Gramians & Ballanced Realizations

Given the time-varying system:

$$\underline{\dot{x}} = A(t)\cdot\underline{x} + B(t)\cdot\underline{u} \quad ; \quad \underline{x}(\emptyset) = \underline{x}_0$$

This system is controllable if the controllability Gramian

$$G_c(t) = \int_0^t e^{A(\tau)\tau}\cdot B(\tau)\cdot B^*(\tau)\cdot e^{A^*(\tau)\cdot\tau}\,d\tau$$

is non-singular for all values $t > T$, where T is a suitable number $T < \infty$.

For a proof, cf. e.g. Kailath, chapter 9.2.1.

In the time-invariant case:

$$\underline{\dot{x}} = A\underline{x} + B\underline{u} \quad ; \quad \underline{x}(\emptyset) = \underline{x}_0$$

this condition simplifies to:

$$\boxed{G_c = \int_0^\infty e^{At}\cdot B\cdot B^*\cdot e^{A^*t}\,dt}$$

G_c is the controllability gramian.

The above forumla is true for <u>all</u> A and B.

Now, let us look at the special case, where A is stable:

$$\text{Real}\{\text{Eig}\{A\}\} < \emptyset \quad .$$

In this case, the computation of the controllability gramian can be further simplified.

$$\int_a^b U(t) \cdot \frac{dV}{dt} \cdot dt \equiv \left[U(t) \cdot V(t) \right] \Big|_a^b - \int_a^b \frac{dU}{dt} \cdot V(t)\, dt$$

is also true for U, V being matrices!

$$\int_0^\infty \underbrace{e^{At} \cdot B \cdot B^*}_{U(t)} \cdot \underbrace{e^{A^*t}}_{\frac{dV(t)}{dt}} dt$$

$$\Rightarrow \frac{dU(t)}{dt} = A \cdot e^{At} \cdot B \cdot B^*$$

$$V(t) = e^{A^* t} \cdot \underbrace{(A^*)^{-1}}$$

exists, since A assumed stable, i.e. $Eig\{A\} \neq \phi \Longleftrightarrow$ A is non-singular.

$$\Rightarrow G_c = \left[e^{At} \cdot B \cdot B^* \cdot e^{A^* t} \cdot (A^*)^{-1} \right] \Big|_0^\infty - \int_0^\infty A e^{At} \cdot B \cdot B^* \cdot e^{A^* t} \cdot (A^*)^{-1} \, dt$$

$$\lim_{t \to \infty} e^{At} = \emptyset^{(n)}$$ since A is stable

$$\Rightarrow e^{At} = V \cdot e^{\Lambda t} \cdot V^{-1}$$

$$\Rightarrow \lim_{t \to \infty} e^{At} = V \cdot \underbrace{\lim_{t \to \infty} e^{\Lambda t}}_{\phi^{(n)}} \cdot V^{-1}$$

$$\lim_{t \to \emptyset} e^{At} = I^{(n)}$$

$$\Rightarrow G_c = -B\cdot B^* \cdot (A^*)^{-1} - \underbrace{A\cdot\int_0^\infty e^{At}\cdot B\cdot B^*\cdot e^{A^*t}\,dt}_{G_c}\cdot (A^*)^{-1}$$

$$\Rightarrow G_c + B\cdot B^*(A^*)^{-1} + A\cdot G_c\cdot (A^*)^{-1} \equiv \phi$$

$$\Rightarrow \boxed{A\cdot G_c + G_c\cdot A^* + B\cdot B^* \equiv \phi}$$

This is a Lyapunov equation for the unknown G_c. This is a linear equation that can be solved by Gaussian elimination.

Example:

$$\dot{\underline{x}} = \begin{bmatrix} \phi & 1 & \phi \\ \phi & \phi & 1 \\ -2 & -3 & -4 \end{bmatrix} \underline{x} + \begin{bmatrix} \phi \\ \phi \\ 1 \end{bmatrix} u$$

$$\Rightarrow \begin{bmatrix} \phi & 1 & \phi \\ \phi & \phi & 1 \\ -2 & -3 & -4 \end{bmatrix}\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} + \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}\cdot\begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & -3 \\ 0 & 1 & -4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \equiv \phi$$

