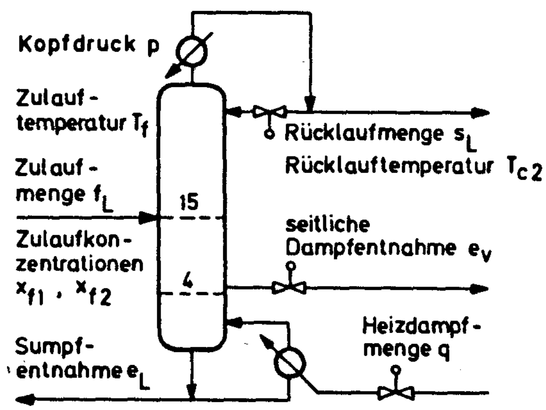
Practical Examples

1) Destillation Column



10 inputs:

$u_1 = x_{f_1} =$ Inflow concentration
$u_2 = x_{f_2} =$ "
$u_3 = q =$ Amount of heating steam
$u_4 = s_L =$ Amount of backflow
$u_5 = e_L =$ swamp outflow
$u_6 = e_V =$ steam outflow
$u_7 = f_L =$ Inflow
$u_8 = T_f =$ Inflow temperature
$u_9 = p =$ pressure
$u_{10} = T_{2c} =$ backflow temperature

state variables:

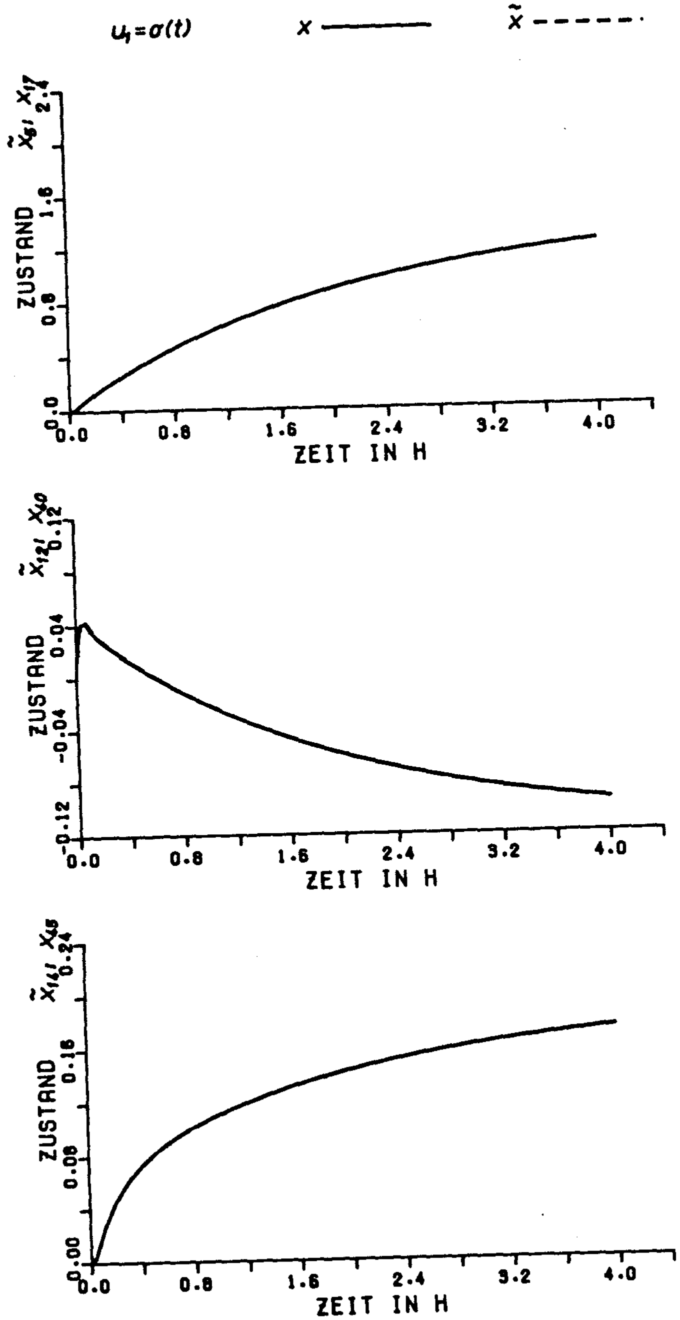
temperature T_i
material concentrations $c1_j$, $c2k$
swamp level h_b

