\infty = \text{norm}(A, \text{inf}) = \text{norm}(A, \text{'inf})$$

$$\|A\|_2 = \|A\|_E = \underline{\text{Euclidean norm}}$$
(is default)

There exists also a non-induced norm that is sometimes used:

$$\|A\|_F = \sqrt{\text{Trace}(A^* A)} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ij}|^2}$$

$\|A\|_F$ is the Frobenius norm

In Matlab:

$$\|A\|_F = \text{norm}(A, \text{'fro'})$$

Useful properties of Norms

1) Given U a unitary matrix:

$$\Rightarrow \quad \|U\|_2 = 1$$

2) $$\|U\underline{x}\|_2 = \|\underline{x}\|_2$$

3) Let $\|\underline{x}\|_2 \equiv \|\underline{y}\|_2$; $\underline{x} \in \mathbb{C}^n$, $\underline{y} \in \mathbb{C}^m$; $n \geq m$

$$\Longleftrightarrow \exists\, U \in \mathbb{C}^{n\times m}, \quad \underline{x} = U\cdot\underline{y}, \quad U^*U = I^{(m)}$$

4) Let $\|\underline{x}\|_2 \leq \|\underline{y}\|_2$; $\underline{x} \in \mathbb{C}^n$, $\underline{y} \in \mathbb{C}^m$; $n \geq m$

$$\Longleftrightarrow \exists\, \Delta \in \mathbb{C}^{n\times m}, \quad \underline{x} = \Delta\cdot\underline{y}, \quad \|\Delta\|_2 \leq 1$$

5) Let $\|\underline{x}\|_2 < \|\underline{y}\|_2$; $\underline{x} \in \mathbb{C}^n$, $\underline{y} \in \mathbb{C}^m$; $n \geq m$

$$\|\Delta\|_2 < 1$$

6) $\|A \cdot U\|_2 = \|A\|_2$
$\|U \cdot A\|_2 = \|A\|_2$ $\quad U$, unitary

7) $\|A \cdot B\|_2 \leq \|A\|_2 \cdot \|B\|_2$

8) $\|U \cdot A \cdot U^*\|_F = \|A\|_F$ $\quad U$, unitary

9) $\|A \cdot B\|_F \leq \|A\|_2 \cdot \|B\|_F$
$\|A \cdot B\|_F \leq \|A\|_F \cdot \|B\|_2$

10) Let

$$A = \underbrace{\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \ddots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{bmatrix}}_{\text{blockpartitioned}} = [A_{ij}]$$

$\Rightarrow \|A\|_p \leq \| [\|A_{ij}\|_p] \|_p$

11) $\|A\|_F = \| [\|A_{ij}\|_F] \|_F$

12) Let $\phi(A) = \max\limits_{1 \leq i \leq n} |\lambda_i|$

be the <u>spectral radius of A</u>.

$\Rightarrow \phi(A) \leq \|A\|_p$
$\phi(A) \leq \|A\|_F$

13) Given $A \in \mathbb{C}^{n \times m}$; $n \geq m$

$$\min_{\|\underline{x}\|_2 = 1} \|A\underline{x}\|_2 \begin{cases} = \sigma_{min}(A) \\ = \underline{\sigma}(A) \end{cases}$$

14) $\|A\|_F = \sqrt{\sum_{i=1}^{r} \sigma_i^2}$; $r = \min\{n, m\}$

15) Let $A \in \mathbb{C}^{n \times n}$; $\rho(A) = \text{Rank}(A) = n$

$$\Rightarrow \quad \overline{\sigma}(A^{-1}) = \underline{\sigma}(A)$$

16) let $A \in \mathbb{C}^{n \times n}$; $\Delta \in \mathbb{C}^{n \times n}$

$$\Rightarrow \quad |\underline{\sigma}(A+\Delta) - \underline{\sigma}(A)| \leq \overline{\sigma}(\Delta)$$

$$\underline{\sigma}(A \cdot \Delta) \geq \underline{\sigma}(A) \cdot \underline{\sigma}(\Delta)$$