$$\Rightarrow \left|\begin{array}{l}
g_{21} + g_{12} = 0 \\
g_{22} + g_{13} = 0 \\
g_{23} - 2g_{11} - 3g_{12} - 4g_{13} = 0 \\
g_{31} + g_{22} = 0 \\
g_{32} + g_{23} = 0 \\
g_{33} - 2g_{21} - 3g_{22} - 4g_{23} = 0 \\
-2g_{11} - 3g_{21} - 4g_{31} + g_{32} = 0 \\
-2g_{12} - 3g_{22} - 4g_{32} + g_{33} = 0 \\
-2g_{13} - 3g_{23} - 4g_{33} - 2g_{31} - 3g_{32} - 4g_{33} + 1 = 0
\end{array}\right|$$

$$\Rightarrow \left[\begin{array}{ccc|ccc|ccc}
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-2 & -3 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & -2 & -3 & -4 & 0 & 0 & 1 \\
\hline
-2 & 0 & 0 & -3 & 0 & 0 & -4 & 1 & 0 \\
0 & -2 & 0 & 0 & -3 & 0 & 0 & -4 & 1 \\
0 & 0 & -2 & 0 & 0 & -3 & -2 & -3 & -8
\end{array}\right] \cdot \left[\begin{array}{c}
g_{11} \\ g_{12} \\ g_{13} \\ \hline g_{21} \\ g_{22} \\ g_{23} \\ \hline g_{31} \\ g_{32} \\ g_{33}
\end{array}\right] = \left[\begin{array}{c}
0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ -1
\end{array}\right]$$

In Matlab:

```
>> Gc = lyap(A, B*B')
```

This problem can be numerically improved (avoiding the multiplication of $B * B'$):

$$B = U \cdot \Sigma \cdot V^* \qquad \text{(singular value decomposition)}$$

$$\Rightarrow B \cdot B^* = U \cdot \Sigma \cdot V^* \cdot V \cdot \Sigma^* \cdot U^*$$
$$= U \cdot \Sigma^2 \cdot U^*$$

$$\Rightarrow A \cdot G_c + G_c \cdot A^* + U \cdot \Sigma^2 \cdot U^* = \emptyset$$

$$\Rightarrow U^* A \cdot G_c \cdot U + U^* \cdot G_c \cdot A^* \cdot U + \Sigma^2 = \emptyset$$

$$\Rightarrow U^* A \cdot \underbrace{U \cdot U^*}_{I^{(n)}} \cdot G_c \cdot U + U^* G_c \cdot \underbrace{U \cdot U^*}_{I^{(n)}} A^* \cdot U + \Sigma^2 = \emptyset$$

$$\Rightarrow \underbrace{(U^* A \cdot U)}_{\hat{A}} \cdot \underbrace{(U^* G_c \cdot U)}_{\hat{G}_c} + \underbrace{(U^* G_c \cdot U)}_{\hat{G}_c} \cdot \underbrace{(U^* A^* U)}_{\hat{A}^*} + \Sigma^2 = \emptyset$$

$$\Rightarrow \hat{A} \cdot \hat{G}_c + \hat{G}_c \cdot \hat{A}^* + \Sigma^2 = \emptyset$$

is again a Lyapunov equation.

$$\hat{G}_c = U^* \cdot G_c \cdot U$$

$$\Rightarrow G_c = U \cdot \hat{G}_c \cdot U^*$$

In Matlab:

```
function [Gc] = gram(A,B)
   [U,S,V] = svd(B);
   Ahat = U'*A*U;
   Gchat = lyap(Ahat, S*S');
   Gc = U*Gchat*U';
end
```

is a function available in Matlab.

Notice that from the symmetry of the Lyapunov equation, it follows that G_c is always Hermitian.

Proof:

$$A \cdot G_c + G_c \cdot A^* + B \cdot B^* = \emptyset$$

$$\Rightarrow (A \cdot G_c + G_c \cdot A^* + B \cdot B^*)^* = \emptyset$$

$$\Rightarrow G_c^* \cdot A^* + A \cdot G_c^* + B \cdot B^* = \emptyset$$

$$\Rightarrow A \cdot G_c^* + G_c^* \cdot A^* + B \cdot B^* = \emptyset$$

Since G_c is unique

$$\Rightarrow \boxed{G_c^* \equiv G_c}$$

q.e.d.