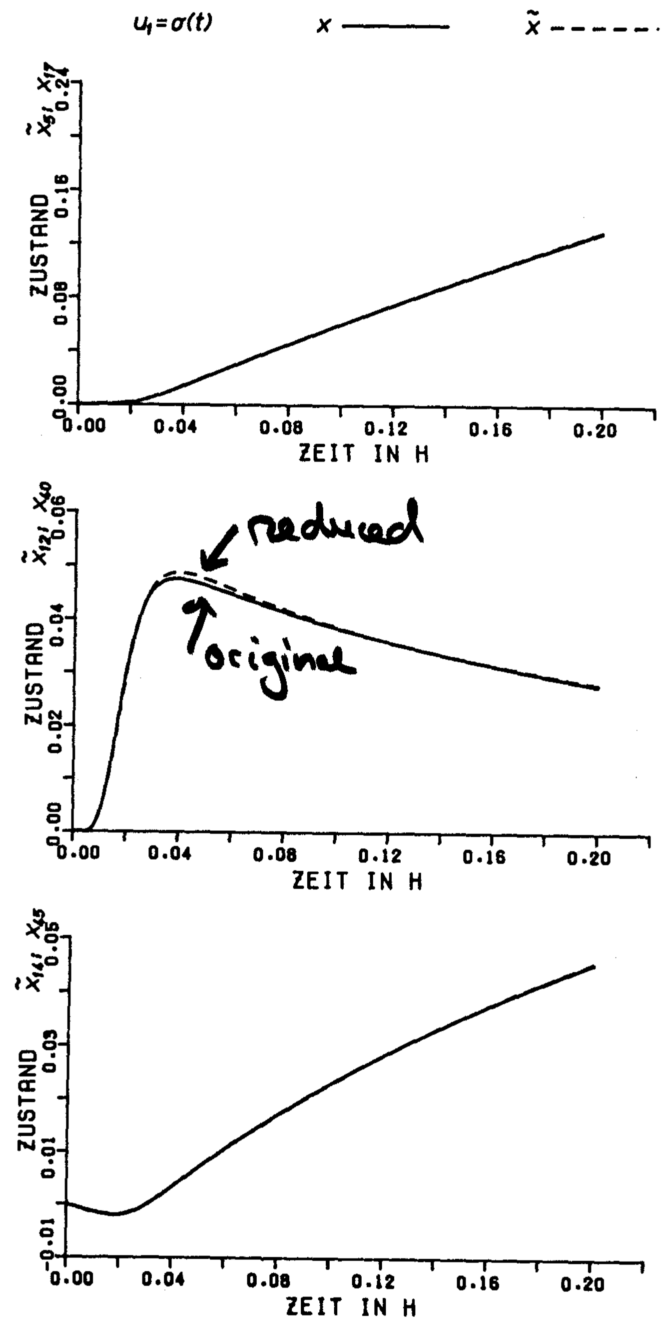
=> 36 state variables

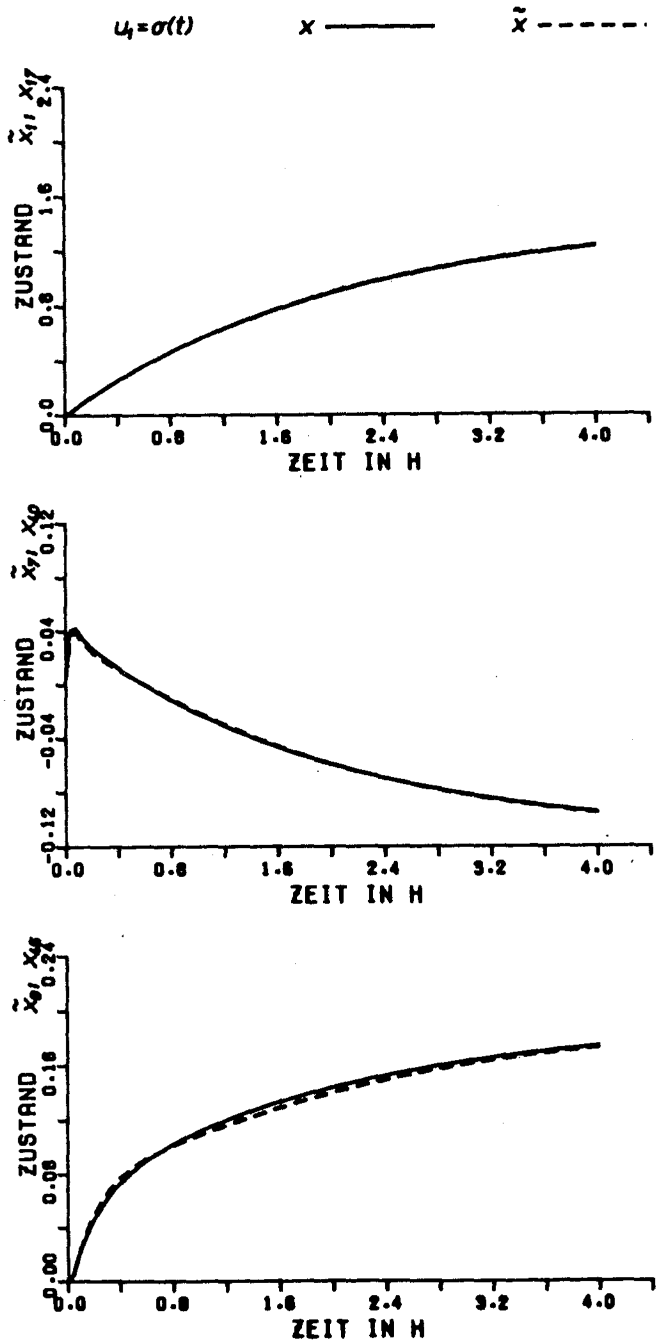
x_4	x_7	x_{11}	x_{14}	x_{17}	x_{18}	x_{19}	x_{21}	x_{24}
$c1_{14}$	$c1_{11}$	$c1_7$	$c1_4$	T_1	$c2_{27}$	T_{26}	T_{24}	T_{21}
x_{28}	x_{31}	x_{33}	x_{36}	x_{37}	x_{40}	x_{42}	x_{44}	x_{45}
T_{17}	T_{14}	T_{12}	T_9	T_8	T_5	T_3	$c2_{28}$	h_b

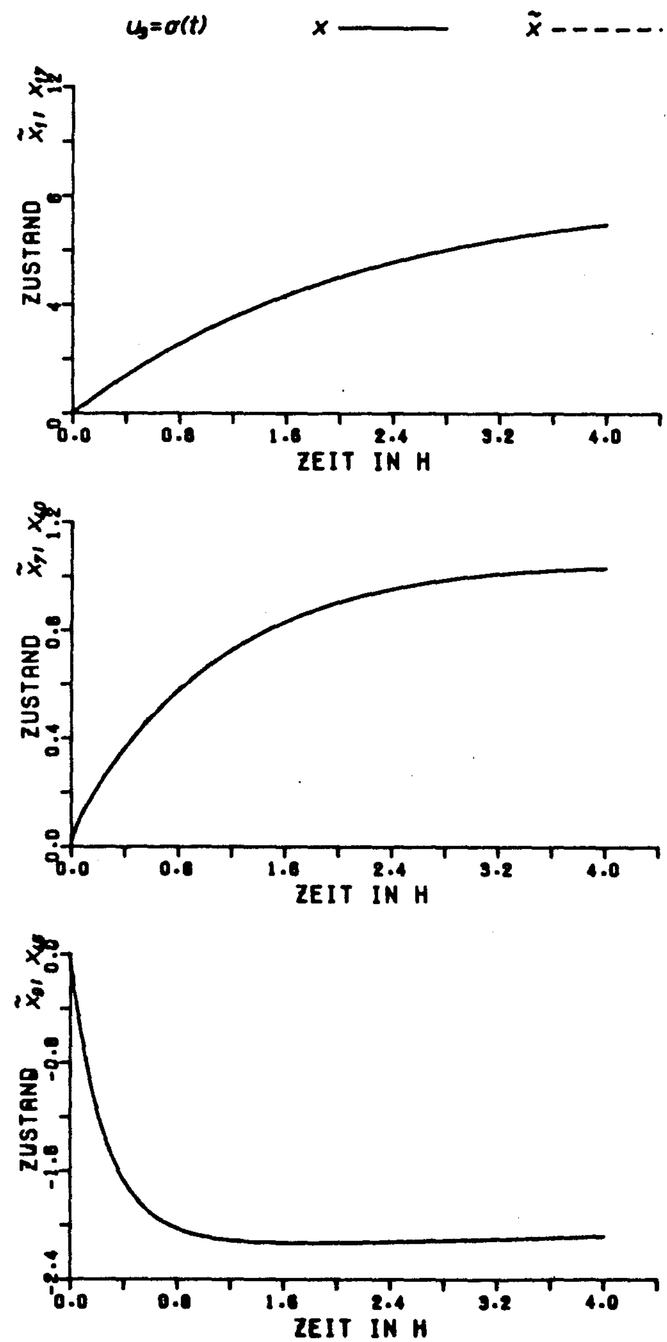
This is a physical process of BASF Corp. They came up with measurements and a state space model.

