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# A TIME-SCALE METHOD FOR MODEL REDUCTION OF DISCRETE-TIME SYSTEMS

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## VIII.7. Minimization of "equation error" for discrete-time systems

A model reduction method based on the minimization of equation error has been proposed by Eitelberg (1978) for continuous-time linear systems. In the following, the version for discrete-time systems will be developed.

Given the n-th order linear-discrete-time system described by

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = cx(k)$$

We assume that the m-th order reduced model is described by

$$\tilde{x}(k+1) = A_r\tilde{x}(k) + B_r u(k)$$

The states of interest $x_r$ could be "picked-out" from the original state vector by means of a (mxn) "masking matrix" R with the elements 1 and zero such that

$$x_r = Rx$$

It's now required to represent the m-dimensional state vector $x_r$ by the m-th order reduced model, i.e. for $\tilde{x} \simeq x_r$, we may write

$$x_r(k+1) \simeq A_r x_r(k) + B_r u(k)$$

We write the equation error as

$$e(k+1) = x_r(k+1) - A_r x_r(k) - B_r u(k)$$

For $x(o) = 0$, $X_r(o) = 0$, $u(k) \equiv$ step function $= \begin{cases} u_0, & k \geq \\ 0, & k < \end{cases}$

the equation error becomes

$$e(k+1) = Rx(k+1) - A_r Rx(k) - B_r U_o$$

$$= (RA-A_r R)x(k) + (RB-B_r)U_o \qquad (8.22)$$

But $x(k) = A^k x(o) + \sum_{i=1}^{k} A^{i-1} Bu(k-i)$

and since, by assumption, $x(o) = 0$, $u(k) = U_o$ for $k \geq 0$, then

$$x(k) = \sum_{i=1}^{k} A^{i-1} B \; U_o \qquad (8.23)$$

substituting from (8.23) into (8.22), we obtain

$$e(k+1) = (RA-A_r R) \sum_{i=1}^{k} A^{i-1} B \; U_o + (RB-B_r)U_o \qquad (8.24)$$

In (8.24), $U_o$ is just a scaling factor and will be droped defining

$$E(k) = (RA-A_r R) \sum_{i=1}^{k} A^{i-1} B + (RB-B_r)$$

$$= (RA-A_r R) \sum_{j=o}^{k-1} A^j B + RB-B_r$$

Assuming A is stable, then the above matrix-series will converge and we may write

$$E(k) = (RA-A_r R)(I-A^k)(I-A)^{-1} B + RB-B_r \qquad (8.25)$$

The stationary value of x is obtained from (8.23) by letting k tends to infinity, i.e.

$$x_{st} = x(\infty) = \sum_{j=o}^{\infty} A^j BU_o = (I-A)^{-1}BU_o$$ which exists for all stable A.

We wish to have

$$x_{r\ st} = Rx_{st}\ ,\quad \text{i.e.}$$

$$R(I-A_r)^{-1}B_rU_o = R(I-A)^{-1}BU_o \tag{8.26}$$

From (8.26) yields $B_r$ that matches the steady-state response of the original states and those of the reduced model.

$$B_r = (I-A_r)R(I-A)^{-1}B \tag{8.27}$$

Substituting from (8.27) into (8.25), and after few algebraic manipulations, we obtain

$$E(k) = [R-(I-A_r)R(I-A)^{-1}]A^kB \tag{8.28}$$

Now, we wish to determine $A_r$ that minimizes the "error measure"

$$q = \sum_{k=o}^{\infty} \| E(k)\|^2,$$ which can be rewritten as

$$q = \sum_{k=o}^{\infty} \text{trace } (E(k)E^T(k)) \tag{8.29}$$

But, $E(k)E^T(k) = [R-(I-A_r)R(I-A)^{-1}]A^kBB^TA^{k^T}[R-(I-A_r)R(I-A)^{-1}]^T$

then

$$q = \text{trace } PSP^T \tag{8.30}$$

where

$$P = [R-(I-A_r)R(I-A)^{-1}]$$

$$S = \sum_{k=o}^{\infty} A^kBB^TA^{k^T}$$

Differentiating q after $A_r$ and letting, $\frac{dq}{dA_r} = 0$, we obtain the optimal matrix $A_r^*$

$$A_r^* = I-RSD^T[DSD^T]^{-1} \tag{8.31}$$

where

$$D = R(I-A)^{-1}$$

S is the solution of the discrete-Lyapunov equation

$$ASA^T-S = -BB^T \tag{8.32}$$

The reduced model is obtained by first solving equation (8.32) for S then substituting in (8.31) to obtain $A_r$. Substitution in (8.27) yields the input matrix $B_r$.