The observability problem is the dual problem to the controllability problem. Thus:

$$G_o = \int_0^\infty e^{A^* t} \cdot C^* \cdot C \cdot e^{At} \, dt$$

must be non-singular. If A is stable:

$$\boxed{A^* \cdot G_o + G_o \cdot A + C^* \cdot C = \emptyset}$$

is the corresponding observability gramian, again computed by means of a Lyapunov equation:

>> $G_c = \text{gram}(A, B)$
>> $G_o = \text{gram}(A', C')$

Of course, G_o is also Hermitian.

<u>Given</u>:

$$\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} + D\underline{u} \end{array} \right|$$

$$A \cdot G_c + G_c \cdot A^* + B \cdot B^* = \emptyset \implies G_c$$
$$A^* \cdot G_o + G_o \cdot A + C^* \cdot C = \emptyset \implies G_o$$

Similarity transformation:

$$\underline{\xi} = T \cdot \underline{x}$$

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{\xi}} = \tilde{A} \cdot \underline{\xi} + \tilde{B} \cdot \underline{u} \\ \underline{y} = \tilde{C} \cdot \underline{\xi} + \tilde{D} \cdot \underline{u} \end{array} \right|$$

where:

$$\left|\begin{array}{l}\tilde{A} = T\cdot A\cdot T^{-1}\\ \tilde{B} = T\cdot B\\ \tilde{C} = C\cdot T^{-1}\\ \tilde{D} = D\end{array}\right|$$

$$\tilde{A}\cdot\tilde{G}_c + \tilde{G}_c\cdot\tilde{A}^* + \tilde{B}\cdot\tilde{B}^* = \emptyset \Rightarrow \tilde{G}_c$$
$$\tilde{A}^*\cdot\tilde{G}_o + \tilde{G}_o\cdot\tilde{A} + \tilde{C}^*\cdot\tilde{C} = \emptyset \Rightarrow \tilde{G}_o$$

$$\Rightarrow T\cdot A\cdot T^{-1}\cdot\tilde{G}_c + \tilde{G}_c\cdot(T^{-1})^*\cdot A^*\cdot T^* + T\cdot B\cdot B^*\cdot T^* = \emptyset$$

$$\Rightarrow A\cdot T^{-1}\cdot\tilde{G}_c\cdot(T^*)^{-1} + T^{-1}\cdot\tilde{G}_c\cdot\underbrace{(T^{-1})^*}_{(T^*)^{-1}}\cdot A^* + B\cdot B^* = \emptyset$$

Comparison with original problem:

$$G_c = (T^{-1})\cdot\tilde{G}_c\cdot(T^{-1})^*$$

$$\Rightarrow \boxed{\tilde{G}_c = T\cdot G_c\cdot T^*}$$

$$(T^{-1})^* A^* T^*\cdot\tilde{G}_o + \tilde{G}_o\cdot TAT^{-1} + (T^{-1})^* C^*\cdot CT^{-1} = \emptyset$$

$$\Rightarrow A^*\underbrace{T^*\tilde{G}_o T}_{G_o} + \underbrace{T^*\tilde{G}_o T}_{G_o} A + C^*\cdot C = \emptyset$$

$$\Rightarrow G_o = T^* \cdot \tilde{G}_o \cdot T$$

$$\Rightarrow \boxed{\tilde{G}_o = (T^{-1})^* \cdot G_o \cdot (T^{-1})}$$

Special case: unitary transformation:

$$\underline{\tilde{x}} = U \cdot \underline{x}$$

$$\Rightarrow \left| \begin{aligned} \tilde{A} &= U \cdot A \cdot U^* \\ \tilde{G}_c &= U \cdot G_c \cdot U^* \\ \tilde{G}_o &= U \cdot G_o \cdot U^* \end{aligned} \right|$$

In this case (and in this case only!)

$$\left| \begin{aligned} &\tilde{G}_c \text{ is similar to } G_c \\ &\tilde{G}_o \text{ is similar to } G_o \end{aligned} \right|$$

It is possible to make either $\tilde{G}_c$ or $\tilde{G}_o$ diagonal by a unitary transformation.

Let:

$$G_c = R_c \cdot R_c^*$$

be a Cholesky decomposition of the Hermitian matrix G_c.