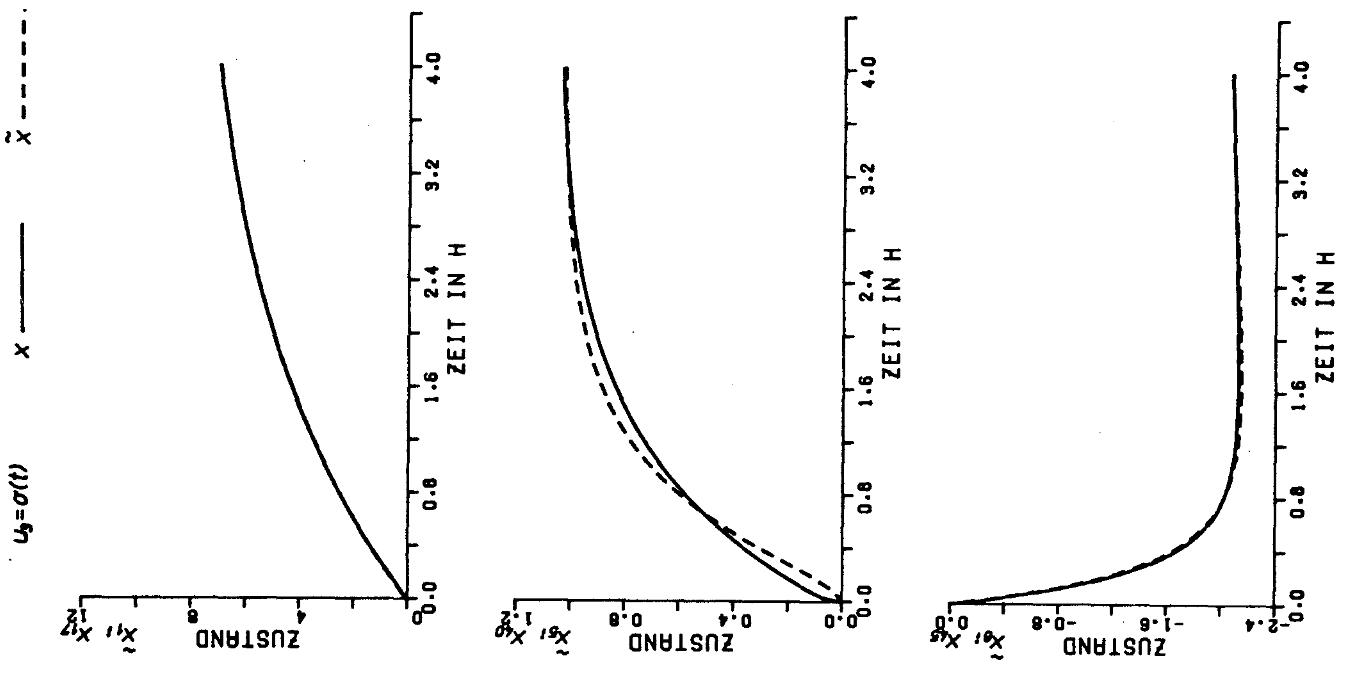
In the following, Eitelbergs technique applied to this process is illustrated for various orders of the reduced model:

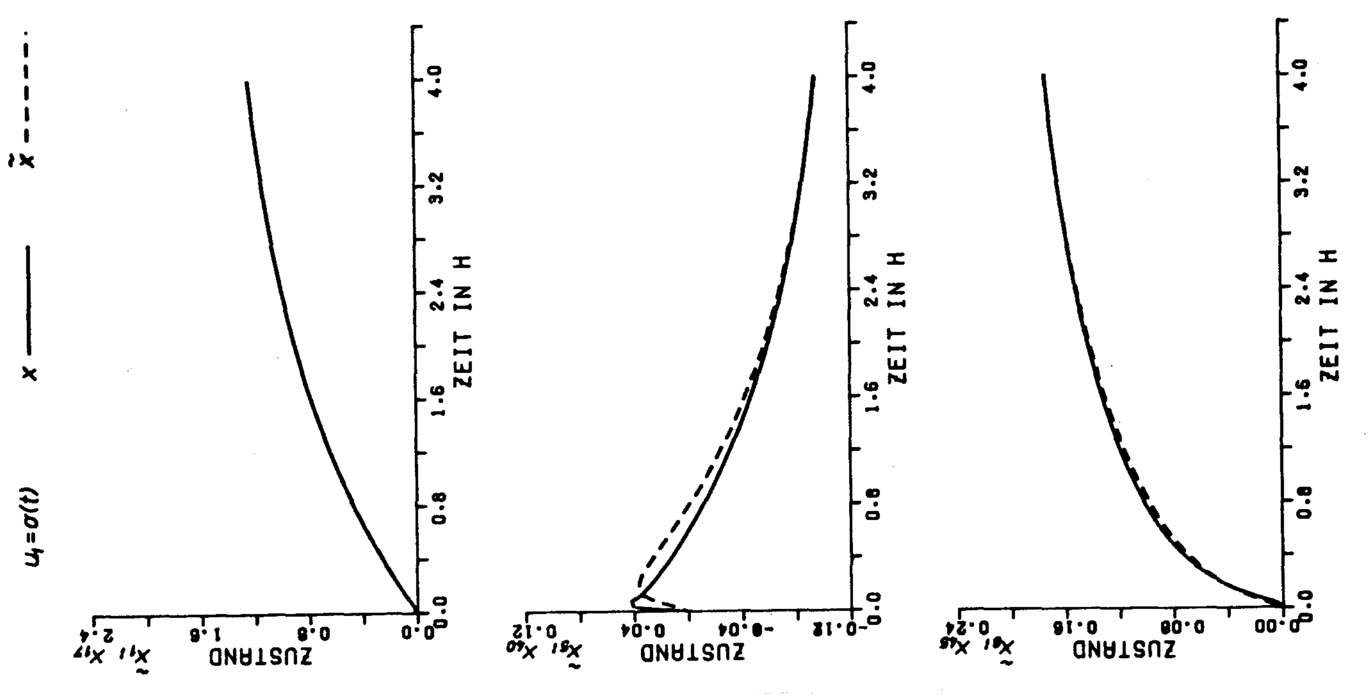

$u_1=\sigma(t)$
x
$\tilde{x}$
ZUSTAND
$\tilde{x}_{51}$ x_{17}
$\tilde{x}_{121}$ x_{40}
$\tilde{x}_{141}$ x_{45}
ZEIT IN H
0.0 0.8 1.6 2.4 3.2 4.0

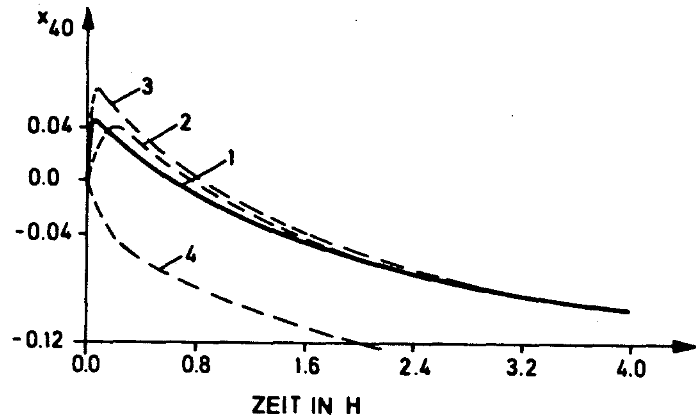

$u_1=\sigma(t)$
x
$\tilde{x}$
ZUSTAND
$\tilde{x}_{51}$ x_{17}
$\tilde{x}_{121}$ x_{40}
$\tilde{x}_{141}$ x_{45}
reduced
original
ZEIT IN H
0.00 0.04 0.08 0.12 0.16 0.20


$u_1 = \sigma(t)$
x ———
$\tilde{x}$ - - - - -
ZUSTAND
ZEIT IN H
0.0
0.8
1.6
2.4
3.2
4.0
0.04
-0.04
-0.12
0.12
0.00
0.08
0.16
0.24


$u_3 = \sigma(t)$
x ———
$\tilde{x}$ - - - - -
ZUSTAND
ZEIT IN H
0.0
0.8
1.6
2.4
3.2
4.0
4
8
12
0.4
1.2
-0.8
-1.6
-2.4