In Matlab:

```
>> Rc = chol(Gc')
```

Use svd:

$$R_c = U_c \cdot \Sigma_c \cdot V_c^*$$

$$\Rightarrow G_c = R_c \cdot R_c^* = U_c \cdot \Sigma_c^2 \cdot U_c^*$$

is the spectral decomposition of G_c. Thus:

```
function [Y]=chol(X)
    [v, l2] = eig(X');
    l2= diag(l);
    l = sqrt(l2);
    l = diag(l);
    Y= v*l*v';
return
```

Since G_c is Hermitian, computing its spectral decomposition is numerically harmless.

$$\Rightarrow A \cdot G_c + G_c \cdot A^* + B \cdot B^* = \emptyset$$

$$\Rightarrow A \cdot U_c \cdot \Sigma_c^2 \cdot U_c^* + U_c \cdot \Sigma_c^2 \cdot U_c^* \cdot A^* + B \cdot B^* = \emptyset$$

$$\Rightarrow \underbrace{U_c^* \cdot A \cdot U_c}_{} \cdot \Sigma_c^2 + \Sigma_c^2 \cdot U_c^* \cdot A^* \cdot U_c + U_c^* B \cdot B^* U_c = \emptyset$$

$$\overline{A} \cdot \Sigma_c^2 + \Sigma_c^2 \cdot \overline{A}^* + \overline{B} \cdot \overline{B}^* = \emptyset$$

$$\overline{A} = U_c^* \cdot A \cdot U_c = T \cdot A \cdot T^{-1}$$

$$\overline{B} = U_c^* \cdot B = T \cdot B$$

$$\Rightarrow \boxed{T = U_c^*}$$

is a unitary transformation that makes:

$$\boxed{\overline{G_c} = \Sigma_c^2}$$

to be diagonal.

Similarly:

$$G_0 = R_0 \cdot R_0^*$$

$$R_0 = U_0 \cdot \Sigma_0 \cdot V_0^*$$

$$\Rightarrow G_0 = R_0 \cdot R_0^* = U_0 \cdot \Sigma_0^2 \cdot U_0^*$$

$$A^* \cdot G_0 + G_0 \cdot A + C^* \cdot C = \phi$$

$$\Rightarrow A^* U_0 \Sigma_0^2 U_0^* + U_0 \Sigma_0^2 U_0^* \cdot A + C^* C = \phi$$

$$\Rightarrow U_0^* A^* U_0 \Sigma_0^2 + \Sigma_0^2 U_0^* A U_0 + U_0^* C^* C U_0 = \phi$$

$$\Rightarrow \bar{\bar{A}}^* \cdot \Sigma_0^2 + \Sigma_0^2 \cdot \bar{\bar{A}} + \bar{\bar{C}}^* \cdot \bar{\bar{C}} = \phi$$

$$\bar{\bar{A}} = U_0^* A U_0 = T \cdot A \cdot T^{-1}$$

$$\bar{\bar{C}} = C \cdot U_0 = C \cdot T^{-1}$$

$$\Rightarrow \boxed{T = U_0^*}$$

$$\Rightarrow \boxed{\bar{G}_0 = \Sigma_0^2}$$

We wish to find a similarity transformation that makes:

$$\hat{G}_c = \hat{G}_o = \Sigma^2 \text{ diagonal.}$$

Unfortunately, this cannot be accomplished using a unitary transformation.

$$\Sigma^2 = \hat{G}_c = T \cdot G_c \cdot T^* = \hat{G}_o = (T^{-1})^* \cdot G_o \cdot (T^{-1})$$

$$G_c = (T^{-1}) \cdot \Sigma^2 \cdot (T^{-1})^*$$

$$G_o = T^* \cdot \Sigma^2 \cdot T$$

$$\Rightarrow \underline{\underline{G_o \cdot G_c = T^* \Sigma^4 (T^{-1})^*}}$$

$$\Rightarrow \Sigma^4 = \mathrm{Eig}\{G_o \cdot G_c\}$$

$$(T^{-1})^* = \mathrm{Modal}\{G_o \cdot G_c\}$$

Although Σ^4 is always diagonal, T is <u>not</u> unitary, because

$G_o \cdot G_c$ is <u>not</u> Hermitian !