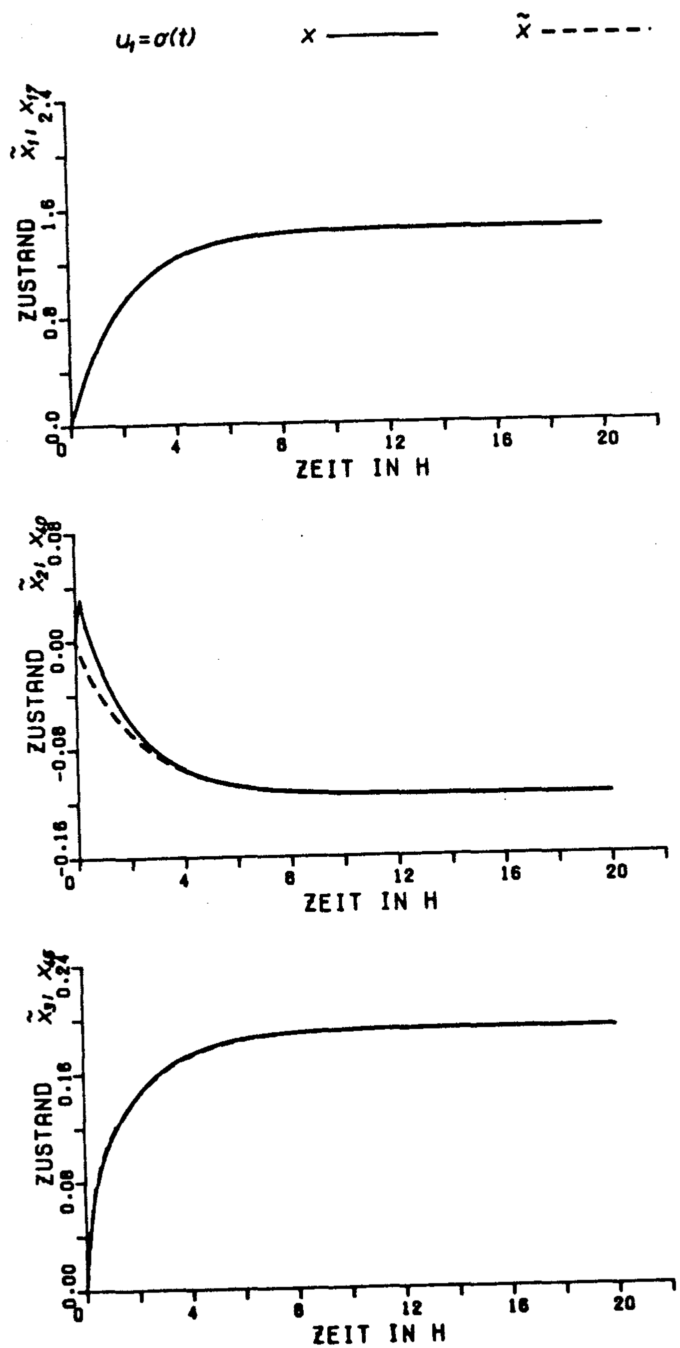

$u_3=\sigma(t)$
x
$\tilde{x}$
ZEIT IN H
ZUSTAND


$u_1=\sigma(t)$
x
$\tilde{x}$
ZEIT IN H
ZUSTAND

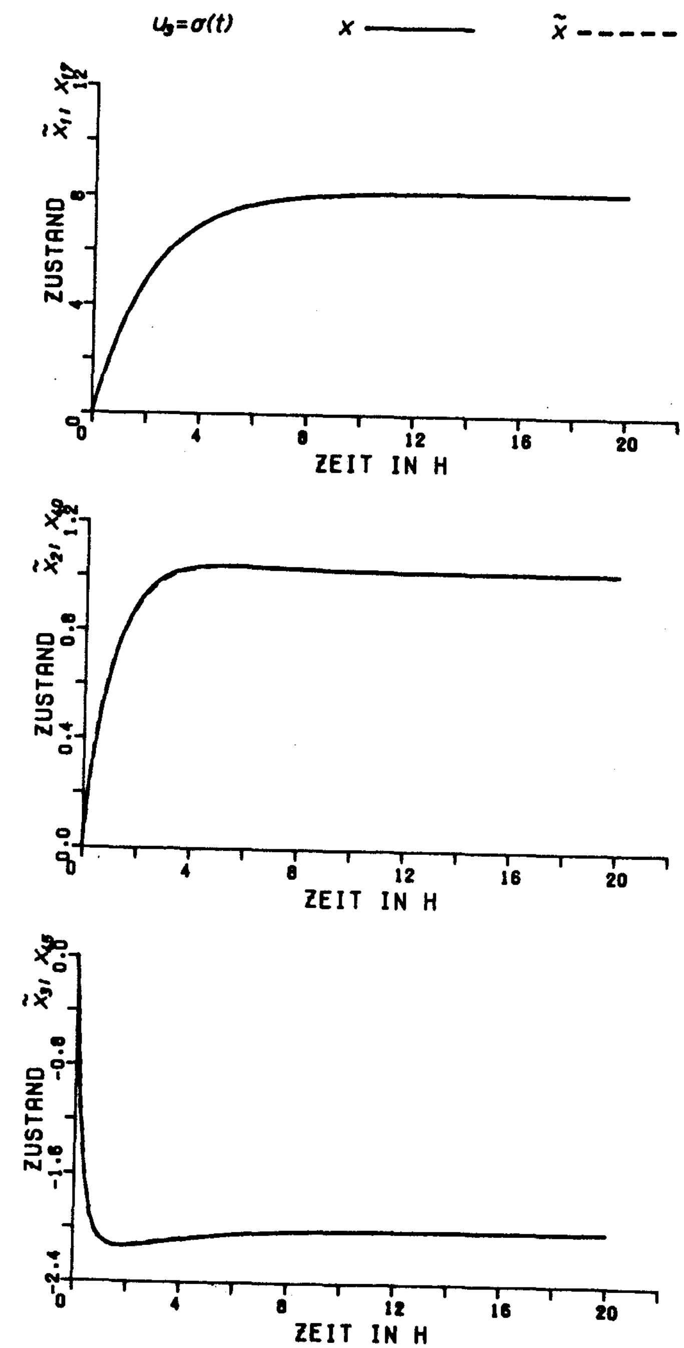


Comparison with other "known" techniques for 6th order reductions (the worst variable only):

1) original
2) Eitelberg
3) Marshall
4) Davison



$u_1=\sigma(t)$
x
$\tilde{x}$
ZUSTAND
$\tilde{x}_{11}\ x_{17}$
$\tilde{x}_{21}\ x_{40}$
$\tilde{x}_{31}\ x_{46}$
ZEIT IN H



$u_3=\sigma(t)$
x
$\tilde{x}$
ZUSTAND
$\tilde{x}_{11}\ x_{17}$
$\tilde{x}_{21}\ x_{40}$
$\tilde{x}_{31}\ x_{46}$
ZEIT IN H

We can learn from these results that:

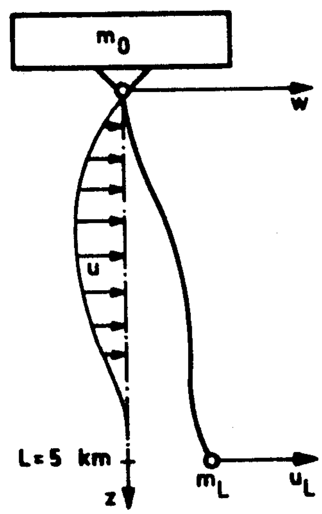
(1) Identification of high-order models is very problematic. As a low-order model can represent the data so beautifully, obviously I do not have enough information to identify parameters of a higher order model in a unique manner (always true !!!)

(2) For control purposes in particular, the low order approximation is just as good as the complete model (due to the feedback).

2) Deep Sea Pumping Equipment.

Applications: Pump oil out of north sea, pump mineral containing thermal leach out of red sea, etc.

Usually, length of pipe > 2000 m.



We want to control the position of the free end by the shipsmotor.

Assume: (1) Connection ship/pipe is torque free

(2) mass of ship >> mass of pipe

→ The displacement $w(z,t)$ is described by the hyperbolic PDE:

$$\frac{\partial^2 w}{\partial t^2} = 2\,\frac{\mu_F v_F}{\mu}\,\frac{\partial^2 w}{\partial z\,\partial t} - \frac{\beta_1}{\mu}\,\frac{\partial w}{\partial t} - \frac{\alpha}{\mu}\,\frac{\partial^4 w}{\partial z^4} + \left(\gamma - \frac{\mu_F v_F^2}{2\mu}\right)\cdot\frac{\partial^2 w}{\partial z^2} - \frac{\bar{\mu}_R g}{\mu}\,\frac{\partial w}{\partial z} + \frac{1}{\mu}\,u$$

where:

$$w\big|_{z=0} = 0\,,$$

$$\left.\frac{\partial^2 w}{\partial z^2}\right|_{z=0} = 0\,,$$

$$\left.\frac{\partial^2 w}{\partial z^2}\right|_{z=L} = 0 \quad \text{und}$$

$$\left.\frac{\partial^2 w}{\partial t^2}\right|_{z=L} = \frac{\alpha}{m_L}\,\frac{\partial^3 w}{\partial z^3} - g\,\frac{\partial w}{\partial z} + \frac{1}{m_L}\,u_L\,.$$

with:

$$\gamma = \frac{g}{\mu}\,(m_L + \bar{\mu}_R(L-z))$$

$$\mu = \mu_R + \mu_F\,.$$

From the pipe data:

ϕ = 0.5 m

density = 7830 $\frac{kg}{m^3}$

Thickness of wall ζ = 0.015 m

elasticity module = 200124 $\frac{N}{mm^2}$

length = 5000 m

we find:

$$\mu_R = \text{mass of pipe} = 173\ \frac{kg}{m}$$

$$\bar{\mu}_R = \text{reduced mass of pipe} = 150\ \frac{kg}{m}$$

$$\mu_F = \text{mass of liquid in pipe} = 180\ \frac{kg}{m}$$

$$\alpha = \text{torsion stiffness} = 142 \cdot 10^6\ \frac{kg}{m^2}$$

The following coefficients are variable:

$$\beta_1 = \text{damping} \in [10, 50]\ \frac{kg}{m \cdot s}$$

$$V_F = \left\{ \begin{matrix} \text{velocity of liquid} \\ \text{in pipe} \end{matrix} \right\} \in [0, 10]\ \frac{m}{s}$$

$$m_L = \text{ship mass (active)} \in [0, 10^5]\ kg$$

Assume:

$$\beta_1 = 20\ \frac{kg}{s \cdot m}$$

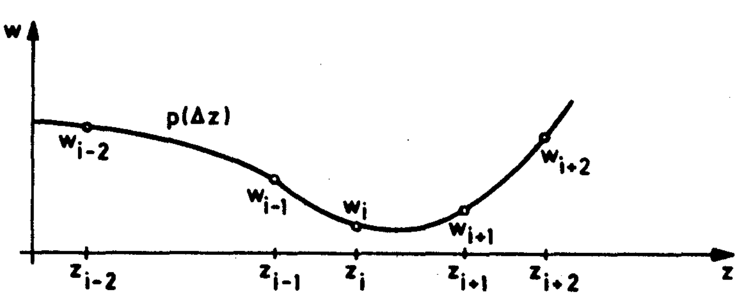
$$V_F = 5\ \frac{m}{s}$$

$$m_L = 10^4\ kg$$

Applying finite differences for the spatial derivatives leads unfortunately to very limited accuracy

⟹ requires very high order model for accurate approximation.

Instead, use approximation polynomia and an unequally distributed grid (basically finite elements !). As we have fourth derivative, the approximating splines must be of at least fourth order.



$$p(\Delta z) = c_o + c_1 \Delta z + c_2 \Delta z^2 + c_3 \Delta z^3 + c_4 \Delta z^4 ,$$

In the neighborhood of z_i :

$$w \approx p(\Delta z)$$

$\Rightarrow$

$$\left.\frac{\partial w}{\partial z}\right|_{z_i} \approx c_1 ,$$

$$\left.\frac{\partial^2 w}{\partial z^2}\right|_{z_i} \approx 2c_2 ,$$

$$\left.\frac{\partial^3 w}{\partial z^3}\right|_{z_i} \approx 6c_3 ,$$

$$\left.\frac{\partial^4 w}{\partial z^4}\right|_{z_i} \approx 24c_4 .$$

Coefficients c_1, c_2, c_3, c_4 are still unknown (different for each spline). $\Rightarrow$ Choose them such that we obtain regular difference formulae

$$p(0) = c_o \overset{!}{=} w_i ,$$

$$p(z_{i-2}-z_i) = p(\Delta_{-2,i}) =$$

$$= c_o+c_1\Delta_{-2,i}+c_2\Delta^2_{-2,i}+c_3\Delta^3_{-2,i}+c_4\Delta^4_{-2,i} \overset{!}{=} w_{i-2} ,$$

$$p(z_{i-1}-z_i) = p(\Delta_{-1,i}) =$$

$$= c_o+c_1\Delta_{-1,i}+c_2\Delta^2_{-1,i}+c_3\Delta^3_{-1,i}+c_4\Delta^4_{-1,i} \overset{!}{=} w_{i-1} ,$$

$$p(z_{i+1}-z_i) = p(\Delta_{1,i}) =$$

$$= c_o+c_1\Delta_{1,i}+c_2\Delta^2_{1,i}+c_3\Delta^3_{1,i}+c_4\Delta^4_{1,i} \overset{!}{=} w_{i+1} ,$$

$$p(z_{i+2}-z_i) = p(\Delta_{2,i}) =$$

$$= c_o+c_1\Delta_{2,i}+c_2\Delta^2_{2,i}+c_3\Delta^3_{2,i}+c_4\Delta^4_{2,i} \overset{!}{=} w_{i+2} .$$

$\Rightarrow$

$$\underline{C}_i \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} w_{i-2} \\ w_{i-1} \\ w_{i+1} \\ w_{i+2} \end{bmatrix} - w_i \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

where $\underline{C}_i$ is the Vandermonde matrix:

$$\underline{C}_i = \begin{bmatrix} \Delta_{-2,i} & \Delta^2_{-2,i} & \Delta^3_{-2,i} & \Delta^4_{-2,i} \\ \Delta_{-1,i} & \Delta^2_{-1,i} & \Delta^3_{-1,i} & \Delta^4_{-1,i} \\ \Delta_{1,i} & \Delta^2_{1,i} & \Delta^3_{1,i} & \Delta^4_{1,i} \\ \Delta_{2,i} & \Delta^2_{2,i} & \Delta^3_{2,i} & \Delta^4_{2,i} \end{bmatrix}$$

which is regular:

$$\Rightarrow \underline{CI}_i = \underline{C}_i^{-1}$$

$\Rightarrow$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \underline{CI}_i \begin{bmatrix} w_{i-2} \\ w_{i-1} \\ w_{i+1} \\ w_{i+2} \end{bmatrix} - w_i \, \underline{CI}_i \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

This corresponds to the central difference formulae:

$$\left.\frac{\partial^k w}{\partial z^k}\right|_{z_i} \approx d_{-2,k,i} w_{i-2} + d_{-1,k,i} w_{i-1} + d_{o,k,i} w_i + \\ + d_{1,k,i} w_{i+1} + d_{2,k,i} w_{i+2}$$

where:

$$d_{-2,k,i} = k!\ \underline{CI}_i(k,1)\ ,$$

$$d_{-1,k,i} = k!\ \underline{CI}_i(k,2)\ ,$$

$$d_{o,k,i} = -k! \sum_{j=1}^{4} \underline{CI}_i(k,j)\ ,$$

$$d_{1,k,i} = k!\ \underline{CI}_i(k,3)\ ,$$

$$d_{2,k,i} = k!\ \underline{CI}_i(k,4)\ .$$

We use the biased formulae for the boundary conditions:

$$\left.\frac{\partial^2 p}{\partial z^2}\right|_{z_{N+2}} = 2c_2+6c_3\Delta_{1,N+1}+12c_4\Delta^2_{1,N+1} \overset{!}{=} 0\ .$$

which leads to:

$$\underline{C}_{N+1}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} w_{N-1} \\ w_N \\ w_{N+2} \\ 0 \end{bmatrix} - w_{N+1}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

where:

$$\underline{C}_{N+1} = \begin{bmatrix} \Delta_{-2,N+1} & \Delta^2_{-2,N+1} & \Delta^3_{-2,N+1} & \Delta^4_{-2,N+1} \\ \Delta_{-1,N+1} & \Delta^2_{-1,N+1} & \Delta^3_{-1,N+1} & \Delta^4_{-1,N+1} \\ \Delta_{1,N+1} & \Delta^2_{1,N+1} & \Delta^3_{1,N+1} & \Delta^4_{1,N+1} \\ 0 & 2 & 6\Delta_{1,N+1} & 12\Delta^2_{1,N+1} \end{bmatrix}$$

Problem still to be solved: How do we determine appropriate supporting points?