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}}_{\text{symmetric}} \cdot \underbrace{\begin{bmatrix} 3 & 4 \\ 4 & 1 \end{bmatrix}}_{\text{symmetric}} = \underbrace{\begin{bmatrix} 11 & 6 \\ 18 & 11 \end{bmatrix}}_{\underline{\underline{\text{not}}}\ \text{symmetric}} \quad !$$

Let me prove that Σ^4 is indeed diagonal.

$$\left.\begin{aligned} G_c &= U_c \cdot \Sigma_c^2 \cdot U_c^* \\ G_o &= U_o \cdot \Sigma_o^2 \cdot U_o^* \end{aligned}\right\} \text{(using svd)}$$

$$\Rightarrow G_o \cdot G_c = U_o \cdot \Sigma_o^2 \cdot U_o^* \cdot U_c \cdot \Sigma_c^2 \cdot U_c^*$$

$$= (U_o \cdot \Sigma_o) \cdot \underbrace{(\Sigma_o \cdot U_o^* \cdot U_c \cdot \Sigma_c)}_{H} \cdot (\Sigma_c \cdot U_c^*)$$

$$H = U_H \cdot \Sigma_H^2 \cdot V_H^* \quad \text{(using svd)}$$

$$\Rightarrow G_o \cdot G_c = U_o \Sigma_o \cdot U_H \cdot \Sigma_H^2 \cdot V_H^* \cdot \Sigma_c \cdot U_c^*$$

Let us choose:

$$T = \Sigma_H^{-1} \cdot U_H^* \cdot \Sigma_o \cdot U_o^*$$

$$\Rightarrow \begin{cases} T^* = U_o \cdot \Sigma_o \cdot U_H \cdot \Sigma_H^{-1} \\ T^{-1} = U_o \cdot \Sigma_o^{-1} \cdot U_H \cdot \Sigma_H \\ (T^{-1})^* = \Sigma_H \cdot U_H^* \cdot \Sigma_o^{-1} \cdot U_o^* \end{cases}$$

$$\Rightarrow \hat{G}_c = T \cdot G_c \cdot T^*$$

$$= \Sigma_H^{-1} \cdot U_H^* \cdot \Sigma_o \cdot U_o^* \cdot U_c \cdot \Sigma_c^2 \cdot U_c^* \cdot U_o \cdot \Sigma_o \cdot U_H \cdot \Sigma_H^{-1}$$

$$= \Sigma_H^{-1} \cdot U_H^* \cdot (\Sigma_o \cdot U_o^* \cdot U_c \cdot \Sigma_c) \cdot (\Sigma_c \cdot U_c^* \cdot U_o \cdot \Sigma_o) \cdot U_H \cdot \Sigma_H^{-1}$$

$$= \Sigma_H^{-1} \cdot U_H^* \cdot (U_H \cdot \Sigma_H^2 \cdot V_H^*)(V_H \cdot \Sigma_H^2 \cdot U_H^*) \cdot U_H \cdot \Sigma_H^{-1}$$

$$= \Sigma_H^{-1} \cdot \Sigma_H^2 \cdot \Sigma_H^2 \cdot \Sigma_H^{-1} \equiv \Sigma_H^2$$

is diagonal!

$$\hat{G}_o = (T^{-1})^* \cdot G_o \cdot (T^{-1})$$

$$= \Sigma_H \cdot U_H^* \cdot \Sigma_o^{-1} \cdot U_o^* \cdot U_o \cdot \Sigma_o^2 \cdot U_o^* \cdot U_o \cdot \Sigma_o^{-1} \cdot U_H \cdot \Sigma_H$$

$$= \Sigma_H \cdot U_H^* \Sigma_o^{-1} \cdot \Sigma_o^2 \cdot \Sigma_o^{-1} \cdot U_H \cdot \Sigma_H$$

$$= \Sigma_H \cdot U_H^* \cdot U_H \cdot \Sigma_H = \Sigma_H^2$$

$$\Rightarrow \quad \underline{\underline{\hat{G}_c \equiv \hat{G}_o \equiv \Sigma_H^2}}$$

q.e.d.

Because of the occurrence of Σ_o^{-1} and Σ_H^{-1}, the algorithm only works for systems that are fully controllable and observable.

<u>In Matlab:</u>

```
function [Abal, Bbal, Cbal, Mode, T] =
    balreal (A, B, C)
Gc = gram(A,B);
Go = gram(A',C');
[Uc, Sc, Vc] = svd(Gc);
[Uo, So, Vo] = svd(Go);
 Sc = diag(diag(Sc)^0.5);
 So = diag(diag(So)^0.5);
 H = So * Uo' * Uc * Sc;
[UH, SH, VH] = svd(H);
T = diag(diag(SH)^0.5)\UH' * So * Uo';
 Abal = T*A/T;
 Bbal = T*B;
 Cbal = C/T;
 Mode = diag(SH);
 return
```