We have (N+2) points (2 boundar[y], N internal).

Eitelberg chose NA uppermost interval (NA still unknown) equidistant, the NE+1 (NE = N−NA) lowermost intervals in a geometric chain

$$\Delta z_i = c_d \cdot \Delta z_{i-1} \; ; \; \forall i = N_A+3, \ldots, N$$

In between:

$$\Delta z_{NA+2} = c_{ii} \cdot \Delta z_{NA+1}$$

where:

$$c_{ii} = 1$$

$$c_d \leq 1$$

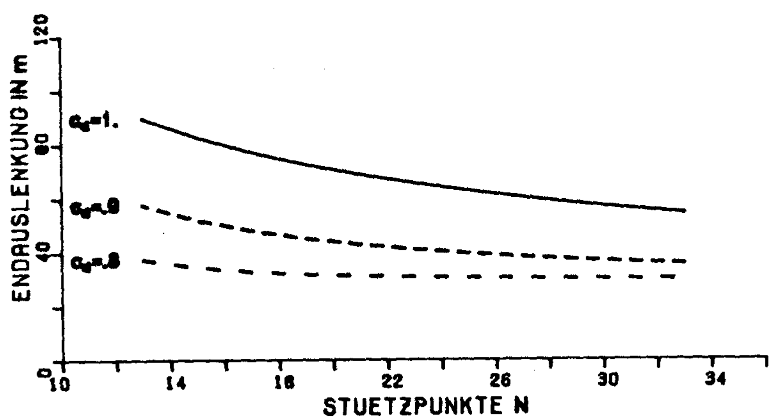
$$\Rightarrow \quad \Delta z_2 = \frac{L}{(N_A+1) + \sum_{k=1}^{N_E} c_d^{\,k}}$$

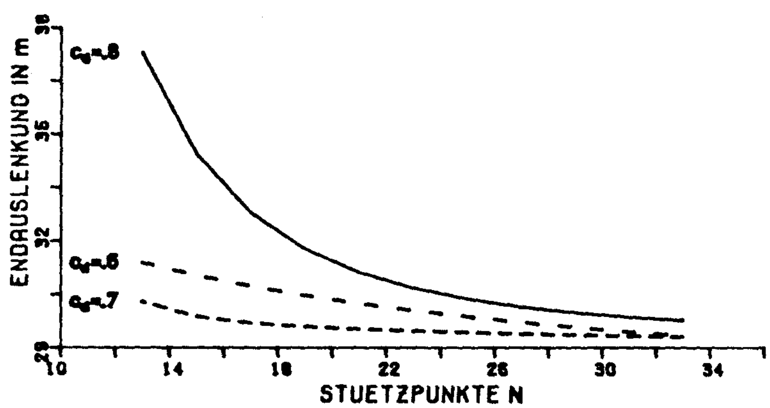
$$\Delta z_i \equiv \Delta z_2 \quad , \forall i = 3, \ldots, N_A+2$$

$$\Delta z_j = c_d \cdot \Delta z_{j-1} \quad ; \forall j = N_A+3, \ldots, N+2$$

He found by experimentation:

$$N_{E\,opt} = 11$$





Convergence of end-displacement depends heavily on c_d.

Optimum: $c_d = 0.7$ $\Longrightarrow$ smallest number of supporting point needed.

$\Longrightarrow$ $N_A = 8$; $N_E = 11$ $\Rightarrow$ $N = 19$

$c_d = 0.7$; $c_{ii} = 1$

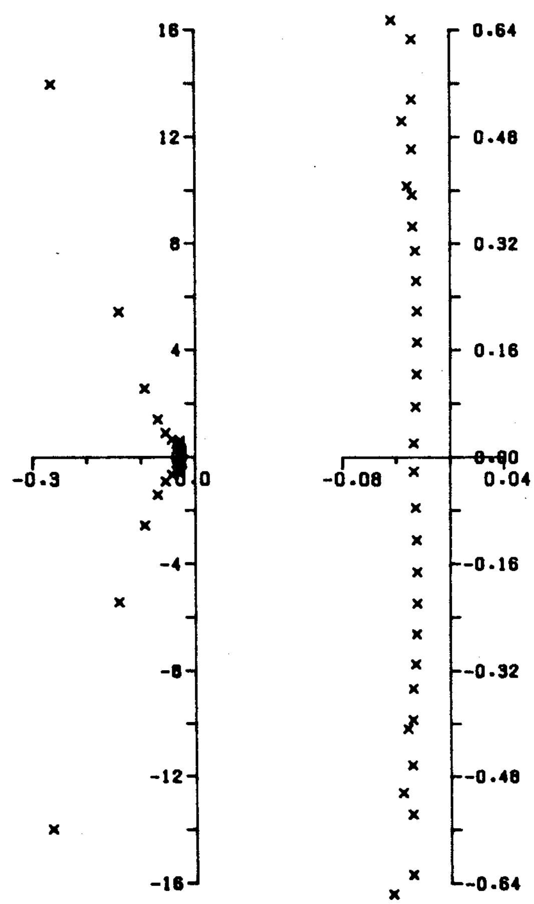
$\Longrightarrow$ supporting values at depths:

443
886
1329
1772
2215
2658
3101
3544
3987
4297
4514
4666
4772
4847
4899
4935
4961
4979
4991
5000

It is interesting to look at the eigenvalues of the 40th order model (each supporting value gives rise to 2 state variables:

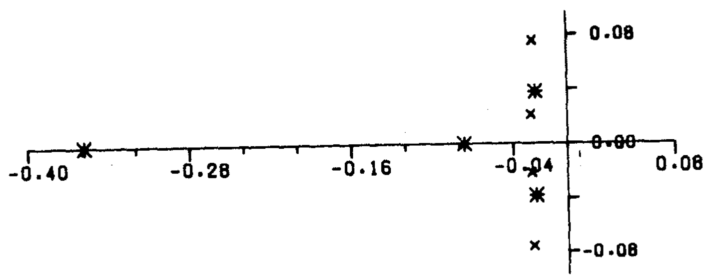
$$\left\{\begin{array}{l} x_i = w_j \\ x_{i+1} = \dot{w}_j \end{array}\right\}$$

)

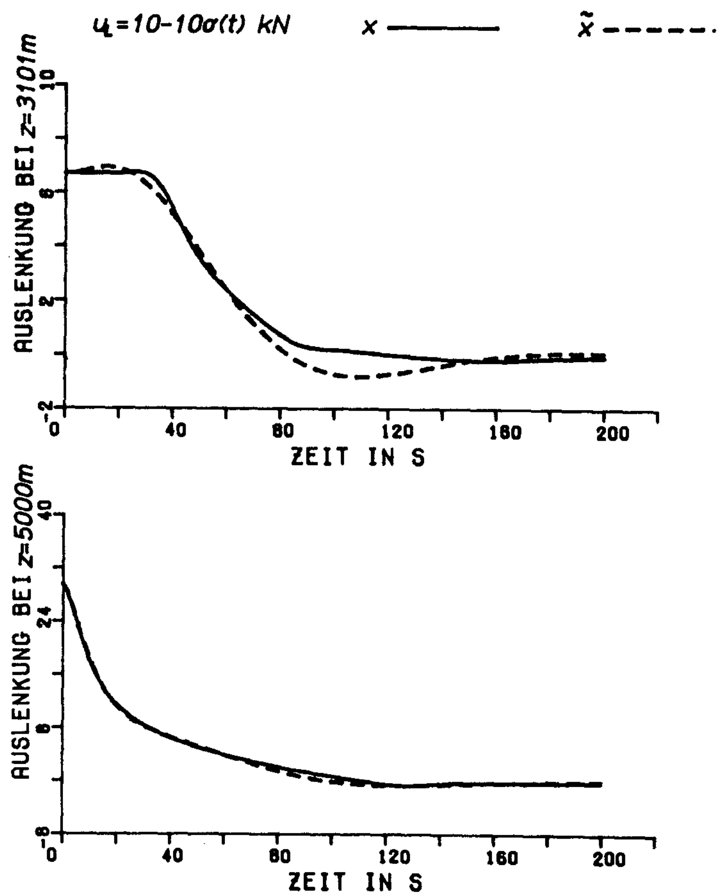


Reduction to fourth order:

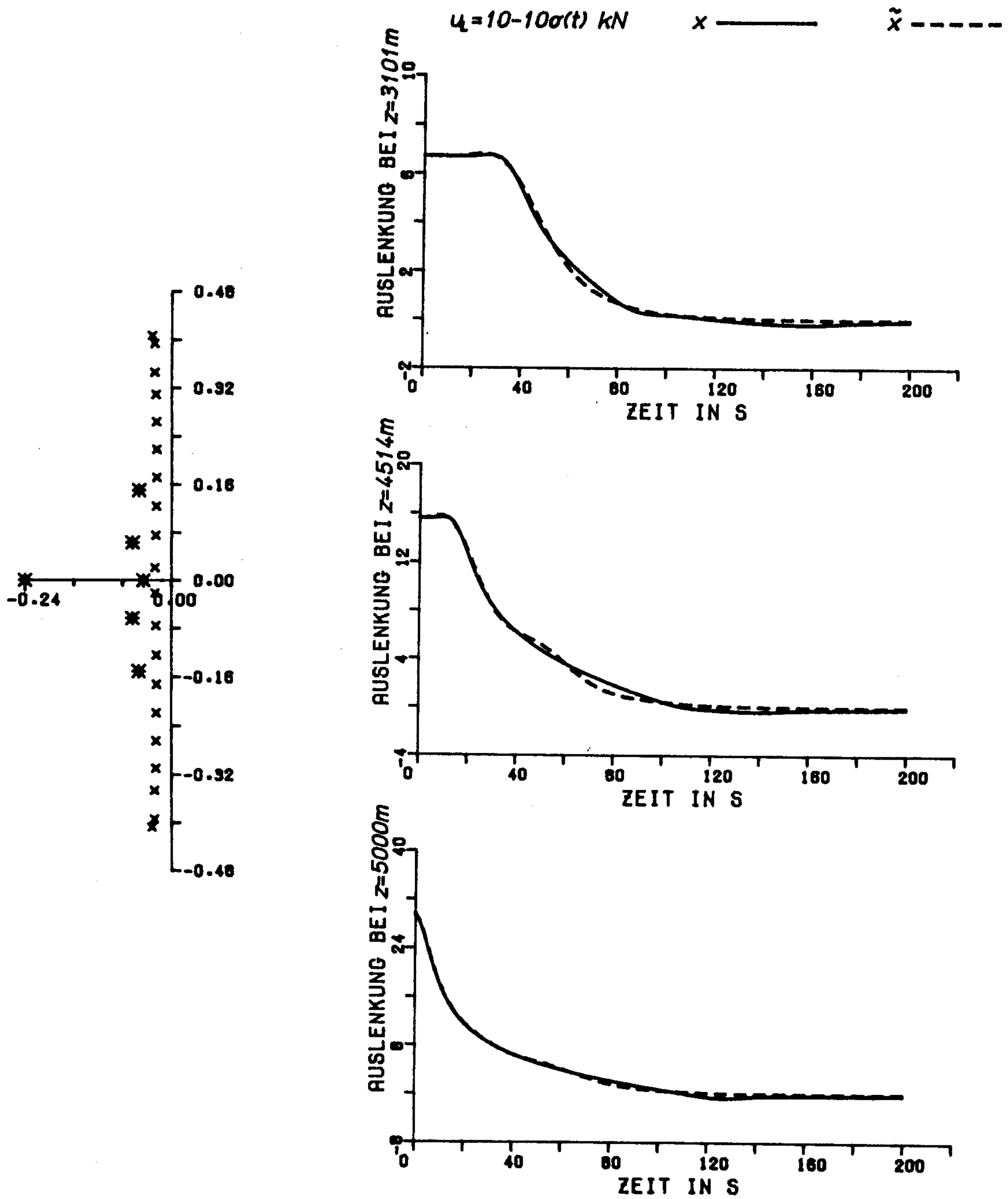
Eigenvalues:



Displacement:



Reduction to 6^{th} order:



Reduction to 8th order: